TIME MEASUREMENT AS AN INTERDISCIPLINARY SUBJECT IN MATHEMATICS EDUCATION

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ABSTRACT

Nowadays measuring time is considered almost instinctively a subject trivially known and well established in our everyday life. However it has been interesting, scientifically fascinating and mathematically nontrivial throughout the ages, constituting a "meeting point" of disciplines epistemologically very remote from each other, based on quite different motivations and having different objectives. It is an exceptionally rich subject that could be beneficial and stimulating in the context of mathematics education at various instructional levels, in a variety of situations, and be approached from many different perspectives. This paper aims at providing evidence to support this point in the context of an appropriate HPM framework, by means of some characteristic examples, which - either explicitly or implicitly - are related to the calendar as we know it today or has been developed historically.

1 Introduction

1.1 The context

Unlike space, time can be conceived and delimited only if represented in terms of symbols, which themselves require and/or are susceptible to different interpretations. As a result, understanding time (more than understanding space) implies the imperative need to rationalize its representation in terms of arithmetical or geometrical symbols of a most unequivocal meaning. Perhaps this is the generic element underlying the relation between the concept of time and its mathematical elaboration (cf. Borst, 1993, pp. 5-6). Therefore, as a fundamental formative category of human thought and perception of the world, conceiving time is indissolubly connected to its quantification through measurement.

On the other hand, inherent to the concept of measurement is the act of comparison; namely, to compare objects with respect to a certain characteristic they possess in common, by agreeing to choose and choosing one among them as a standard of comparison; the unit of measurement of this common characteristic. This is true for any kind of measurement. In the case of time, this means specifying and using time units that are "stable" enough. This, in turn, is realized by focusing on **periodic** (cyclic) phenomena. (Enough) "stability" in this context means the existence of periodic phenomena **compatible** with each other; i.e. **their periods' ratios do not change** (appreciably) in the course of human life, society's existence, or generally, during a specified corpus of (individual and/or collective) human experience (cf. Fraser, 1987, ch. 2 pp. 58-59). And it is a fundamental **empirical fact** that phenomena do exist for which this condition is (approximately) fulfilled. Somewhat loosely, they can be called "clocks".

From a historical point of view two points should be particularly stressed: (i) it was first perceived and understood that astronomical and physical clocks compatible in the above sense do exist. Therefore, they were preferably selected for quantifying (thus, measuring) time; (ii) it was gradually (and slowly) realized that their compatibility was only approximate; that the more human observations and experimentations were becoming finer, the more rough this approximate compatibility was getting. As we shall argue, this development was strongly interrelated with that of the required mathematics.

1.2 The "*three basic clocks*"

It is a fact that since early in history, time measurement has been based on what henceforth is called the "*three basic clocks*" (Borst, 1993, ch. 2, p.9; Whitrow, 1988, ch. 2, pp. 14-17; Richards, 1998, ch. 1, pp. 7-8):

- The *day*: Earth's rotation (around its axis) causing the day-night alternation;
- The *month* (and the week): Moon's revolution (around the earth) causing the (four) lunar phases.
- The *year*: Earth's revolution (around the sun) causing the succession of the seasons and the return of the fixed stars to the same position in the sky;

For later use and for comparison, the **current** values of their periods' ratios are given below, using the **modern** definition of (atomic) *second* in SI units, the currently accepted system of units in physics (International System of Units, n.d.):

$$1 \text{ (atomic) sec} := \frac{9{,}192{,}631{,}770}{\nu}$$

where ν is the frequency of radiation emitted due to the transition between the two hyperfine energy levels of the ground state of the Caesium isotope ${}^{133}_{55}\text{Cs}$. In this unit, the "day" is by definition 86,400sec

$$t_{\text{D}} = \frac{\text{day (SI)}}{\text{sec}} := 86{,}400 \tag{1.1}$$

whereas, the three basic clocks' periods are to a reasonably good degree of approximation (Richards, 2013, §15.1.3; for the terms "tropical" and "synodic" see §2.3).

$$t_{\text{Y}} = \frac{\text{year}}{\text{day}} := \frac{\text{(mean) } \textbf{tropical} \text{ year}}{\text{day (SI)}} = 365.2421897 = 365^{\text{d}}\, 5^{\text{h}}\, 48'45'',19 \tag{1.2}$$

$$t_{\text{M}} = \frac{\text{month}}{\text{day}} := \frac{\text{lunar (mean) } \textbf{synodic} \text{ period}}{\text{day (SI)}} = 29.5305885 = 29^{\text{d}}\, 12^{\text{h}}\, 44'2'',88 \tag{1.3}$$

$$t'_{\text{D}} = \frac{\text{day}}{\text{day (SI)}} = 86{,}400.003 \tag{1.4}$$

We note that (i) the expression of the above periods as **decimal** fractions of the **day** and **sexagesimal** fractions of the **hour** already underlies the complexity of the measurement problem[1]; (ii) the *second* is not a concept as simple as one may think, but is founded on the current theory of **microphysics**, far beyond any "naïve" approach to the measurement of the periods of the basic clocks (Auerbach, 1995; Nelson *et al*, 2001).

1.3 The *social* context and the fundamental *mathematical* constraint

These "innocent-looking" and taken for granted to be simple notions (day, month, year), will be found to be very complicated both conceptually and technically.

Firstly, their implementation is strongly delimited by the **social context**: For several political, religious and economic reasons social life is requested to be based on "simple" temporal cycles; i.e. **integral** periods of appropriate periodic phenomena.

Secondly, it is a **mathematical fact** that there are no simple **integer** relations among the periods of the **empirically** determined three basic clocks; i.e. their periods' ratios

[1] E.g. that sun's annual revolution in the sky is not 12 lunar cycles, or 365 days as originally thought. Due to the complexity of these periodic phenomena it took a long time in history to realize that these are not good approximations for long time intervals, and to find better approximations of their relative periods.

cannot be expressed as rational numbers sufficiently simple to be practical for civil purposes in a straightforward manner. Evidently, given any prescribed degree of accuracy (determined by social or scientific criteria) and (implicitly) conceiving (real) numbers as a continuum, such rational relations may in principle result by choosing sufficiently small time units. However, this presupposes an ever increasing accuracy of the corresponding measurement processes.

To summarize, all this indirectly points to the (in principle) importance of specific mathematics both conceptually and technically in relation to the measurement of time by means of the three basic clocks. Therefore, this paper is organized as follows. Section 2 gives evidence that measuring time has been a multifaceted, interdisciplinary and intercultural issue throughout its historical development, based on mathematical knowledge available at the time, possibly stimulating its further development as well. It is also argued that aspects of this development can be beneficial for mathematics education. In section 3 a general framework for integrating history in mathematics education is outlined. Finally, in section 4 this main point is exemplified by five specific examples, at the same time commenting on their placement in the general framework of section 3.

2 Motivation

2.1 The basic questions

Two basic questions have been implicit so far:
(1) **Why** was/is it **important** to determine accurate and stable **time units**?
(2) **How** was/is it **possible** to determine accurate and stable **time-keeping** in terms of such **time units**?
In other words: In the context of which domains of human activity and knowledge, and in relation to which questions and problems was this important?

These two interrelated questions are related to mathematical issues, either elementary or advanced when judged by current standards. In fact, measuring time is closely related, touches upon and addresses interconnected issues in the context of several distinct disciplines, leading to questions and problems that finally led to important developments both in mathematics and in these disciplines. However, because time (as a quantified concept) is so deeply rooted into our (modern) civilization, its accurate measurement seems to be an elementary subject, at least conceptually. As a consequence, its emergence (hence, the awareness of the time concept itself; see §1.1) and the way it has been interrelated deeply with many intellectual, practical, political and religious aspects of human history is hardly appreciated in general, and in education in particular (for a concise overview see Whitrow, 1972a).

Therefore, in relation to question (1), it is helpful to comment briefly on some of the reasons **why** accurate time-keeping and time units have been important:

History: Already in antiquity it was realized that it is important to have trustful means to reckon historical facts (i.e. some sort of accurate chronology) by carefully considering the appearance of each historical fact in relation to others' (Smyntyna, 2009; Borst, 1993, ch. 2); e.g. Herodotus' and Thucydides' historical accounts in ancient Greece, Varro's chronology in the Roman republic period starting "from the founding of the City [Rome]" (AUC; *Ab Urbe Condita*), Flavius Josephus' account of Jewish history (1st century AD), Bede's "Ecclesiastical History of the English People" (8th century AD) etc.

Politics: Early in history it became necessary to have a measure of the duration of the term of persons and collective bodies holding public positions (e.g. Athenian Archons, Roman Consuls etc), or/and to reckon events by means of regnal periods or eras. This was achieved by using regularly occurring or important and exceptional events; e.g. the Olympiads in ancient Greece; counting years from the founding of Rome (AUC) in ancient Rome; the Diocletian era (or the era of the Martyrs – *Anno Diocletiani*) used by the Alexandria's early Christian church and starting on the first regnal year of the Roman emperor Diocletian (284AD); the AD (*Anno Domini*) era introduced in the 6th century by Dionysius Exiguus replacing the Diocletian era and used in the Christian world since then; the *Hegira* era in the Islamic world and starting with Mohammed's migration to Mecca (622AD); the republican era of the French Revolution (starting on 1792 AD), etc (Hannah, 2005, chs. 3, 5; Richards, 1998, ch. 6, pp. 104-109; Fraser, 1987, pp. 91-95).

Economy: For economic reasons, temporal cycles were specified already in antiquity; e.g. the division of the month by the Romans into *calends*, *nones* and *ides*[2], the calends indicating both a month's first day (originally starting with a new moon) and the day on which debts should be paid according to the accounting books called *kalendaria* (Hannah 2005, ch. 5; Holford-Strevens, 2005, pp. 28-31; Richards, 1998, p. 210; Whitrow, 1988, p. 68); or, longer temporal cycles like the Roman 15-year *indictions* - a fiscal period for the agricultural or land taxes' reassessment adopted by the Byzantines, used in medieval Europe and kept until late in modern times (Whitrow, 1988, p. 67; Richards, 1998, p. 101). Moreover, from the late Middle Ages to the industrial revolution, the gradual rise of a "money economy" led to the need of measuring and paying for the human work and wages, which led to the conception of time's uniformity, thus requesting its measurement in terms of appropriate, well-defined, "unchanging" units (Whitrow, 1988, pp. 108-110)[3]. It is interesting that already *The Treviso Arithmetic* (the earliest known printed arithmetic book; 1478) concerning commercial arithmetic for the general public, includes calendrical calculations for finding the date of Easter Sunday (Swetz, 1987, pp. 164-168). Though looking strange nowadays, these were important for merchants because civil holidays and religious feast days (greatly determined by the celebration of Christian Easter; see below) put constraints on activities related to trade and human labour (Swetz, 1987, pp. 248-253).

Theology: Since early Christianity – especially after it was no longer persecuted in the roman empire from the early 4th century AD onwards – the variety of habits and rituals adopted by different early Christian groups in relation to their worship led to an imperative request to establish a **uniform** religious canon, to be followed by the clergy all over the Christendom. Such a canon presupposed the ability of a sufficiently accurate determination of the time when each religious activity is done. In this connection developing methods for the indisputable determination of the Christian Easter day was of immense importance; a theological problem that acted as a catalyst in the development of the western civilization's concept of time, its measurement, and effective methods for doing complicated numerical calculations. It is worth noting that "computation" comes from the Latin "*Computus*", which - since the *Computus Paschalis* of Cassiodorus' disciples in the mid 6th century AD - signified the calculation method to determine the calendar date of the Christian Easter

[2] In Latin: *Kalendae*, *Nonae*, *Ides*, the modern word *calendar* coming from the first.

[3] Except the astronomers in the Hellenistic period, dividing the day into 24 **equal** hours stems from this, contrary to the unequal hours in earlier periods (12 *temporal* hours for the daylight period, varying with the season and geographical latitude; 12 *equinoctial* hours for the night period; the vague *canonical* hours of the Christian monasticism for religious and other duties (Richards, 1998, p. 44, Whitrow, 1988, pp. 28, 108).

(Borst, 1993, pp. 28-29; Swetz, 1987, p. 33). Ever since in the Middle Ages, finding Easter Sunday's date on any given year, and the AD time reckoning (introduced by Cassiodorus' contemporary, Dionysius Exiguus) were interlinked (Borst, 1993, ch. 4; Duncan, 1998, ch. 5). Along the same lines *St Benedict's Rule* by Benedict of Nursia (early 6th century AD) contains precepts for his monks on their daily occupations, duties and worship, which gradually were spread out beyond the clergy to the entire society. By dividing the day into canonical hours, "Benedictine monasticism in the early Middle Ages formed the basis of the modern European measurement and discipline of time"[4] (Borst, 1993, p. 3; see also Richards, 1998, ch. 30; Duncan, 1998, ch. 5).

Geography and ***Navigation***: Although finding the geographic latitude of a place on earth is simple (being the inclination of the celestial pole(s) to the place's horizon), the accurate determination of geographic longitude requires a sufficiently accurate determination of time in order to be able to compare the geographic longitude of two different places by observing the position of the sun (or other celestial bodies) at a given moment[5]. This was very important for explorers, especially during the great expeditions in the Renaissance. It greatly motivated and stimulated technical methods and their theoretical background for developing high accuracy time-keeping devices; especially, marine chronometers for determining the position at sea (because of the unavoidable movement of any object on a ship, accurate marine chronometers required much more elaborate techniques; Whitrow, 1988, ch. 9; Newton, 2004, chs. 4, 5).

Similarly, several disciplines are involved in relation to question (2) (**how** accurate time keeping and time units have been determined):

Astronomy: Through the systematic study of the periodic motion of celestial bodies and their periods' determination as accurately as possible (cf. §1.2).

