# NOTHING LEFT TO BE DESIRED

## The naming of Complex Numbers

**Gavin HITCHCOCK**
SACEMA, University of Stellenbosch, 19 Jonkershoek, Stellenbosch, South Africa
aghitchcock@gmail.com

**ABSTRACT**

This paper aims to motivate and exemplify the use of theatre to exhibit the human drama behind mathematics-making, by communicating a sympathetic understanding of mathematicians in their historical context. (In this way, the ESU8 talk served also as an introduction and motivation for a workshop incorporating a play about Niels Abel.) Historical processes of profound transformation in human numerical cognition are illustrated by John Wallis, speaking in the late 17$^{th}$ century on negative numbers, and by Carl Friedrich Gauss and Augustin-Louis Cauchy in dialogue in the 1830s on the re-naming of 'impossible numbers' as 'complex numbers'. Such theatrical devices can help teachers respond to the conceptual difficulties faced by learners. By evoking the intellectual adventure through which notions like 'irrational', 'fictional', 'false', 'real', 'imaginary', and 'impossible', were negotiated, refined, named and re-named, learners may be encouraged to see negatives and complex numbers as wonderful products of a human quest.

## 1 Introduction

When I first started teaching analysis, beginning with extensions of number systems, I had little idea of the great historical transitions negotiated by the early pioneers in developing and naming the number systems: natural numbers, fractions, negatives, integers, rationals, irrationals, complex numbers. But it soon became clear to me that similar enormousintellectual gulfs have to be crossed in mathematics education today.Our students may repress their anxiety and perplexity, and appear to swallow without painour over-hurried treatment and purely logical development.But we can get a feel for their inevitable struggle by exploring the parallel historical struggle in the evolution of those intriguing names: false, fictional, negative, impossible, imaginary, complex.Appreciating this human intellectual adventure is critical for both history and pedagogy. In this paper I aim to show how the devices of theatre can help to bring alive the colourful story behind the concepts–to show, as vividly as possible, how definitions emerged from passionate debate, concepts were forged and re-forged, names were proposed and argued over and changed –nothing was instantly conceived or distilled ready-made from the intellectual air. Introducing concepts in this way, learners may be drawn into the excitement of the human adventure, and coaxed into welcoming ideas that otherwise appear alien and threatening. This may occur at all levels of education, from primary to tertiary, but the play featured in this talk is especially relevant to the introduction of complex numbers, which may happen informally in secondary (high) schools or colleges (particularly when quadratic equations are encountered), or more formally at university level. The first monologue in the play (by John Wallis) may even be used effectively in primary school when negative numbers are introduced, with suitable explanation or editing of Wallis' seventeenth century English expressions.

In my talk at ESU8, three other participants kindly agreed to help the audience get inside the heads of three historical characters who were instrumental in negotiating the concept and fixing the name, complex numbers. In the final full acceptance of ‘impossible’ numbers, and their more respectful re-naming as ‘complex numbers’, the authority of Carl Friedrich Gauss (1777–1855) was central. In contrast, the long drawn-out resistance of Augustin-Louis Cauchy (1789–1857) shows that skill in formal manipulation and willingness to make fruitful application are only partial stages on the journey towards embracing mathematical objects in and for themselves. But it was John Wallis (1616–1703), much earlier, who first pointed the way towards representing numbers as relations, grounding his relations in concrete geometrical ideas.The establishment of the negative number system was achieved historically (and is approached today in the classroom) by blending the ideas of geometrical number line and algebraic rules. Similarly, the final step in the establishment of the complex number system, for the pioneers and for numerical cognition of students today, is achieving a fully blended conceptualization of the complex plane and the algebra of complex numbers. This blending is accompanied by a transformation in perception of numbers, from representing objects to representing relations.

Many mathematicians, from Bombelli to Euler, made great use of the so-called impossible or ‘phantom’ numbers without believing fully in them, and even Cauchy struggled, well into the 19th century, to accept them as numbers.The dawning of geometrical representation was crucial, but it came slowly. Opinions differed widely, especially across the English Channel, and communications were bad. But one mathematician, based in Brunswick, Germany, was universally respected. Gauss decided in 1831 to go public with his strong conviction that the so-called impossible numbers were as meaningful as the real numbers. Much earlier hehad made great use of imaginaries in his first proof of the fundamental theorem of algebra. It was in his first paper of 1799, and in correspondence with Bessel in 1811, that he indicated his conviction that these things might be more than just strangely useful phantoms. Finally,two decades later he produced a new and respectful name that would stick, insisting that bad names had been a curse, and asserting that geometry was the handle by which to grasp these things.

It was John Wallis, back in the 17th century, who first proposed a geometrical picture of the square root of minus one as a unit perpendicular to a number line. In his Algebra, Wallis first makes great play with the analogy of the negatives – they can be seen as translations backwards along the accepted positive number line.

## 2 John Wallis introduces negative quantities

*[ENTER Wallis with stick/cylindrical ruler]*

WALLIS:[1] It is impossible (everyone agrees) that any quantity can be negativesince it is not possible that any magnitude (or geometric length) can be less than nothing, or any number fewer than none.Yet [...] that supposition (of negative quantities) is not either unuseful or absurd, when rightly understood. And though, as to the bare algebraic notation, it imports a quantity less than nothing, yet, when it comes to a physical application, it denotes as real a quantity as if the sign were ‘plus’ [gestures + in the air with his stick], but to be interpreted in a contrary sense.