Physics: Through the study of specific periodic phenomena and the determination of the physical laws governing them. These range from mechanical systems of great mathematical and historical interest (like Galileo's *simple pendulum*, or Huygens *cycloidal pendulum*; §4.5), to modern crystal and atomic clocks based on understanding microscopic periodic phenomena (oscillations of atoms and nuclei).

Technology: Through the construction of devices operating accurately as artificial periodic phenomena, mutually compatible in the sense of §1.1; from devices based on macroscopic phenomena like water clocks, sundials and mechanical clocks (that use springs or pendulums and some type of "escapement mechanism"; Appendix B), to modern high-precision clocks based on microscopic vibrations of crystals and nuclei like those of SiO_4, ${}^{133}_{55}Cs$, ${}^{1}_{1}H$, ${}^{87}_{37}Rb$ (Fraser, 1987, ch. 2 pp. 45-75; Whitrow, 1972b, ch. 4).

This outline of the social and scientific domains in which accurate time measurement has been important, at least indirectly suggests that educationally it can also be beneficial.

2.2 Examples: A short list

From an educational perspective, the above outline of the social and scientific domains in which accurate time measurement has been important, suggests that the

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[4] In this context, an hour indicated not a fixed period of time in today's sense, but less precisely specified parts of the day devoted to religious and other duties (Whitrow, 1988, p. 108). It is worth noting that modern *siesta* comes from Benedict's *sexta hora* that included a midday break for rest (Borst, 1993, pp. 26-27).

[5] A 4' time difference corresponds to 1^{o} difference in longitude; about 111km along the equator. Finding the geographic longitude in this way is a mathematically nontrivial astronomical problem (Smart, 1971, ch. XIII).

study of its development in history and in different cultures, constitutes a multifaceted, strongly interdisciplinary area touching upon a variety of subjects. Or, aspects of it provide insightful examples that in elaborated form could illuminate and reveal the crucial role of (nowadays considered classical, elementary, or even trivial) mathematics in addressing and tackling problems in several different disciplines and shaping man's ever-changing view of the world. An indicative list of interrelated examples and their not always obvious relation to (often deep) mathematical issues is:

(a) Measuring the **compatibility** of the three "basic clocks" (§1.1) requires the use of rational numbers. Since this involves increasingly more accurate measurements, hence successive approximations, this problem relates to fractions and their decimal[6] expansions.

(b) **Temporal cycles**, i.e. regularities among the three "basic clocks", require looking for their common multiples. This may involve deeper mathematics, e.g. congruences in Number Theory (§4.1).

(c) The accurate determination of the **periods** of the three "basic clocks" and their ratios, especially for periods very long compared to the duration of an individual's life, is related to the search for a **calendar** both physically correct and computationally simple enough to be understood and used by the laymen, hence convenient for civil purposes. This may involve much of the theory of continued fractions (§4.2).

(d) Finding the **week day on a given date** (especially **Easter Sunday**) involves clever **tabulation and treatment** of **data**. This is greatly facilitated by data parameterization using algebraic representations, symbolism and operations, ranging from elementary school algebra manipulations, to more sophisticated algebraic modelizations appropriate for algorithms to be used by modern computers (§§4.3, 4.4).

(e) Specifying and constructing **accurate** (mechanical) **clocks** greatly stimulated the development of important parts of mathematics. Seen in a modern context, it involves a lot of mathematics: from Calculus and differential equations (e.g. Galileo's simple pendulum), to the geometry of plane curves (e.g. Huygens' cycloidal pendulum); §4.5.

In all these cases (except (e)), the *leitmotiv* is the existence and acquaintance with the **positional number system**, automatically taken for granted nowadays, though it is not so either historically or didactically!

2.3 Temporal cycles: A mixture of astronomical facts & social conventions

Before outlining an HPM framework and considering in its context these examples, it helps to present the *temporal cycles* involved, thus giving hints into why and how mathematical issues are also involved. Determining such adequate cycles consists of searching for integer common multiples of the basic clocks' periods, and difficulties result because there are no simple rational relations among the *tropical year*[7], the *lunar synodic month* (or *lunation*)[8] and the *civil month* (from 28 to 31 days), when measured in *days*!

2.3.1 The *Metonic* cycle

Geminus' (*Introduction to the Phenomena* - Elementa Astronomiae) and Ptolemy

[6] Or using other bases; e.g. sexagesimal expansions were used in medieval astronomical calculations (§4.2).
[7] The time between two successive passages of the sun from the *vernal equinox*, which is the intersection point of the celestial equator and the *ecliptic* (sun's (apparent) annual orbit around the earth) when the sun passes from the southern to the northern celestial hemisphere (Smart, 1971, §86).
[8] The time moon takes to return to the same position relative to the earth-sun line; i.e. lunar phases to be repeated (Smart, 1971, §83).

(*Almageste* - Syntaxis Mathematica) mention that earlier astronomers in Athens had observed that to a very good approximation lunar phases are repeated on the same date every 19 years; i.e. 19 tropical years t_Y are nearly equal to 235 lunations t_M, or approximately 6940 days (though doubtful, this is attributed to Meton, 5th century BC; Heath, 1991, pp. xvii, 140-142). With t_Y the value of the Julian year (t_Y=365.25=365+1/4) established as the official duration of the civil year later by Julius Caesar (§4.2) and t_M in two decimals (t_M=29.53=29+1/2+1/33; §1.2), we get 19t_Y=6939.75 and 235t_M=6939.55. With [x] the integer part of x, this means

19-year ***Metonic* (*lunar*) *cycle***: [19t_Y] = [235t_M] = 6940^d

The tiny discrepancy of about 0^d.2 per cycle was taken into account later by considering longer periods. Callipus of Cyzicus (4th century BC) noticed that a better approximation results if one day is omitted every four lunar cycles (4×19=76 years) because $\frac{6940}{19} = 365 + \frac{5}{19} = 365 + \frac{1}{4} + \frac{1}{4\times 19}$ (Heath, 1991, *op.cit*; Hannah, 2005, pp. 55-58; Richards, 1998, pp. 33, 96, 198; Whitrow, 1988, pp. 45, 189).

It is insightful to consider this from a different perspective using *continued fractions*, a historically much later concept closely related to Euclid's algorithm for the greatest common divisor of two integers (Khinchin 1964; Vinogradov 1954, §I.4): By (1.2), (1.3)

$$\frac{t_Y}{t_M} = 12.36827 = 12 + 0.36827$$

to five decimals. Developing the decimal part as a continued fraction

$$0.36827 = \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{17 + \cfrac{1}{287/218}}}}}}}$$

gives its *convergents* as decimal and fractional approximations (Table 2.1)

Convergents of 0.36827	1/2	1/3	3/8	4/11	7/19	123/334
t_Y/t_M	25/2	37/3	**99/8**	136/11	**235/19**	4131/334
Decimal approximation	12.5	12.333	12.375	12.364	12.3684	12.3682
			Octaeteris		**Metonic** cycle	

Table 2.1

including not only the Metonic cycle, but also the earlier less accurate *Octaeteris* mentioned by Geminus and introduced by Cleostratus of Tenedos c.6th century BC (Heath, 1991, pp. xvi, 137-138; Hannah, 2005, pp. 35-39; Richards, 1998, pp. 94-95; Whitrow, 1988, p. 45); an 8-year cycle used by the Greeks to reckon time in terms of Olympiads (50 lunar months for one Olympiad and 49 for the next).

2.3.2 The *Solar* cycle

Because 365 = 52×7+1 ≡ 1 mod7, in a 365-day year divided into 7-day weeks each date on a given year moves forward by 1 weekday from one year to the next. Therefore, if there were no leap years, calendars would repeat every 7 years. But because there is one leap year every 4 years and 28 is the least common multiple of 7, 4, this happens every 28 years; i.e. every 28 years each date of the year falls on the same weekday. This is the *Solar cycle* (valid for the Julian calendar; §4.2) with the convention, that cycles **start** on a **leap year** with **January 1** being a **Monday** (Richards, 1998, p. 303).

2.3.3 The *Indiction* cycle

As mentioned in §2.1, an *Indiction* (in Latin: *Indictio*, meaning "declaration", "statement", "tax") was a fiscal period for reassessing agricultural or land taxes', originally introduced in Roman Egypt as a 5-year cycle. Constantine the Great extended it into a 15-year period for imposing taxes, starting on 313AD. Since 313AD was the first year of an indiction, any AD year Y is the year of an indiction cycle given by

$$\text{Indiction Year of } Y \equiv (Y+2) \bmod 15 + 1 \qquad (2.1)$$

3 Measuring Time as a subject of/in Mathematics Education: An *HPM* framework

In the previous sections enough evidence was given to support that time measurement is a multifaceted interdisciplinary subject, whose detailed treatment certainly touches upon mathematical issues presupposing a great deal of what is considered to be elementary or trivial mathematical knowledge nowadays. Identifying those aspects of this subject that could be beneficial in the context of mathematics education is facilitated by considering it in the context of an appropriate educational framework. In this section this is done with reference to an HPM framework presented in more detail in previous work and which refers to six interrelated aspects of integrating historical issues in mathematics education (Tzanakis, 2016, §3; Clark *et al*, 2016, §2.3; Clark *et al*, 2018, §1.3).

More specifically, this framework is structured along the following questions: In the context of mathematics education, **which history** is suitable, pertinent and relevant? Having **which role** and **objective**? Serving in **which way**? By following **which approach** and implementing **which methodological scheme(s)**?

Although these questions point to the key issues to be addressed while integrating history in mathematics education and provide the spectrum of possible relevant aspects to be considered, no unique answer is to be expected in each particular case, because approaches may vary in size and scope according to the specific didactical aim, the subject matter, the level and orientation of the learners, the available didactical time, and external constraints (curriculum regulations, number of learners in a classroom etc). With this in mind, these questions are considered in relation to time measurement and possible answers are outlined to be further detailed by considering specific examples in section 4.

3.1 Which *history*?

The question of ***which history** is suitable, pertinent, and relevant for didactical purposes* has been a permanently debated issue among historians and educators with no easy answer (Fried, 2001; Clark *et al*, 2018 §1.3.1; Clark *et al*, 2019 §4.2). Grattan-Guinness' proposed distinction of what he calls *history* and *heritage* was an important step towards clarifying conflicts and tensions among mathematicians', educators' and historians' view of mathematical knowledge. In brief, a *history* perspective focuses on what happened in the past and why did it happen (or did not happen), whereas a *heritage* perspective focuses on what impact (new) knowledge had on later work and the ways it was embodied in later contexts. These are equally important perspectives for understanding the development of mathematics comprehensively. However, they are complementary to each other in the sense that although both are legitimate, they are incompatible because muddling them is not permissible since this may lead to a distorted view of the past (Grattan-Guinness, 2004a, b; Tzanakis, 2016, §3.1, Tzanakis & Thomaidis, 2012, §12.2).

In this context, though a *history* perspective is certainly possible for problems related to time measurement (e.g. by elaborating on the mathematics underlying many of the points outlined in section 2 placed in the appropriate historical context), adopting a *heritage* perspective seems to be more suitable for didactical purposes; e.g. when exploring the development of the calendar or the clock, to stress the fact that modern life is almost unthinkable without them and raise the question whether they "were always there?", aiming to help learners get aware of "why and how did we get to the present situation".

3.2 With which *role* and *objective*?

(a) The question of ***which role*** *the history of mathematics can play in mathematics education* has been discussed and analyzed considerably on the basis of a priori theoretical and epistemological arguments and empirical research. Nowadays there is consensus that generally, history can play one or more of three mutually complementary roles or functions (Barbin, 1997; Furinghetti *et al*, 2006, pp. 1286-1287; Jahnke *et al*, 2000, §9.1; Jankvist, 2013, §7; Clark *et al*, 2019 §4.3):

Replacement: To replace mathematics as usually understood (a corpus of results consisting of finished and polished intellectual products), by something richer (not only such intellectual products, but also a vivid intellectual activity including the mental processes leading to these products).

Reorientation: To look at what is familiar and taken for granted, from a different perspective as something that has not always been existing in its currently established form; hence to make it appear less familiar. Thus modifying the conventional perception of mathematical knowledge as something "time-independent", into the deeper awareness that mathematics is an evolving human intellectual activity and that mathematical knowledge is potentially subject to changes; i.e. *historicity* is one of its ontological characteristics.

Cultural: To help appreciating mathematical knowledge as an integral part of human intellectual history in the development of society; hence, perceiving mathematics from perspectives that lie beyond its currently established boundaries as a discipline.

Though in principle all three roles may be relevant in the context of problems related to time measurement (e.g. by appreciating the significance of using a positional number system, which is considered as something instinctively familiar to us today), the cultural role can be dominant: Time measurement problems in historical perspective can help to appreciate that facts and customs taken for granted nowadays, emerged via mathematics (often) simple by modern standards, under the strong influence of factors, problems and questions of social, political or religious origin and focus.

(b) In connection with the ***objective*** of integrating the history of mathematics in mathematics education, there are five main areas in which this could be beneficial:

(i) The learning of mathematics;

(ii) The development of views on the nature of mathematics and mathematical activity;

(iii) Teachers' didactical background and pedagogical repertoire;
(iv) The affective predisposition towards mathematics;
(v) The appreciation of mathematics as a cultural-human endeavour.

Each of them can be analyzed into finer objectives, providing in this way a more detailed description of history's role(s) in the educational process (Tzanakis *et al*, 2000, §7.2; Tzanakis & Thomaidis, 2012, §12.3; Clark *et al*, 2018 §4.3).

Implicit to the discussion in section 2 (to become clearer in section 4), is that time measurement problems in historical perspective could be beneficial
- for (iii); by interrelating mathematics with other disciplines, providing interesting, non-trivial recreational problems, and enriching the teaching of mathematics with historically important questions from other domains;
- and for (v); e.g. by considering the historical background for the emergence of the calendar and the mathematics needed and/or developed for this purpose; the non-mathematical (but socially important theological) problem of finding Easter Sunday; the problem of determining geographic longitude; the mathematics underlying the construction and operation of accurate clocks and their importance etc.

3.3 In which *way*?

In relation to the ***way*** history could serve in mathematics education, Jankvist (2009) made an important distinction between history serving (i) as a *tool* for assisting the actual learning and teaching of mathematics; and (ii) as a *goal* in itself for the teaching and learning of the historical development of mathematics (a similar distinction was made by Furinghetti, 2004; 2019, §5). These are complementary ways to integrate history in mathematics education, in the sense that they may co-exist, though not necessarily being of equal weight, depending on the other factors analyzed in this section. In case (i) history functions as a motivational, cognitive or affective tool to assist and support learning mathematics. In case (ii) history poses questions and suggests answers about the development of mathematics, identifies and explores the (intrinsic or extrinsic to mathematics) driving forces of this development and its cultural and societal aspects.

In a historical perspective of time measurement problems, history serving as a *goal* is expected to be dominant while considering particular cases related to the two basic questions of §2.1; e.g. the questions why it was important and how it was possible to construct accurate clocks as a source of stimulation for developing the mathematics required for their answer (§4.5). Or, by elaborating on the religious and political importance of an accurate calendar and the difficulties encountered without the (nowadays) "simple" mathematics; the positional number system and the algorithms of arithmetic operations, the algebraic symbolism and its elementary use, basic concepts and methods in number theory (like congruences and their properties), etc.

3.4 Following which *approach* and implementing which *methodological scheme*?

(a) Following Tzanakis *et al* (2000, §7.3) for the classification of the approaches to integrate history in mathematics education, there are three broad possibilities that may be combined (thus complementing each other), each one putting emphasis on a different issue: (i) *providing direct historical information*, with emphasis on learning history; (ii) *implementing* *a teaching approach* (explicitly or implicitly) *inspired by history*, with emphasis on learning mathematics; (iii) *focusing on mathematics as a discipline and the cultural and social context* in which it has been evolving, with emphasis on developing awareness of its evolutionary character, epistemological characteristics, relation to other disciplines and the influence exerted by factors both intrinsic and extrinsic to it.

Though for time measurement problems all three approaches are possible (for (ii) see e.g. Anderson n.d.), it is more direct to follow a combination of (i) and (iii). For example, to discuss the search for an accurate calendar and elaborate on different proposals in different historical periods and cultures and their relative advantages or disadvantages either mathematical or non-mathematical; e.g. the Julian calendar and its amended successors like the Gregorian calendar and its long forgotten but more accurate rival proposed by Omar Khayyam several centuries earlier (§4.2).

(b) The methodological schemes to be employed for integrating history in mathematics education, are classified by Jankvist (2009, §6; cf. Clark *et al*, 2018, §1.3.3) into three broad categories: (i) *Illumination approaches* in which teaching and learning is supplemented by historical information of varying size and emphasis; (ii) *Module approaches* in the form of instructional units devoted to history, often based on specific cases; (iii) *History-based approaches* in which history shapes the sequence and the way of presentation, often without history appearing explicitly, but rather being integrated into teaching.

As mentioned at the beginning of this section, implementing any particular approach and methodological scheme in a specific case depends on several additional factors. Having this in mind and the interdisciplinary and multifaceted character of time measurement problems, (i) and (ii) seems better suited to these problems. E.g. indicative examples of a *module approach* could be: (1) The calendar in the western world: Its history and mathematical background (§4.4; cf. Anderson, n.d.); (2) The clock, its history and the underlying physico-mathematical basis (cf. §4.5). Similarly, indicative examples of an *illumination approach* could be: (1) To find the week day of a given date, focusing on presenting various methods and their history, or/and emphasizing their modelisation using the algebra of congruences in number theory (§4.3); (2) The astronomers' Julian date, its historical origin and number-theoretic basis (§4.1).

In section 4, the above framework is illustrated by means of specific examples.

4 Examples

Below five examples illustrate in more detail what has been presented in sections 1 to 3.

4.1 The astronomers' Julian date: Its origin, Gauss & the Chinese remainder theorem in Number Theory

4.1.1 The historical issue

The *Julian date* (JD) is a time reckoning, measuring in days the time elapsed since 1 January 4713BC. It was introduced in 1583 by the French classical scholar Joseph Justus Scaliger and is still indispensable in Astronomy (Smart, 1971, §90; Richards, 2013, §15.1.10; IAU, 2017, §2.3). Scaliger considered this date as the beginning of a very long temporal cycle of 7980 years! These mysterious at first glance numbers were not chosen arbitrarily, however. On the contrary, their justification provides an example rich in interrelations among number theory, astronomy, the history of

Chinese mathematics, and historical-theological considerations. Here we touch upon some of them briefly.

One year after the Gregorian reform of the calendar, Scaliger in his *Opus novum de emendatione temporum* (*New treatise on amending time* [chronology]), he made elaborate calendrical calculations (which he called *Computi Annales*) and by starting to count sufficiently backwards in time, he introduced a new long temporal cycle aiming to disentangle chronology from the absoluteness of religious creeds and unreliable records, and to avoid the difficulties of reconciling the 3 "clocks"[9] (Borst, 1993, pp. 104-106). The argumentation had theological motivation and was based on placing the year 1AD within the three temporal cycles of §2.3:

(i) Dionysius Exiguus had introduced the AD era in 525AD (§2.1). Knowing that there was a new moon on 23/3/323AD, he readily concluded by simple counting that there was a new moon on 1/1/325AD as well. He considered this as an important theological coincidence, because this was the year of the *Council of Nicaea* which created the (first part of the) Nicene Creed and the Christian Easter celebration canon. Because of this, he considered 323AD as the 1st year of a Metonic cycle. But 323 ≡ 0 mod19, which implies that "year zero", that is 1BC[10] was also the start of a Metonic cycle (Richards, 1998, pp. 350-351), hence

1AD is a 2nd year of a Metonic cycle

(ii) Moreover, by eq(2.1),

1AD is a 4th year of an Indiction cycle

(iii) It is known that in the Julian calendar (§4.2), the year of the *Council of Nicaea* started on Friday; i.e. 1/1/325AD was a Friday. Therefore, 1/1/328AD was a Monday of a leap year, hence by the definition of the 28-year solar cycle (§2.3.2), this was a 1st year of a solar cycle. Since 328 ≡ 20 mod28

1AD is a 10th year of a Solar cycle

4.1.2 The mathematical problem

Instead of going into Scaliger's early elaborate approach, we outline Gauss' treatment in his *Disquisitiones Arithmeticae* (Gauss, 1801, Part II §36; Ore, 1988, pp. 245, 247).

The mathematical problem consists of finding the year x (1AD) which is the 2nd year of a Metonic cycle, the 4th year of an Indiction cycle, and the 10th year of a Solar cycle; i.e.

$$x \equiv 10 \bmod 28, \quad x \equiv 2 \bmod 19, \quad x \equiv 4 \bmod 15 \tag{4.1}$$

Since (28,19,15) are pair-wise relatively prime, by the *Chinese Remainder Theorem* (Dence & Dence, 1999, §4.5; Ireland & Rosen, 1982, §3.4; Vinogradov, 1954, §IV.3) a solution x exists, **unique** modulo 28×19×15=7980

$$x \equiv (10x_1 + 2x_2 + 4x_3) \bmod 7980 \tag{4.2}$$

where (x_1, x_2, x_3) is the solution of the auxiliary system of congruences

$$x_1 = 19\times15y_1 \equiv 1 \bmod 28,\ x_2 = 28\times15y_2 \equiv 1 \bmod 19,\ x_3 = 19\times28y_3 \equiv 1 \bmod 15 \tag{4.3}$$

This can be solved easily to get (y_1, y_2, y_3) = (45, 10, 28), and therefore

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[9] Reiner in Paderborn Germany (12th century), making one of the first uses of Arabic numerals and the decimal system, had argued already that all time reckoning methods introduced significant errors over long time periods (Borst, 1993, pp. 73-74).

[10] There is no zero AD. In astronomy years BC are given by non-positive integers, with 0 for 1BC; Richards, 2013, §15.1.9.

$$x \equiv 196234 \bmod 7980 = 4714 \quad (4.4)$$

as the **unique** solution in a long temporal cycle of 7980 years, starting on 4713BC; the temporal cycle on which the JD is based.

This is a rich example that can be extended in several directions in a *heritage*-like perspective (§3.1), ranging from insights into the history of Chinese mathematics[11], the appearance of similar problems in Fibonacci's *Liber Abbaci* (Sigler, 2002, pp. 402-403) and the importance of this book for the emergence of arithmetical concepts and methods, or the significance of Gauss' *Disquisitiones Arithmeticae*[12], to subjects less central to mathematics like the origin and significance of the temporal cycles of §2.3, etc. Here, history has a cultural role that helps both to appreciate the cultural aspects of (otherwise abstract) mathematical problems and to provide teachers' with resourceful material (§3.2). Clearly, in this example, the historical development (both extrinsic and intrinsic to mathematics) is the main goal to a large extent (§3.3) by focusing on the cultural and societal aspects of the problem and implementing an illumination approach, e.g. in the context of a course in number theory and its applications (§3.4).

4.2 The optimal *leap year rule* & continued fractions: Julius Caesar, Omar Khayyam & Pope Gregory XIII

4.2.1 The problem in historical perspective

This example concerns the historically long struggle to reconcile two of the basic clocks (the year and the day), in a way applicable for civil purposes and easy enough to be understood by people. This has been a difficult problem for several reasons:

(i) There is **no unique definition** of the three basic "clocks" because of the relative motions of the sun, moon and earth, and because these motions are quite complicated as a result of the complicated interaction dynamics among these bodies and the other planets. This gives rise to several periodic phenomena with slightly different periods. However, these differences cannot be ignored over long time intervals and/or if high precision is required, leading to many refinements of the basic clocks. For the *year*: sidereal, (mean) tropical, anomalistic, lunar; for the *month*: sidereal, (mean) synodic, anomalistic, nodical; for the *day*: sidereal, apparent solar, mean solar, day (SI). Conventionally, the *tropical year* t_Y (for the seasons), the *synodic month* t_M (for the lunar phases) and the *day* (SI) t_D (§§1.2, 2.3) are used (for details see e.g. Smart, 1971, ch.VI and §§24, 28, 81-83, 86).

(ii) For social reasons, **both** the day & the (mean) tropical year are important.

(iii) The ratio *tropical year/day* is **not** an **integer**, eq(1.2). This required introducing additional days, done in diverse ways in different historical periods and/or cultures (Richards, 1998, part II). E.g. the calendar in ancient Egypt (util the 1st century BC) consisted of 12 30-day months plus 5 additional (or "intercalation") days, yielding a year of 365 days; i.e. $[t_Y]$, the lowest-order approximation to t_Y (Richards, 2013, §15.2.1).

(lowest approximation) ancient Egyptian calendar: $t_Y = 365^d = 12\times 30^d + 5^d$ (intercalation)

During the Roman republic a more complicated and considerably less symmetrical variant was used (Richards, 1998, ch.16). However, since t_Y is longer by about ¼ of a day (eq(1.2)), the seasons' periodicity lagged behind the civil year by almost 1 month within

[11] For the history of the Chinese remainder theorem see Katz, 1998, pp.197-199; Dauben, 2007, p. 302.

[12] E.g. close to the context of Goldstein *et al*, 2007, especially chs. I.1, I.2.

one century! On the advice of the Alexandrian astronomer Sosigenes, Julius Caesar introduced in 46BC a better approximation: 4-year cycles consisting of 3 common years and one leap year (the Julian calendar)

(first approximation) Julian year (JY): $t_{\mathrm{Y}} = 365^{\mathrm{d}}.25$,

Julian calendar (4-year cycles): $4\times365.25^{\mathrm{d}} = 3\times365^{\mathrm{d}}$ (common year) $+1\times366^{\mathrm{d}}$ (leap year)

(iv) Table 4.1 shows that this was again an approximation, amounting to a discrepancy of about 10′/year between the JY and the tropical year according to the Alfonsine tables[13] (late middle Ages). This amounts to the loss of 1 day in about every 134 years.

t_{Y} year current value	t_{JY} Julian year	t_{M} month (mean synodic) current value	t'_{Y} Alfonsine tables (1252/1492)	t_{D} day current value
$365^{\mathrm{d}}.2422$	$\mathbf{365^{d}.25}$	$29^{\mathrm{d}}.53059$	$\mathbf{365^{d}.242546}$	86,400 sec(SI)
$365^{\mathrm{d}}\,5^{\mathrm{h}}\,48'46''$	$365^{\mathrm{d}}\,6^{\mathrm{h}}$	$29^{\mathrm{d}}\,12^{\mathrm{h}}\,44'2''$	$365^{\mathrm{d}}\,5^{\mathrm{h}}\,49'16''$	
$t_{\mathrm{JY}} - t'_{\mathrm{Y}}$ =10′ 4″ = 0.0074537 days/year, or **1 day** lost every **134.16 years**				

Table 4.1

This tiny **measurement errors** due to the limited accuracy of observations produced **observable** effects only over **long** time intervals (centuries). Nevertheless this was important for several reasons: (1) *religious*: to have regular and strict celebration of festivities (especially the Christian Easter); (2) *historical*: to reckon correctly facts in different epochs at different places; (3) *political*: to coherently realize seasonal activities related to society, economy, agriculture etc.

Many attempts for corrections were made in the middle Ages, greatly enhanced by the gradual establishment of the Arabic numerals and the positional number system in the Renaissance. Thus the French cardinal P. d’Ailly (1412) proposed to omit 1 day every 134 years; more importantly, the Italian astronomer and mathematician P. Pitatus (1568) proposed to omit 3 days every 400 years, because

$$\frac{3}{400} = \frac{1}{133 + \frac{1}{3}} \cong \frac{1}{134}$$

(Dutka, 1988, p. 60). Probably influenced by this proposal and upon request of Pope Gregory XIII, a group of scholars with central figures the Italian physician and astronomer A. Lilius and the Jesuit mathematician and papal astronomer C. Clavius proposed the currently used rule; namely, Pitatus’ rule omitting from the leap years the centurial years not divisible by 4, thus keeping only 97 leap years in 400 years. This was the Gregorian reform of the calendar officially declared and implemented in 1582 (Richards, 1998, ch. 19; 2013, pp. 598-599; Dutka, 1988; Duncan, 1998, ch. 13).

Later, in his *Introductio in analysin infinitorum* Euler developed t_{Y} as a continued fraction. This approach, understood in a *heritage*-like perspective (§3.1) with history playing a strong *cultural* role (§3.2), provides interesting insights into the underlying mathematics and its cultural significance. This enriches the teachers’ didactical background by interrelating mathematics with other non-mathematically oriented domains

[13] In honor of Alfonso X of Castille (1252-1284), who supported the conduction of accurate astronomical tables during his regnal period (Richards, 1998, pp. 38-39).

and deepens the awareness of mathematics as an intellectual endeavour having strong permanent bonds with culture and society. We proceed to reveal this point more clearly.

4.2.2 Continued fractions

In view of the historical outline above, two questions naturally arise (Rickey, 1985):

- Since there is always a difference between the tropical and the civil year, **is there any better leap-year-rule**?
- Can the Julian and Gregorian calendar rules be mathematically described **uniformly?**

Here, continued fractions enter the scene; a historically and mathematically interesting and rich concept introduced by Bombelli, Huygens, Wallis, Euler and others (cf. §2.3.1): Any (rational) number *a* is represented by a (finite) continued fraction by subtracting the integral part from it, taking the reciprocal, recording the integral part and subtracting it, and repeating the procedure:

$$a = [a] + \frac{1}{\frac{1}{a-[a]}} = [a] + \frac{1}{[a'] + \frac{1}{\frac{1}{a'-[a']}}} = \cdots \text{etc}, \quad a' = \frac{1}{a-[a]} \tag{4.5}$$

This leads to the continued fraction representation of the rational number. If truncated at the n^{th} step, it gives its *convergents* $\frac{A_n}{B_n}$ obeying recursive relations already proved by Wallis (1655) and Euler (1737) (Hairer & Wanner, 1996, §I.6, theorem 6.1):

$$a = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{..+\cfrac{1}{..+\cfrac{1}{b_n}}}}}} = \frac{A_n}{B_n} \tag{4.6}$$

with

$$A_n = b_n A_{n-1} + A_{n-2}\ ,\ B_n = b_n B_{n-1} + B_{n-2} \tag{4.7a}$$

$$A_{-1} = 1,\ A_0 = b_0 = [a],\ A_1 = b_1 b_0 + 1\ ,\ B_{-1} = 0, B_0 = 1, B_1 = b_1 \tag{4.7b}$$

With t_Y -[t_Y] = 0.2422 to four decimals (Table 4.1) and b_0=0, b_1=4, we get the continued fraction expansion (cf. Rickey, 1985; Eisenbrand, 2012; Grabovsky, n.d.)

$$0.2422 = \frac{2422}{10000} = \frac{1211}{5000} = \cfrac{1}{4 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}}}}}} \tag{4.8a}$$

and its convergents giving possible calendar options shown in Table 4.2.

$$\frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{31}{128}, \frac{132}{545}, \frac{163}{673}, \frac{295}{1218}, \frac{458}{1891}, \frac{1211}{5000} \tag{4.8b}$$

Table 4.2 includes both the JY (1st row) and another proposal by the 11th century Persian scholar and poet Omar Khayyam (2nd row), largely forgotten but more accurate than the Gregorian one (3rd row), which is shown for comparison even though it does not correspond to any convergent of t_Y -[t_Y] (Rickey, 1985):

- 2nd row (*Omar Khayyam*): **7** consecutive 4-year cycles with 1 leap year per cycle, followed by one 5-year cycle having 1 leap year

But there are other even more accurate options:

- 4th row: 32 consecutive 4-year Julian cycles with no leap year in the last one;
- 5th row: the Gregorian rule applied to centurial years divisible by 5 (i.e. 96+25=121 leap years in 5 centuries), with the 5000th year not being a leap year (Rickey, 1985).

convergent	Proposal	Value	temporal cycles of JY (1 leap year per cycle)
1st	Julian calendar (46 BC)	**1**/**4** = 0.25	**4**y = 1×4y leap years: **1**
3rd	Omar Khayyam (c. 1076 AD)	**8**/**33** = 0.2424…	**33**y = 7×4y + 1×5y leap years: **8** = 7+1
	Gregorian Calendar (1582 AD)	**97**/**400** = 0,2425	**400**y = 3×(25×4y-1d) + 1×(25×4y) leap years: **97** = 3×(25-1) + 1×25
4th	Other possibilities	**31**/**128**= 0.24219	**128**y = 31×4y + 1×4y-1d leap years: **31** = 32-1
9th		**1211**/**5000**= 0.2422	**5000**y = 10×[4×(25×4y-1d)+1×(25×4y)] + 1d leap years: **1211** = 10× [4×(25-1)+1×25] + 1

Table 4.2

Non-mathematical arguments of a social or practical nature for or against these alternatives can be a stimulating part of an interdisciplinary *teaching module* on this problem, with emphasis on historical information about the mathematics required, used, or underlying it and the cultural and social issues that called for its solution and stressed its significance (§3.4). In this context, while exploring the driving forces behind this problem, historical issues are dominant, thus serving mainly as a *goal* in themselves (§3.3).

4.3 The weekday on a given date: From old *dominical letters* to modern computer algorithms

4.3.1 Outline of the problem and its history

Finding the weekday on a given date has attracted considerable attention throughout history for social, political and religious reasons (§§2.1, 2.2(d); e.g. it is crucial for finding the date of Easter Sunday; §4.4). Its treatment involves three temporal cycles of different origin: the *year*, of astronomical origin, the civil *month*, and the *week*, both being determined by a mixture of astronomical and civil factors[14]. Though mathematically elementary (simple counting is sufficient in principle), it is a non-trivial problem if a sufficiently quick method is requested, and/or dates with a long time separation are considered. This implies that an as much as possible formalized procedure is desirable or necessary. Therefore, this problem has considerable recreational value (e.g. Kraitchik, 1953, ch.5; Ore, 1967, ch.8; Beveridge, n.d.), reinforced by its “asymmetric” aspects that allow for **no simple** solution because of various reasons of diverse character:

(1) Since 365≡1(mod7) and 366≡2(mod7), each date moves forward by 1 or 2

[14] For historical issues see Richards, 1998, chs. 15-17, 21; 2013 §§15.1.6, 15.3.2, 15.3.3; Duncan, 1999, ch. 2; Whitrow, 1988 pp. 32, 55, 68-70.

weekdays per common or leap year respectively;
(2) There is 1 leap year every 4 years (Julian calendar), plus the correction for centurial years (Gregorian calendar). This is due to the lack of a simple rational relation between the year and the day, and the obvious request that socially important temporal cycles should be expressed by relations involving integer numbers only;
(3) For historical or even incidental reasons, the number of days is distributed irregularly among civil months; 4 having 30 days, 7 having 31, 1 having 28 or 29.

Due to the interest it arose throughout history, it has been considered by different people (including many eminent mathematicians) in various ways, several of which are interesting from a *heritage-like* educational perspective (§3.1). No comprehensive presentation will be given. We simply note that the problem can be tackled and solved in different, progressively formalized ways roughly reflecting the historical development:
(i) In the early middle Ages, due to the lack of sufficiently developed mathematical background, the problem was treated by simple counting, possibly using the four *arithmetical operations* and **extensive** *data tabulation*. In this connection the 28-year solar cycle was central (§2.3.2), which is not directly applicable to the Gregorian calendar however, because some centurial years are common (§4.2.2).
(ii) With the development of (elementary modern) mathematics - the *arabic numerals*, the *positional number system*, and the emergence of *elementary algebraic symbolism* and *methods* - it became possible to use these tools to parameterize dates and weekdays, thus reducing the use of *tabulated data*.
(iii) Further development of mathematics led to the invention of more compact methods and their algebraic formulation that minimized the need for data tabulation, and by a progressively more *elaborate formalization* the development of numerical algorithms admitting computer implementation nowadays.

The key historical steps in this development could form the basis of a *teaching module*, focusing on the *religious* and *social* need to obtain a sufficiently simple and practical solution (§3.4). Identifying and elaborating on these steps means that history will serve mainly as a *goal* in itself (§3.3), helping to realize both the *cultural significance* and the *evolving character* even of the most elementary mathematical knowledge that today is often considered as "something that was always there" (e.g. the positional number system, or the existence of algebraic symbolism; §3.2(a)). On the other hand, since there are several different **equivalent** formulations and solutions of the problem (Appendix A), it is mathematically interesting and insightful to check and prove their equivalence. This ranges from simple computational exercises, to more sophisticated mathematical elaborations, thus helping to get acquainted with specific pieces of mathematics and enriching the teachers' didactical background (§3.2(b)).

4.3.2 A semi-formalized method of solution

To illustrate some points of §4.3.1, a "hybrid" solution method is outlined; a semi-formalized one (close to Richards' 1998, ch.24) minimizing the need of data tabulation.

(1) Each method uses some mapping of the weekdays to an arithmetical set. Here and in Appendix A we use the following numbering and notation:

Sun	Mon	Tue	Wed	Thu	Fri	Sat	*Weekday number W*
1	2	3	4	5	6	7	

Table 4.3

Year AD		Month	Day of month	date	*Weekday number*
Y	Y=100*c*+*y*	*m*	*d*	d/m/Y	*W*

Table 4.4

(2) The dates of the year 1/1, 2/1 …, 31/12 are numbered consecutively from 1 to 365, assigning to each one its remainder when divided by 7, its so-called *calendar number*; that is, the calendar numbers are given by the

$$\text{canonical mapping: } \{1/1, 2/1 \dots 31/12\} \leftrightarrow \{1, 2, 3, \dots, 365\} \to \mathbb{Z}_7 \qquad (4.9)$$

with the important **convention** that 29/2 and 1/3 are both mapped to 60 ≡ 4(mod7) (a year's 60th day is 1st March (common year), or 29th February (leap year)).
To avoid confusion with the weekday numbers, identify calendar numbers with calendar letters:

$$\{1, 2, 3, 4, 5, 6, 0\equiv 7\} \equiv \{A, B, C, D, E, F, G\}$$

(3) Though obvious, it is important to note that **each weekday** in a (common) year has the **same** calendar **number/letter** throughout the year (i.e. the same image in $\mathbb{Z}_7$). For a leap year, each weekday has two calendar letters; one for January and February and one for the other months, this one being identical with the calendar letter of that day in the next year.

(4) Therefore, define the *dominical letter/number* of the year Y as

$$N_Y = \textit{calendar letter/number} \text{ of Sundays of year Y} \qquad (4.10)$$

Little inspection shows that N_Y determines the **weekday of 1/1/Y**. By (3), there are 14 alternatives for N_Y, shown in Table 4.5, which readily implies that

$$W_{1/1/Y} + N_Y \equiv 2 \text{ mod7} \qquad (4.11)$$

		Dominical Number/Letter				
Week day, 1/1	*W*	*N* (Common Year)		Leap Year (*N* \ *N* -1)		Forward in time
Sunday	1	1	A	1\7	A\G	↑
Saturday	7	2	B	2\1	B\A	
Friday	6	3	C	3\2	C\B	
Thursday	5	4	D	4\3	D\C	
Wednesday	4	5	E	5\4	E\D	
Tuesday	3	6	F	6\5	F\E	
Monday	2	7	G	7\6	G\F	
		For the **whole** year		Jan, Feb\ Mar-Dec		

Table 4.5

common year	leap year
$W_{1/1/Y+1} \equiv (W_{1/1/Y} + 1) \bmod 7$	$W_{1/1/Y+1} \equiv (W_{1/1/Y} + 2) \bmod 7$
$N_{Y+1} = N_Y - 1$	$N_{Y+1} = N_Y - 2$

Table 4.6

Therefore if N_Y is known, this relation gives the week day of 1/1/Y (Table 4.6). This in turn gives the weekday of any date in the year Y, as shown below. This is one of the reasons for which dominical letters played an important role in the past (especially before the development of algebraic symbolism and operations) and were extensively tabulated (Richards, 1998, ch.24; Dominical Letter, n.d.); e.g. Table 4.5 immediately gives the succession of years in the 28-year solar cycle, thus providing a constructive proof that every 28 years, **each** date of the year – and for all dates - falls on the same weekday, as shown in Table 4.7 (solar cycles start on a leap year with 1/1 being Monday; §2.3.2).

Year of the cycle	**N**	**Year of the cycle**	**N**
1	G/F	15	C
2	E	16	B
3	D	17	A/G
4	C	18	F
5	B/A	19	E
6	G	20	D
7	F	21	C/B
8	E	22	A
9	D/C	23	G
10	B	24	F
11	A	25	E/D
12	G	26	C
13	F/E	27	B
14	D	28	A

Table 4.7

(5) Define the *regular of month m*, R_m by

$$R_m = (\textit{calendar number of } 1/m) \bmod 7 \qquad (4.12)$$

By (4.9), R_m is **independent** of the year (common or leap) and given in Table 4.8

m	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
$\boldsymbol{R_m}$	1	4	4	0	2	5	0	3	6	1	4	6

Table 4.8

It is readily seen that $W_{1/m} \equiv (W_{1/1} - 1 + R_m) \bmod 7$; hence, for any day d of that month:

$$W_{d/m} \equiv (W_{1/1} - 1 + R_m + d - 1) \bmod 7 \qquad (4.13)$$

Therefore, the problem reduces to determining $W_{1/1/Y}$. Given that passing from Y to Y+1 each date moves forward by 1 or 2 weekdays per common or leap year respectively, it is readily seen that for the Gregorian calendar and for a common year there have been Y-1 +

[Y/4] – [Y/100] + [Y/400] shifts forward from 1/1/1 to 1/1/Y[15]. For a leap year, 1 should be subtracted from the r.h.s. for $m \leq 2$ (i.e. for January, February) since its leap day has already been added in the above sum. 1AD can be replaced by any other reference year Y_0, adding a constant depending on $Y-Y_0$ and Y_0 being a common or leap year. Choosing Y_0 to be a common year for which $W_{1/1/Y_0}$ is known and combining the above number of shifts with (4.13) gives the weekday number for any date

$$W_{d/m/Y} \equiv (d - 1 + R_m + Y - 1 + [Y/4] - [Y/100] + [Y/400] + a) \bmod 7$$

with a specified by Y_0; e.g. 1/1/2018 was Monday, hence $W_{1/1/2018} = 2$ (Table 4.3). Therefore $W_{1/1/2018} = 2 \equiv (2507+a) \bmod 7 \equiv (1+a) \bmod 7 \Rightarrow a = 1$ and $W_{d/m/Y}$ is:

$$W_{d/m/Y} \equiv (d + R_m + Y - 1 + [Y/4] - [Y/100] + [Y/400] - \delta) \bmod 7 \qquad (4.14a)$$

$$\delta = \begin{cases} 1\ ; \text{ for Y leap year \& } m \leq 2 \\ 0\ ; \text{ otherwise} \end{cases} \qquad (4.14b)$$

Though what precedes gives only the basics of the method, it is clearly susceptible to considerable elaboration in various directions; discussion on the role of algebraic/symbolic modelization vs. using exclusively tabulation methods (the only thing medieval scholars could do); comparison with other methods (e.g. those in Appendix A); finding other equivalent formal expressions and developing algorithms appropriate for computer implementation; further discussion on calendars different from the Gregorian, and conversion procedures from one to another (e.g. Richards, 1998, part III) etc.

4.4 *Computus* & Easter Sunday: The struggle for reconciling astronomical and civil temporal cycles

Finding the date of Christian Easter Sunday has been central for Christendom since the early middle Ages, closely related to finding the weekday on any given date (§4.3). Though by modern standards the problem in principle requires only very basic mathematics (the positional number system, the four arithmetical operations, the decimal representation of fractions, and an elementary algebraic symbolism and modelization), it is difficult to be dealt without the mathematical sophistication of §4.3. In fact, it is even more difficult because **four** temporal cycles are involved instead of three: two of the "basic clocks" (the (tropical) year and the lunar (synodic) period), and two of social origin (the civil month and the week), all measured in days (the third "basic clock"). The complications inherent to this problem for astronomical and civil reasons, and the lack of sufficiently adequate mathematical background prohibited an accurate - hence definite – solution. This made it a subject of ceaseless interest since the early Christian era, greatly influencing social life, theological debates and religious customs, and calling for more and more accurate astronomical observations and ever increasing mathematical sophistication. The interference of nonmathematical (basically theological) issues as the motive force, with the treatment of complicated astronomical data by means of mathematical techniques developed for that purpose, led to the medieval *computus*; the particular art of calculating Easter Sunday's date developed by specialized scholars and clergymen, and whose basics had to be familiar to the clergy in medieval Europe (cf. §2.1; for historical details see

[15] The number of years that passed since 1AD, plus the additional days because of the leap years, minus the centurial years that are common (for the Julian calendar, the last term [Y/400] is missing).

Borst, 1993, chs. 4-10; Duncan, 1998, chs. 5-7, 11; Whitrow, 1988, Appendix 3).

Being a meeting point of ancient Greek astronomy, time reckoning in the Roman Empire, and the Jewish lunar calendar and festivities, it is a historically fascinating and strongly intercultural and interdisciplinary subject related to elementary mathematics. Therefore, from an educational point of view, it is ideal for perceiving pieces of elementary mathematics from a non-mathematical perspective (§3.2(a)) by revealing the cultural factors that influenced the development of mathematics and the social problems solved in this way (§3.2(b)). In a *heritage*-like perspective (§3.1) the emphasis is on learning about the historical origin and development of some nowadays very basic mathematics (§3.4(a)). By following a *history-based* approach, the key steps of this development will shape the presentation (§3.4(b)) in the form of the questions and problems that were the driving forces for exploring and solving the problem (§3.3).

Some issues related to this problem were discussed in the previous sections. Therefore, only certain basic points are presented with reference to the literature for further details.

4.4.1 The historical milieu

Since its original appearance as a historical religion, Christianity considered Easter's celebration as the most important part of its worship. Originally, this was dependent on the Jewish *Passover* because according to the Gospels, Jesus' Crucifixion took place one day after the Passover feast on the 14th day of Nisan, the first month of the religious Jewish year, starting near the vernal equinox (Richards, 1998, ch.17). Given that the Jewish calendar is lunar, but the Julian calendar (hence, the Gregorian as well) is solar, Christian Easter could not be celebrated on a fixed day of the year. Therefore, after Constantine the Great allowed Christians to follow their faith without oppression in 313AD, he convened the first *Ecumenical Council* in Nicaea, Asia Minor, in 325AD, which – among other things (§4.1.1) – set out the framework for Easter's celebration by requesting: (i) independence from the Jewish calendar, as it was done by the Alexandria's church (namely that it must be celebrated on a Sunday necessarily after the vernal equinox); and (ii) celebration on the same date everywhere. After a transient period in which the Alexandrine computational method was stabilized into its final form, this led to the following convention used as a normative rule throughout Christendom since long ago[16]:

- *Christian Easter* is celebrated on the *first Sunday after the first full moon that coincides with or follows the 21st of March* (the ecclesiastical vernal equinox).

It is here assumed that the full moon occurs on the 14th day after the new moon (counting new moon's occurrence as the first day). The lunar cycle in the above rule is called the *paschal moon* and its full moon is the *paschal full moon.*

4.4.2 The astronomical background and the resulting complications

(a) Easter's celebration rule clearly involves the combination of three temporal cycles: The (Julian) *civil year* via the vernal equinox; the *lunar synodic period* (or lunation; §2.3) via the occurrence of a full moon; and the *7-day week* via the request of Easter's celebration being on a Sunday. Therefore, its actual implementation meets several difficulties for the following reasons (cf. §1.2):

(1) The (Julian) civil year and the tropical year differ and the difference accumulates, even if one leap year is inserted every 4 years. Moreover, leap years introduce further

[16] This is an oversimplified picture of the complicated historical development (Richards, 1998, ch.24).

computational complications similar to those in §4.3;
(2) There exist no simple rational relations among the above three temporal cycles;
(3) Civil months (the 4th cycle involved) differ considerable from the lunar synodic period;
(4) Because of (2), the full moon does not occur exactly 14 days after the new moon;
(5) The vernal equinox does not occur exactly on March 21st (March equinox, n.d.).

Hence, the result of applying the above rule for Easter's celebration deviates from the occurrence of the astronomical phenomena it is supposed to describe. It corresponds rather to a notional picture close but not coincident with physical reality. This gets clearer below.

(b) The Alexandrians employed the 19-year Metonic cycle (§2.3.1) to determine Easter Sunday. Based on this, Victorius of Acquitaine (5th century AD) realized that the dates of Easter Sunday repeat every 532 years which was named after him (*Victorian cycle*). But this cycle is the least common multiple of 19, 28 and it is illuminating to note in Table 4.9 its interrelation with other cycles (§2.3).

cycle	**Solar**	**Metonic**	**Callipic**	**Victorian**
duration (years)	28×19 =	$(19 \times 4) \times 7$	= 76×7 =	532
origin	Astronomical & civil conventions	Astronomical		Astronomical & civil conventions

Table 4.9

By combining the Victorian cycle with the AD era introduced in 525AD (§§2.1, 4.1.1), tables with the Easter days were produced for two Victorian cycles (i.e. until 1064AD), originally by Isidore of Seville (early 6th century AD) and then by Bede in his influential *De temporum ratione* (725AD); Borst, 1993, ch. 5, Richards, 1998 ch. 28. But because of the reasons in (a) above, these calculations led to results incompatible with the physical phenomena over long time periods: The use of (i) the **Julian year** ($365^d.25$) instead of the slightly shorter tropical year $t_Y = 365^d.2422$, eq(1.2), and (ii) the **notional** lunation obtained as the mean value of the lunar synodic month in one Metonic cycle (§2.3.1) i.e. $6940^d/235 = 29^d.5319$, instead of $t_M = 29^d.5306$, eq(1.3), produced a cumulative effect: the vernal equinox was falling progressively **earlier** than 21 March, and astronomical new moons appeared progressively **earlier** than the calendar dates of notional new moons.

It is obvious how difficult it was to deal with this problem without the nowadays elementary mathematical knowledge of the positional number system, the arithmetic operations and fractions' decimal representation. It also gives hint why the *computus* (based on simple counting and extensive use of tables) required special training and was accessible to a limited circle of educated people (Fraser, 1987, p. 81).

At the same time, it is illuminating to consider the problem from a modern perspective and comment briefly on why a possible solution was never implemented. Table 4.10 gives the values of the physical and notional quantities involved (approximated to four decimals; cf. §§1.2, 2.3). Imposing the condition that the civil year should have 12 (civil) months, yields

Metonic cycle = (19×12) = 228 civil months = 235 notional lunations = 6940^d

t_Y current value	t_{JY} Julian year	t_M lunar synodic period
$365^d.2422$	$365^d.25$	$29^d < 29^d.5306 < 30^d$
$19t_Y = 6939^d.60$	$19t_{JY} = 6939^d.75$	$235t_M = 6939^d.69$
rounded to **6940^d** (**Metonic** cycle)		
	physical fact	$29^d < t_M < 30^d$ [17]
		$12\, t_M < t_{JY} < 13\, t_M$
never used	civil (common) year 365^d = 12 months $= 7 \times 30^d + 5 \times 31^d$	$t_{JY} = 29^d.5 \times 12 + 11^d.25 = 354^d + 11^d.25$
original (roman)	$= 5 \times 30^d + 6 \times 31^d + 1 \times 29^d$	
final (current)	$= 4 \times 30^d + 7 \times 31^d + 1 \times 28^d$	

Table 4.10

If it is further required that notional lunations contain an integer number of days (29^d or 30^d), then the Metonic cycle admits a unique partition into notional lunations; namely

$$\left.\begin{array}{l} 29x + 30y = 6940 \\ x + y = 235 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = 110 \\ y = 125 \end{array}\right.$$

However, this solution is incompatible with the additional requirement of civil years having 12 months, in the sense that no such convenient division exists!

4.4.3 The adopted solution and a modern formulation

Because of the many "asymmetric" features of the problem (partly due to social constraints or historical incidents) the solution finally accepted is as follows (for details see Richards, 1998, ch. 29):

(a) The Metonic cycle is used, consisting of 235 notional lunations, or "months" m of either 30^d or 29^d.

(b) These lunations are distributed among the 19 solar years as follows: 12 years are *common* in the sense that they contain 6 lunations of 30^d and another 6 of 29^d. The remaining 7 are *embolismic* in the sense that they contain a 13th (embolismic) lunation, which has 30^d for the six of these year and 29^d for the last one. Finally the remaining 5^d were added by inserting 1^d every 4 years into whichever lunation contains 24 February. Or, in symbols

$$\begin{aligned} 235\text{m} &= \mathbf{12}\text{y} \times \mathbf{12}\text{m/y} + \mathbf{7}\text{y} \times \mathbf{13}\text{m/y} = \\ &= \mathbf{12}\text{y} \times \mathbf{6}\text{m/y} \times (30\text{d/m} + 29\text{d/m}) + \mathbf{6}\text{y} \times [\mathbf{7}\text{m/y} \times 30\text{d/m} + \mathbf{6}\text{m/y} \times 29\text{d/m}] + \\ &\quad + \mathbf{1}\text{y} \times [\mathbf{6}\text{m/y} \times 30\text{d/m} + \mathbf{7}\text{m/y} \times 29\text{d/m}] + \mathbf{5}\text{d} = 6940\text{d} \end{aligned}$$

(c) The paschal full moon is on the14th day of the notional lunation **on** or **after** 21/3 and this notional lunation should always have 29^d. This can be accounted for by arranging lunations so that the first one has 30^d for all years of the Metonic cycle. This means that the earliest is on 21/3, and the latest on 18/4 (full moon on 20/3 + 29d). Therefore, Easter Sunday falls from 22/3 to 25/4 (paschal full moon on Sunday 18/4).

Clearly, this is not a mathematically elegant solution. For centuries it was necessary to

[17] A purely lunar calendar of 12 lunar synodic periods equals 354.3672^d. It can be approximated by six 30-day and six 29-day months corresponds to a lunar year of 354 days, with some years having one additional day to account for the resulting discrepancies. The Islamic calendar is of this kind (Richards, 1998, ch. 18).

construct appropriate tables satisfying the above conditions and in some way depicting the notional lunations in the calendar of the 12 civil months. To this end further parameters were defined for each year; the *golden number* and the *epact* (see below). With the development of algebraic symbolism and operations it became possible to formalize the solution further, so that nowadays this formalization is complete admitting computer implementation. This procedure is nontrivial and can constitute a nice project in discrete mathematics, computer algorithms etc. No details are given, but we define the additional concepts needed and simply state the final result (Richards, 1998, ch. 29).

(d) We define:

(i) The *Golden Number* G_Y of the AD year Y (the place of Y in the Metonic cycle; §4.1.1):

$$G_Y \equiv 1+Y \bmod 19 \qquad (4.14)$$

(ii) The *Epact*[18] of Y: The age of the notional moon on 1/1/Y (ranging from 0 for a new moon, to 28).

(iii) Having in mind (c) above, we define the mapping

$$\{21/3, 22/3 \ldots 25/4\} \rightarrow R = \{21, 22, \ldots, 56\} \qquad (4.15)$$

and consider the Easter Sunday number, $\mathbf{S}_Y$ and the next day of the Paschal full moon, r_Y. By (c) above, $\mathbf{S}_Y$, $r_Y \in R$ and it can be shown that the following relations hold:

The *calendar number* C of r_Y (§4.3.1) is

$$C \equiv (r_Y + 3) \bmod 7 \qquad (4.16)$$

Moreover,

$$r_Y \equiv (75 - E_Y) \bmod 30 \qquad (4.17)$$

$$E_Y \equiv (11\, G_Y - 3) \bmod 30 \qquad (4.18)$$

Writing (4.11) in the equivalent form

$$N_Y \equiv (9 - W_{1/1/Y}) \bmod 7 \qquad (4.19)$$

it can be proved that the date of Easter Sunday for the year Y is given by

$$\mathbf{S}_Y \equiv r_Y + (7 + N_Y - C) \bmod 7 \qquad (4.20)$$

Clearly the above relations can be combined to give several other equivalent forms of the final result (4.20). The important thing to note here is that by means of some careful algebraic modelization, any reference to tabulated data has been dispensed with. Following in detail the procedure leading to the above result provides a nice opportunity of getting insight into the way an appropriate mathematical formalization replaces more empirical and common sense methods to deal with a problem of non-mathematical origin.

4.5 Accurate clocks, the escapement mechanism & the underlying physico-mathematical theories: Galileo & Huygens

This is a vast subject in which interesting mathematics, (partly) advanced physics and complicated technology interact in a multifarious way. The discussion will be confined to comments on Galileo's and especially Huygens' contribution.

According to the previous examples there was a gradually increasing need for more and more accurate time-keeping, further enhanced by the nontrivial problem of determining geographic longitude (especially at sea) during and after the great geographical expeditions (Whitrow, 1988, ch. 9; Newton, 2004, chs. 5 & pp. 59-61). To this end, devices were invented based on continuous processes (e.g. water clocks), or oscillatory processes (e.g. mechanical clocks), with a gradual shift from the first to the second

[18] Introduced by the Alexandrian church in the 3rd century AD (Holford-Strevens, 2005, ch. 4 & Appendix B).

because of the invention of the *verge escapement* (Appendix B). Originally these were inaccurate because of swinging across wide angles (at least ~50^0). Without going into details, we comment on two historically important cases in which technology and mathematics are interconnected:

-The *simple pendulum clock*: Although Galileo's discovery of the simple pendulum isochronism (c.1602; Drake, 1995, pp. 68-70, 72-73) motivated him in his later years to think of it as the basis of a possible time-keeping mechanism described posthumously (1659) by his biographer Viviani (Fig. A.1; Drake, 1995, pp. 378-379, 399, 419-421), such a clock was constructed by Coster in 1657 only after Huygens' theoretical investigations (Fig. A.2; van Helden, 1995; Whitrow, 1988, pp. 122-124; Richards, 1998, p. 58).

-The *Cycloidal pendulum*: Also invented by Huygens in 1659 and described in his influential *Horologium Oscillatorium* (1673/2013, Part I), which is an ingenious construction based on the mathematical properties of the cycloid that he also proved (Fig. 4.5; Whitrow, 1988, p. 123; Horologium Oscillatorium, n.d.).

4.5.1 The simple pendulum clock

The first sufficiently accurate clocks (17th century) were based on the isochronism of the simple pendulum[19]. Though discovered empirically (Newton, 2004, pp. 87-88; cf. Drake, 1995, p.397), it can be deduced as a simple application of Newtonian mechanics (see e.g. Sommerfeld, 1964, §III.15): Since the pendulum bob of mass m moves under its own weight mg and the tension $\mathbf{T}$ along the massless rod or string of length l (g being the gravity acceleration), applying Newton's second law gives the equation of motion in terms of the oscillation angle, accents denoting differentiation with respect to time t (Figure 4.1):

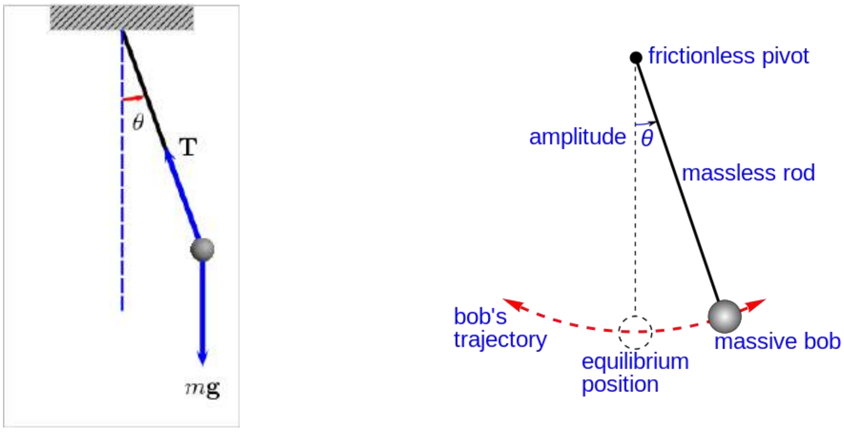

Figure 4.1: The physics of the simple pendulum

$$\theta''(t) = \frac{g}{l}\sin\theta \qquad (4.21)$$

This is a non-linear 1st-order differential equation, whose solution can be obtained in terms of elliptic integrals (Sommerfeld, 1964, *ibid*). It is well-known that for very small oscillation amplitudes, i.e. to lowest order in the oscillation amplitude (maximum of $|\theta|$), (4.21) becomes linear and its solution is a simple periodic function independent of this amplitude with period T given by (4.21').

$$\theta''(t) \approx \frac{g}{l}\theta, \quad \Rightarrow T = 2\pi\sqrt{\frac{l}{g}} \qquad (4.21')$$

As an interesting application of elliptic functions and integrals, it is insightful to solve

[19] For earlier constructions see Newton, 2004, ch.3; Whitrow, 1988 pp. 120-122; 1972, ch.4; Clock, n.d.

(4.21) and get T in (4.21') as the lowest order approximation of the elliptic integral's modulus as a function of the pendulum's parameter $(l/g)^{1/2}$.

4.5.2 The cycloidal pendulum clock

(a) The next very important development both theoretically and technically, was Huygens cycloidal pendulum and the underlying mathematical properties and physical theory. Huygens sought and conceived (more) accurate clocks (Huygens, 1673/2013). The key steps of his approach are schematically

- The study of the *cycloid*;
- The introduction of the novel **geometrical** concept of the *evolute* (and implicitly its dual concept, the *involute*) of a curve (Evolute, n.d.);
- The proof of the cycloid's important properties: its *tautochrone* (oscillations along it are isochronous) and its *self-involuteness* (its *involutes* are again *cycloids identical* to it);
- The design of in principle more accurate clocks[20].

Though Huygens' approach is geometrical (in the Euclidean tradition of that period), its outline below is analytical (based on Newton's laws), and it is insightful to be compared in detail with Huygens' approach and proofs, in a *heritage*-like perspective (§3.1). This comparison points to the evolutionary character of the same mathematical results, by looking at them from a different, less familiar perspective (§3.2(a)), thus learning specific pieces of mathematics and developing a wider and richer view of mathematics (§3.2(b)).

(b) A cycloid is the trajectory of the point of contact of a circle and a straight line, as the circle is rolling along the line without slipping (Fig. 4.2).

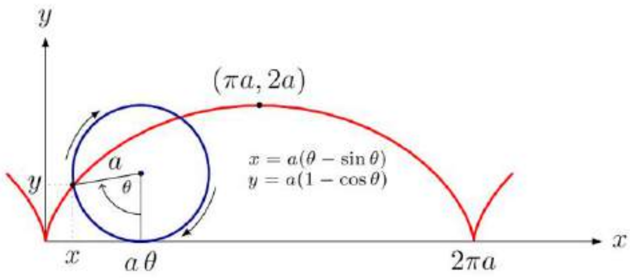

Figure 4.2: The geometry of the cycloid

By Newtonian mechanics, a point mass moving along a cycloid under its weight obeys the equation $\frac{\mathrm{d}v}{\mathrm{d}t} = g\frac{\mathrm{d}y}{\mathrm{d}s}$; v being its tangential speed and s the arc length along the cycloid. Expressing s in terms of x, y and noting that $v = \frac{\mathrm{d}s}{\mathrm{d}t}$, recasts this equation as a linear differential equation in $\cos(\theta/2)$ **formally identical** to (4.21') (Sommerfeld, 1964, §III.17).

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\cos\frac{\theta}{2} = -\frac{g}{4a}\cos\frac{\theta}{2} \tag{4.22}$$

Therefore, the period of oscillation is independent of the point's initial position, given by

$$T = 4\pi\sqrt{\frac{a}{g}} \tag{4.23}$$

i.e. strict isochronism holds **irrespective** of the **oscillation amplitude**. This is the cycloid's *tautochrone* property (Huygens, 1673/2013, Part II, proposition XXV).

(c) In his investigation of the cycloid, Huygens introduced the geometrical concepts of the *evolute* (Latin: *evolutione*) and *involute* (or *evolvent*) of a curve (without naming the

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[20] Huygens' design was based on the *verge escapement*, soon surpassed by the much better *anchor escapement* (Figure A.3; Whitrow, 1988, p. 123; Escapement, n.d.).

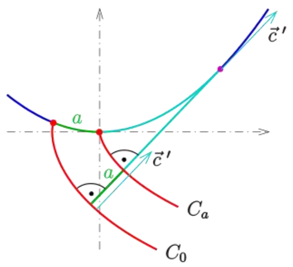
latter explicitly; Huygens, 1673/2013, Part III, Definitions I & III). They can be described in **two mathematically equivalent** but **conceptually different** ways (one geometrical and one physical) and they are dual to each other in the sense below:

- Physical description: An *involute* of a curve c_E is the locus c_I of the free end of a taut string attached to a point of the curve c_E as the string is wound along c_E.[21]
- Geometrical description: The *evolute* of a curve c_I is the locus c_E of the center of curvature of its points.

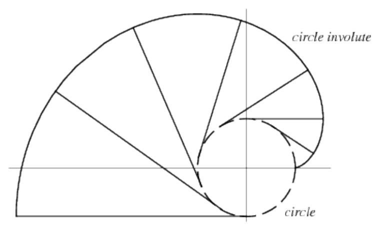
Clearly, each curve has infinitely many involutes (depending on the initial point chosen for the string's fixed end), but only one evolute. It can be shown that an involute of a curve is orthogonal to the curve's tangents (Huygens, 1673/2013, Part III, Proposition I; Stillwell, 1989, §16.2; Evolute, n.d.; Involute, n.d.) implying that these concepts are **dual** to each other in the sense that **the evolute of an involute of a curve is the curve itself**, hence the use of the same symbols in the above definition (Table 4.11; Figure 4.3).

curve c_I	curve c_E
Locus of **free end** of a taut **string** attached to a point of curve c_E as the string is wound along c_E	**Locus** of **center of curvature** of the points of curve c_I
Involute of c_E is c_I	**Evolute** of c_I is c_E
c_I orthogonal to the tangents to c_E	

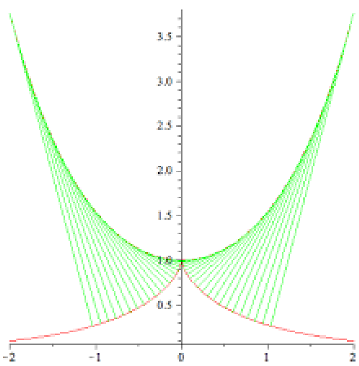
Table 4.11: The involute-evolute dual concepts



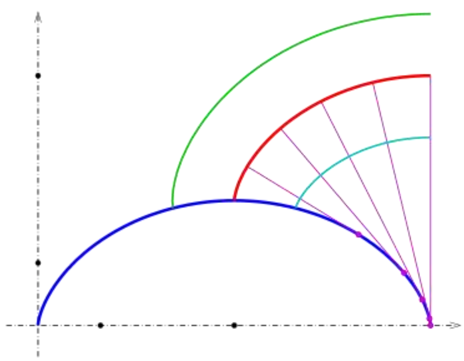
(a) The involutes c_I are orthogonal to the tangents to the evolute c_E



(b) The involutes of a circle c_E are spirals c_I; The evolute of a spiral c_I is a circle c_E



(c)The involutes of a catenary c_E are tractrices c_I; The evolute of a tractrix c_I is a catenary c_E



(d) The involutes of a cycloid c_E are cycloids c_I identical to it; the evolute of a cycloid c_I is an identical cycloid c_E

Figure 4.3: Examples of the involute-evolute dual concepts

[21] This formulation appears already in Huygens' (1673/2013) proof of Proposition I, Part III.

The important result here is the "*self-involuteness***"** of the cycloid (Figure 4.3(d)), proved by Huygens (1673/2013, Part III, proposition VI).

(d) The above two **mathematical** properties of the cycloid were combined by Huygens to give an appropriate clock based on a cycloidal pendulum; i.e. a pendulum constrained to oscillate between two cycloidal cheeks: Because of the cycloid's *self-involuteness*, a point mass (pendulum's heavy bob) suspended by a weightless (inelastic) string, constrained to move under its own weight between two cycloidal cheeks, moves along a cycloid. Therefore, by the *tautochrone* property, its oscillation period is amplitude-independent; that is, it takes the same time for all pendulums to arrive at the lowest point, irrespective of their initial position, i.e. the oscillation amplitude (Figure 4.4).

Fig. 4.5(a) is Huygens' own, showing both the cycloidal cheeks constraining the pendulum's motion and the "verge escapement" converting the pendulum's oscillations into pulses measuring time (cf. Fig. A.3(a)). Fig. 4.5(b) is a modern reconstruction.

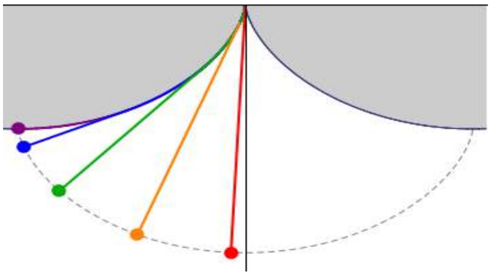
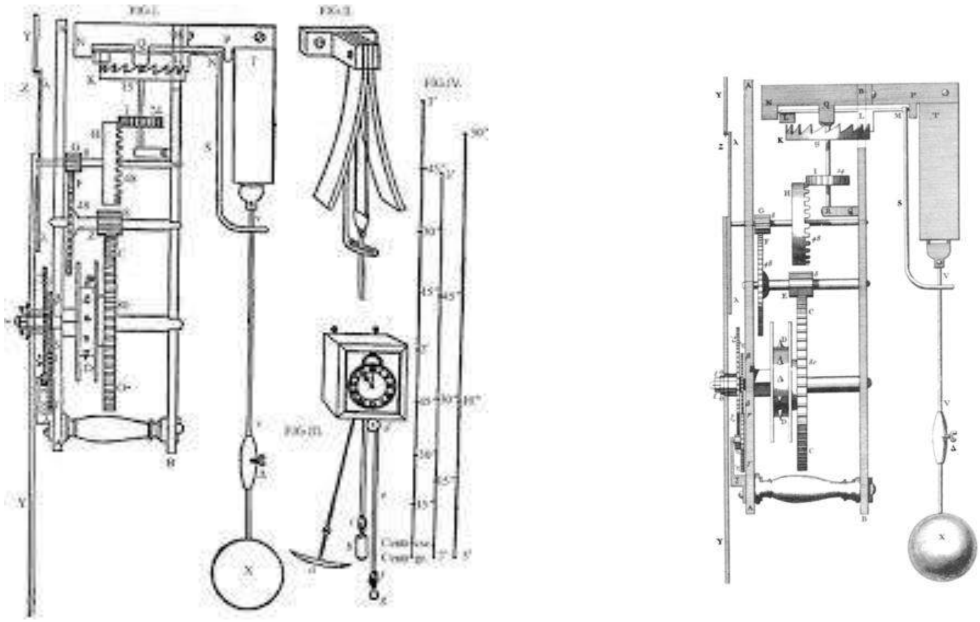


Figure 4.4: Schematic drawing and function of the cycloidal pendulum.



(a) Huygens' cycloidal pendulum clock (Huygens, 1673/2013, p.4)

(b) A modern drawing showing the *verge escapement* (Pendulum, n.d.)

Figure 4.5

The above outline of Huygens achievements related to the cycloidal pendulum, could be the basis of a teaching sequence inspired by history, either as an illumination approach, or as a complete teaching module (§3.4), depending on the time available, the specific didactical objective and other external constraints. History will serve mainly as a tool when following an illumination approach (e.g. in a course on mechanics, calculus, differential geometry, or their combination). On the other hand, in a teaching module devoted to this problem, historical issues (e.g. extensive use of original excerpts) can be treated in more detail and shape teaching explicitly (§3.3). The above presentation could also be used in an interdisciplinary module with emphasis on the mathematical basis of the technological achievements underlying clocks' construction; e.g. by elaborating on the various kinds of escapement mechanisms (see Appendix B), or the industrial applications of the geometrical concept of the involute etc (Involute, n.d.). This will emphasize the social context and its influence on the development of mathematics (§3.4(a)), thus helping to appreciate this development as the result of a cultural-human endeavour (§3.2(b)).

5 Concluding remarks

This paper aimed to provide evidence in support of the following main point: The multifarious aspects of the "*time measurement issue*" constitute an interdisciplinary resource for mathematics education, fruitful and insightful at quite different levels and purposes both for the learners and their teachers. The historical and epistemological facts were considered in the context of a theoretical framework for integrating history in mathematics education and the main point above was illustrated by means of five examples of varying mathematical content and social significance. Their presentation is far from being exhaustive, but was intended to stress the richness of the "*time measurement issue*": its mathematical content per se; its relation with other disciplines; and the social significance of the mathematics involved. A detailed presentation of these examples as teaching sequences and/or resource material will be given elsewhere.

APPENDIX A: Formalized methods to find the weekday on a given date

Examples of such methods are given below without proof (in the notation of §4.4) but with reference to the literature. The interested reader can try to check and possibly simplify them, explore conversions between them etc.

A1. C. F. Gauss

In an unpublished note, Gauss gave a formula for calculating the weekday number $W_{1/1/\mathrm{Y}}$ of year Y, from which a formula for $W_{d/m/\mathrm{Y}}$ results (for the original, details and proof see Schwerdtfeger, 2010; see also Richards, 1998, pp. 376-378):

$$W_{1/1/\mathrm{Y}} = 2 + (5\mathrm{Y}\bmod 4 + 4\mathrm{Y}\bmod 100 + 6\mathrm{Y}\bmod 400)\bmod 7 \qquad \text{(A.1a)}$$

From this, numbering months from March (m=1) to February (m=12), and using Y-1 instead of Y for January and February, a formula for $W_{d/m/\mathrm{Y}}$ results:

$$W_{d/m/\mathrm{Y}} = 1 + (d + [(13m-1)/5] + y + [y/4] + [c/4] - 2c)\bmod \mathbf{7} \qquad \text{(A.1b)}$$

(A.1b) is a variant of the formula in Weekday, n.d.

A2. C. Zeller

In 1883, the German mathematician C. Zeller gave another formula for $W_{d/m/\mathrm{Y}}$ where months are numbered as in (A.1b) (Zeller's congruence, n.d.; Richards, 1998, chs.23-24);

$$W_{d/m/\mathrm{Y}} = (d + [13(m'+1)/5] + y + [y/4] + [c/4] - 2c)\bmod 7, \quad m'=m+2 \qquad \text{(A.2)}$$

A3. A. De Morgan

In 1845, De Morgan gave a verbally expressed algorithm for finding the *dominical letter* of a given year (§4.3.2; also Richards, 1998, p.377; Dominical Letter, n.d.). It can be adapted to give $W_{1/1/Y}$ for any year Y (then, $W_{d/m/Y}$ is found by the method of §4.3.2)

I. Add 1 to the given year.
II. Take the quotient found by dividing the given year by 4 (neglecting the remainder).
III. Take 16 from the centurial figures of the given year if that can be done.
IV. Take the quotient of III divided by 4 (neglecting the remainder).
V. From the sum of I, II and IV, subtract III.
VI. Find the remainder of V divided by 7, and subtract from 7: this is the weekday number of 1/1.

Or, as a formula:

$$W_{1/1/Y} = 7- (1 + Y + [Y/4] + [(Y\text{-}1600)/400] - [(Y\text{-}1600)/100]) \bmod 7 \qquad \text{(A.3)}$$

A4. C. L. Dodgson (Lewis Carroll)

In 1887, Lewis Carroll published another verbally expressed algorithm (Richards, 1998, ch.24; Weekday, n.d.):

"Take the given date in **4 portions**, viz. the number of centuries, the number of years over, the month, the day of the month. **Compute the following 4 items**, adding each, when found, to the total of the previous items. When an item or total exceeds 7, divide by 7, and keep the remainder only.

- *Century-item*: For '*Old Style*' (which ended 2 September 1752[22]) subtract from 18. For '*New Style*' (which began 14 September 1752) divide by 4, take overplus from 3, multiply remainder by 2.
- *Year-item*: Add together the number of dozens, the overplus, and the number of 4s in the overplus.
- *Month-item*: If it begins or ends with a vowel, subtract the number, denoting its place in the year, from 10. This, plus its number of days, gives the item for the following month. The item for January is '0'; for February or March (the 3rd month), '3'; for December (the 12th month), '12'.
- *Day-item*: The total, thus reached, must be corrected, by deducting '1' (first adding 7, if the total be '0'), if the date be January or February in a leap year, remembering that every year, divisible by 4, is a Leap Year, excepting only the century-years, in 'New Style', when the number of centuries is not so divisible (e.g. 1800).

The final result gives the day of the week, '0' meaning Sunday, '1' Monday, and so on."

A5. Modern methods

It is remarkable that the problem is still attracting modern researchers' interest. E.g. a much simpler algorithmic method was published recently for finding $W_{1/1/Y}$[23]; the so-called "odd-plus-11 method", where a very limited tabulation of data is also needed (Fong & Walters, 2011; Dominical Letter, n.d.):

$$W_{1/1/Y} = \left(7 - \left[\frac{y+11(y \bmod 2)}{2} + 11\left(\frac{y+11(y \bmod 2)}{2} \bmod 2\right)\right] \bmod 7\right) + W_{1/1/100c} + 1 - \delta \qquad \text{(A.4)}$$

where $W_{1/1/100c}$ is the (tabulated) weekday number (Table 4.3) of 1/1 of the century to which Y belongs, and $\delta = 1$ for leap years and 0 otherwise.

Though (A.4) appears complicated, the quantity in parentheses is verbally expressed easily: If and only if y is odd, then add 11 to y. Divide the result by 2. If and only if the

[22] The year Great Britain adopted the Gregorian calendar.
[23] The algorithm (slightly adapted to the present context) was invented for finding the so-called "doomsday" of the year (Doomsday, n.d.).

result is odd then add 11. Compute the result modulo 7 and subtract from 7.

APPENDIX B: Basic characteristics of clocks

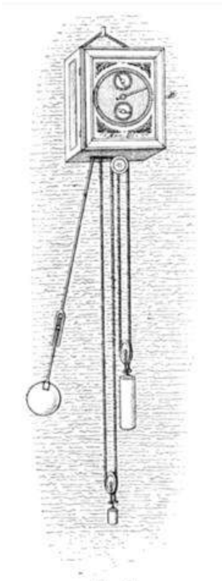
Some more information is given below about the general characteristics of a clock's mechanism, with emphasis on mechanical clocks (§4.5). Further elaboration could lead to an interdisciplinary module with emphasis on the mathematical basis of the technological achievements underlying clocks' construction (cf. §4.5.2(d)).

Figure A.1 shows Galileo's conception of a time-keeping device based on a simple pendulum's oscillations (§4.5). It is essentially an escapement mechanism, since no dials or drive are shown (Drake, 1995, pp. 419-420). Figure A.2 shows Huygens' original pendulum clock, with the *verge escapement* shown on the right. The improved cycloidal pendulum clock is shown in figure 4.5. For more details see Pendulum clock, n.d.

(a) Galileo's pendulum clock drawn by Viviani (Pendulum clock, n.d)

(b) A 19th century reconstruction[24]

Figure A.1



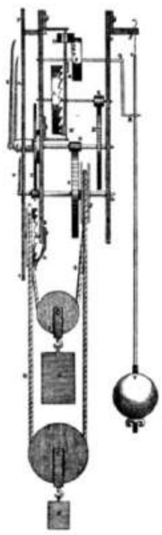


Figure A.2: Huygens' first pendulum clock (Gerland & Traumüller, 1899)

[24] Retrieved from https://cosmolearning.org/images/galileos-pendulum-clock-c-1642/ (11/11/2018).

A clock consists of three main parts (Richards, 1998, pp. 58-60; Clock, n.d.)

I. The *energy source* e.g.

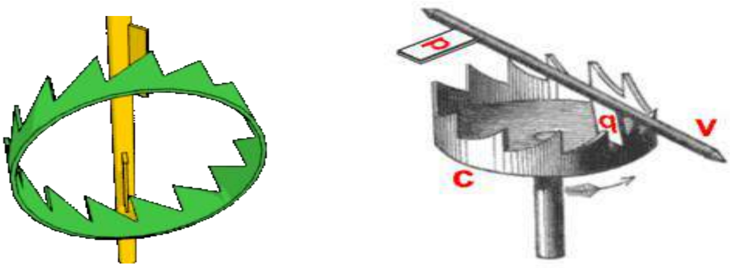
Type of clock	Energy source
Water clocks	water
Mechanical clocks	weight springs (elastic potential energy)
Quartz clocks	electricity

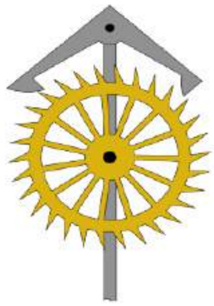
II. The *regulator*; i.e. the time-keeping element (oscillator). It is very important that the regulator possesses a natural frequency to resist vibration at other frequencies and in this way to minimize the effect of external disturbances; e.g.

	Regulator	Type of clock
No natural frequency	periodic filling of scoops with water	water clocks
Possess **natural frequency** (resistance to vibration at other frequencies)	balance wheel (foliot) pendulum	mechanical clocks
	quartz crystal	quartz clocks
	vibrating atom	atomic clock

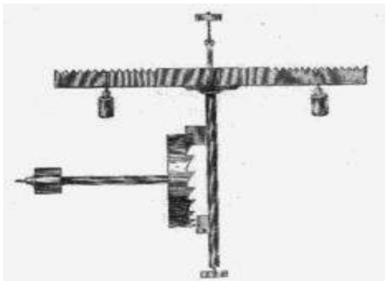
III. The *Escapement mechanism* (control energy release). The decisive step for improving the accuracy of clocks in measuring time was the invention of the *escapement mechanism* (Newton, 2004, ch. 3; Whitrow, 1972a, ch. 4, p. 60; 1988, chs. 7, 8; Fraser, 1987, pp. 49-58; Richards, 1998, ch.3; Verge escapement, n.d.). In general, this is a device that controls the energy release during the oscillatory process, by transferring energy to the oscillator to replace its energy loss due to friction, and allowing its oscillations to be counted; thus measuring time in this way (Escapement, n.d.). The invention of the *verge escapement* was a decisive step, originally combined with a balance wheel (not giving a very accurate clock), then with a spring. Verge escapements are shown in figures 4.5, A.2 of Huygens' clocks, as well as in figure A.3(a), (c), (d) below. It was finally superseded by the *anchor escapement* (figure A.3(b)) in the last quarter of the 18th century (see Verge escapement, n.d.; Anchor escapement, n.d.; Clock, n.d.; Pendulum clock, n.d.)

Escapement mechanism	Type of clock
Verge escapement Anchor escapement	mechanical clocks
Electronic oscillator circuit	electronic clocks
Microwave cavity attached to microwave oscillator, controlled by microprocessor	atomic clocks





(a) verge escapement (b) anchor escapement



(c) Verge with balance wheel (d) Verge with spring

Figure A.3: Escapement mechanisms

REFERENCES

Anchor escapement (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/Anchor_escapement (accessed 11/11/2018).

Anderson, M. (n.d.). The Mathematics of the Calendar (Lecture course MA 110, Colorado College, USA). Retrieved from https://faculty1.coloradocollege.edu/~manderson/calendar/calendar2.htm (11/11/18).

Auerbach, D. (1995). Leap-times and the fall of the second. *European Journal of Physics*, *16*, 42-44.

Barbin, É. (1997). Histoire des Mathématiques: Pourquoi? Comment? *Bulletin de l'Association Mathématiques du Québec*, *37*(1), 20–25.

Beveridge, R. W. (n.d.) *Friday the 13th and the Mathematics of the Gregorian Calendar*. Retrieved from https://www.clatsopcc.edu/sites/default/files/file/Friday_13th_Gregorian_Calendar.pdf (11/11/2018)

Borst, A. (1993). *The Ordering of Time*. Cambridge: Polity Press.

Clark, K.M., Kjeldsen, T.H., Schorcht, S., Tzanakis, C., & Wang, W. (2016). History of mathematics in mathematics education: Recent developments. In L. Radford, F. Furinghetti, T. Hausberger (Eds.), *Proc. of the 2016 ICME Satellite Meeting – HPM 2016* (pp.135-179). Montpellier: IREM de Montpellier.

Clark, K.M., Kjelden, T.H., Schorcht, S., & Tzanakis, C. (2018). Introduction: Integrating history and epistemology of mathematics in mathematics education. In K.M. Clark, T.H. Kjelden, S. Schorcht, & C. Tzanakis (Eds.), *Mathematics, Education and History: Towards a harmonious partnership* (ch.1, pp.1-23). Cham: Springer.

Clark, K.M., Kjelden, T.H., Schorcht, S., & Tzanakis, C. (2019). History of mathematics in mathematics education: An overview. *Mathematica Didactica*, *42*(1), 3-27.

Clock (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/Clock (accessed 11/11/18).

Dauben, J.W. (2007). Chinese Mathematics. In V.J. Katz (ed), *The Mathematics of Egypt, Mesopotamia, China, India and Islam : A Sourcebook* (ch.3, pp. 187–384) Princeton: Princeton University Press.

Dence, J.B, & Dence, T.P. (1999). *Elements of the Theory of Numbers*. San Diego: Academic Press.

Dominical Letter (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/Dominical_letter (accessed 11/11/18).
Doomsday (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/Doomsday_rule (accessed 11/11/18).
Drake, S. (1995). *Galileo at work: His scientific biography*. New York: Dover
Duncan, D. E. (1998). *Calendar: Humanity's Epic Struggle to Determine a True and Accurate Year*. New York: Avon Books.
Dutka, J. (1988). On the Gregorian revision of the Julian calendar. *The Mathematical Intelligencer*, **10**(1), 56-64.
Eisenbrand, F. (2012). Pope Gregory, the Calendar, and continued fractions. *Documenta Mathematica*, Extra volume ISMP, 87-93.
Escapement (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/Escapement (accessed 11/11/18).
Evolute (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/Evolute (accessed 11/11/18).
Fong, C., & Walters, M. K. (2011). Methods for Accelerating Conway's Doomsday Algorithm (part 2). *7th International Congress of Industrial and Applied Mathematics*. Melville (N.Y.): American Institute of Physics. Retrieved from https://arxiv.org/ftp/arxiv/papers/1010/1010.0765.pdf (11/11/2018).
Fraser, J. T. (1987). *Time: The familiar stranger*. Redmond (WA): Tempus Books.
Fried, M. (2001). Can mathematics education and history of mathematics coexist? *Science & Education*, **10**(4), 391–408.
Furinghetti, F. (2004). History and mathematics education: A look around the world with particular reference to Italy. *Mediterranean Journal for Research in Mathematics Education*, *3*(1–2), 1–20.
Furinghetti, F. (2019, in print). History and Epistemology in mathematics education. *International Journal of Mathematical Education in Science and Technology*. DOI:10.1080/0020739X.2019.1565454
Furinghetti, F., Jahnke, H. N., & van Maanen, J. (Eds.). (2006a). *Report No22/2006 on the Mini-workshop on studying original sources in mathematics education*. Oberwolfach: Mathematisches Forschungsinstitut Oberwolfach.
Gauss, C.F. (1801). *Disquisitiones Arithmeticae*. Leipzig: G. Fleischer (in Latin); Retrieved from https://gdz.sub.uni-goettingen.de/id/PPN235993352 (11/11/18).
Gerland, E. & Traumüller, F. (1899) *Geschichte der physikalischen Experimentierkunst*. Leipzig: Wilhelm Engelmann, p.182, fig.182 & p.179, fig. 178. Retrieved from *Pendulum clock* (n.d.) (11/11/2018).
Goldstein, C., Schappacher, N., & Schwermer, J. (eds). (2007). *The Shaping of Arithmetic: after C. F. Gauss's Disquisitiones Arithmeticae*. Berlin: Springer.
Grabovsky, Y. (n.d.) *Modern calendar and continued fractions*, Retrieved from https://math.temple.edu/~yury/calendar/calendar.pdf (11/11/18).
Grattan- Guinness, I. (2004a). The mathematics of the past: distinguishing its history from our heritage. *Historia Mathematica*, **31**, 163–185.
Grattan-Guinness, I. (2004b). History or heritage? An important distinction in mathematics for mathematics education. *The American Mathematical Monthly*, *111*(1), 1–12.
Hairer, E. & Wanner, G. (1996). *Analysis by its history*. New York: Springer
Hannah, R. (2005). *Greek and Roman calendars: Constructions of Time in the classical world*. London: Duckworth.
Heath, T.L. (1991). *Greek Astronomy*. New York: Dover
Holford-Strevens, L. (2005). *The history of time: A very short introduction*. Oxford: Oxford University Press.
Huygens, Ch. (1673/2013). *Horologium Oscillatorium: sive de motu pendulorum ad horologia aptato demonstrationes geometricae* English translation. I. Bruce (transl.) Retrieved from http://www.17centurymaths.com/contents/huygenscontents.html (11/11/2018)
Horologium Oscillatorium (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/Horologium_Oscillatorium (retrieved 8/11/2018).International Astronomical Union (IAU) (2017). *Standards of fundamental*

Astronomy. http://www.iausofa.org/sofa_ts_c.pdf. (accessed 11/11/18).
International System of Units (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/International_System_of_Units (accessed 11/11/18).
Involute (n.d.). In *Wikipedia* https://en.wikipedia.org/wiki/Involute (accessed 11/11/18).
Ireland, K., & Rosen, M. (1982). *A Classical Introduction to Modern Number Theory*. New York: Springer New York.
Jahnke, H. N., Arcavi, A., Barbin, E., Bekken, O., Furinghetti, F., El Idrissi, A., Silva da Silva, C. M., & Weeks, C. (2000). The use of original sources in the mathematics classroom. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: The ICMI study*, New ICMI Study Series, vol. 6 (pp. 291–328). Dordrecht: Kluwer.
Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235–261.
Jankvist, U. T. (2013). History, applications, and philosophy in mathematics education: HAPh—A use of primary sources. *Science & Education, 22*(3), 635–656.
Katz, V.J. (1998). *A History of Mathematics: An Introduction* (2nd ed.), Cambridge (MA): Addison Wesley Longman.
Khinchin, A. Ya. (1964). *Continued Fractions*. Chicago: University of Chicago Press.
Kraitchik, M. (1953). *Mathematical recreations*. New York: Dover
March equinox (n.d.). In *Wikipedia*, https://en.wikipedia.org/wiki/March_equinox (accessed 11/11/18).
Nelson, R.A., McCarthy, D.D., Malys, S., Levine, J., Guinot, B., Fliegel, H.F., Beard, R.L. & Bartholomew, T. R. (2001). The leap second: its history and possible future. *Metrologia 38*, 509-529.
Newton, R.G. (2004). *Galileo's pendulum: From the rhythm of time to the making of matter*. Cambridge (MA): Harvard University Press.Ore, O. (1967). *Invitation to Number Theory*. Washington (DC): The Mathematical Association of America.
Ore, O. (1988). *Number theory and its history*. Washington (DC): The Mathematical Association of America.
Pendulum (n.d.). In *Wikipedia*, https://en.wikipedia.org/wiki/Pendulum (accessed 11/11/18).
Pendulum clock (n.d.). In *Wikipedia*, https://en.wikipedia.org/wiki/Pendulum_clock (accessed 11/11/18).
Richards, E.G. (1998). *Mapping Time: The calendar and its history*. New York: Oxford University Press.
Rickey, V. F. (1985). Mathematics of the Gregorian Calendar. *The Mathematical Intelligencer*, *7*(1), 53-56.
Sawer, C. (2002). Years, Calendars and Continued Fractions. Lecture delivered at Liverpool University, 26/1/2002. Retrieved from http://www.maths.liv.ac.uk/~mathsclub/talks/20020126/talk2/continued.pdf (accessed 11/11/2018).
Schwerdtfeger, B.E. (2010). *Gauss' calendar formula for the day of the week*. Retrieved from http://berndt-schwerdtfeger.de/wp-content/uploads/pdf/cal.pdf (11/11/2018).
Sigler, L. (2002). *Fibonacci's Liber Abaci: A Translation into Modern English of Leonardo Pisano's Book of Calculation*. New York: Springer.
Smart, W.M. (1971). *Textbook on Spherical Astronomy*. Cambridge: Cambridge University Press.
Smyntyna, O. V. (2009). Chronology. In H. James Birx (ed). *Encyclopedia of Time: Science, Philosophy, Theology, & Culture* (Vol.1, pp.183-186). Thousand Oaks (CA): Sage Publications.
Stillwell, J. (1989). *Mathematics and its history*. New York: Springer.
Swetz, F.J. (1987). *Capitalism & Arithmetic: The New Math of the 15th Century*. La Salle (IL): Open Court.
Tzanakis, C., & Thomaidis, Y. (2012). Classifying the arguments & methodological schemes for integrating history in mathematics education. In B. Sriraman (Ed.), *Crossroads in the history of mathematics and mathematics education* (ch.12). Charlotte, NC: Information Age.
Tzanakis, C. (2016). Mathematics & Physics - An innermost relationship: Didactical implications for their teaching and learning. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016*

ICME Satellite Meeting – HPM 2016 (pp.79-104). Montpellier: IREM de Montpellier.

van Helden, A. (1995). Pendulum Clock. Retrieved from http://galileo.rice.edu/sci/instruments/pendulum.html (11/11/2018).

Verge escapement (n.d.) In *Wikipedia*, https://en.wikipedia.org/wiki/Verge_escapement (accessed 11/11/2018).

Vinogradov, I. M. (1954). *Elements of Number Theory*. New York: Dover.

Weekday (n.d.). In *Wikipedia*, https://en.wikipedia.org/wiki/Determination_of_the_day_of_the_week (accessed 11/11/2018).

Whitrow, G.J. (1972a). *The Nature of Time*. Harmondsworth: Penguin Books.

Whitrow, G.J. (1972b). Reflections on the History of the Concept of Time. In J. T. Fraser, F. C. Haber, G. H. Müller (Eds.) *The study of Time*. Berlin: Springer.

Whitrow, G.J. (1988). *Time in history*. Oxford: Oxford University Press.

Zeller's congruence (n.d.). In *Wikipedia*,https://en.wikipedia.org/wiki/Zeller's_congruence (accessed 11/11/2018).