As for instance: supposing a man to have advanced or moved forward (from A to B) 5 yards [*begins pacing*]; and then to retreat (from B to C) 2 yards. Suppose it be asked, how

[1] Based closely on (Wallis, 1685), including his language and spelling for historical flavour.

much he had advanced (upon the whole march) when at C, or how many yards he is now forwarder than he was at A. I find (by subducting 2 from 5) that he is advanced 3 yards. (Because plus 5 minus 2 equals plus 3).

But if, having advanced 5 yards to B*[returns to B]*, he thence retreat 8 yards to D*[he paces them out and stands at D]*; and if it be then asked, how much he is advanced when at D, or how much forwarder than when he was at A: I say minus 3 yards! (Because plus 5 minus 8 equals minus 3). That is to say, he is advanced 3 yards less than nothing!

*[throughout the rest of his speech, he points with his stick while talking]*

Which, in propriety of speech, cannot be (since there cannot be less than nothing). And therefore as to the line AB *forward*, the case is impossible.But if (contrary to the supposition) the line from A be continued *backwards*, we shall find D, 3 yards *behind* A (which was presumed to be before it).
And thus to say, 'he is advanced minus 3 yards', is but what we should say (in ordinary form of speech): 'he is retreated 3 yards', or 'he wants 3 yards of being so forward as he was at A'.

*[EXIT]*

## 3 From negatives to imaginaries to 'complex numbers'

It is clear how new and strange Wallis feels the concept of *negative quantity* might be for his readers. Today we introduce young children to'directed numbers'. Perhaps we can learn from Wallis' concrete, kinetic style here, in giving one of the first intimations of the doubly infinite number line. He goes on to extend this careful justification of negatives to an analogous representation for imaginaries, as located off the line, in a plane. His aim, as he put it, is 'to explicate what we commonly call the Imaginary Roots of Quadratick Equations'.

It took a century longer for mathematicians to begin to accept the full planar geometrical representation of what were still called 'imaginary quantities'.This is not surprising, for even the one dimensional representationof positive and negative numbers was not yet an integral part of mathematicians' cognitive scaffolding, as Wallis' painstaking exposition shows. And he could go no further, for the fully developed Cartesian plane was not part of his world.Then, a century later, a stream of others, notably Wessel, Argand, Buée, and Warren, independently saw how to match imaginariesusefully with points in the Cartesian plane, which was then available to them as a cognitive resource. The name 'imaginary' survives today, in referring to the 'real' and 'imaginary' parts of a complex number. Other apparently disrespectful names survive also: negative, irrational! It has been suggested that these names might discourage learners, who could infer that such concepts are fearsome and inaccessible; some have even campaigned to change 'complex numbers' to'composite numbers', or 'compound numbers'; we would then call imaginary numbers 'perpendicular numbers', and real numbers 'limit numbers'. However, the rudenames can be usedto great pedagogical advantage by introducing the historical adventure and drama of the making of mathematics.

In the 1830s, Carl Gauss finally grew impatient with the ambivalence and caution of the mathematical community, and announced a new name for the 'imaginary' creatures, making a strong claim for their objective existence. In the next section he engages in dialogue with Cauchy, instigator of the theories of complex integration and complex functions,but one who nevertheless took most of his life to accept that the indispensable tools he was using could have realexistence as numbers. The dialogue is the author's invention, but is based on primary sources.[2]

## 4 Cauchy and Gauss in dialogue

*[Cauchy and Gauss walk on in conversation, stopping mid-stage]*

CAUCHY: Herr Gauss, I hear you are seriously proposing to give equal rights to impossible numbers!

GAUSS: *Liberté, égalité, fraternité*, eh, Monsieur Cauchy? I would expect the French to be the first to agree with me!–not you, of course – I believe you are no *Révolutionnaire!*

CAUCHY: My father lost his job after the Revolution, and our family was forced to leave Paris when I was a child. But I hope I have been a mathematical revolutionary.

GAUSS*: Oh wirklich,* you have inspired new directions, and a fine rigour in the analysis. But I have not seen much from you lately?

CAUCHY: Cursed kings and wretched revolutionaries – all bent on disrupting or exploiting the creativity of liberal men of science! I have, since the July Revolution, once again been forced into exile from home and from my wife and daughters –

GAUSS: And, I suppose, lost your position and income. That is regrettable. Where have you been?

CAUCHY: It has not been easy …Fribourg, then Turin,and now Prague, where I am tutor to the Duke of Bordeaux, an exceedingly dull boy who will never learn any science. What's more, he insults me incessantly –

GAUSS: Insults you? A teacher is to be respected!

CAUCHY: Ah, I committed the folly of telling him that as an engineer I once repaired the sewers in Paris, and he delights in spreading the lie that M. Cauchy began his career in the sewers!

GAUSS: *Himmel*! I would not tolerate such behaviour from a pupil. Can you not return to Paris where you can be once more at the heart of science, and work in peace?

CAUCHY: *Non* – they would demand that I swear an oath of allegiance to regain my position. At least I now have my family with me in Prague. One day we will return …

GAUSS: And you will get back to inventing revolutionary mathematics!

CAUCHY: Ah, *mon ami*, for that I long! But let's sit down and discuss your own revolutionary mathematics – you can tell me why you seek to elevatecertain mathematical symbols beyond their station. I am told you even propose a new name for them!

*[during the narrator's speech they take seats at a café table, each partially facing the audience; the following dialogue may have optional ad-libbed interludes while wine is ordered from a waiter]*

NARRATION: So, now our characters have some human background, context and

[2] Mostly based on (Gauss, 1831), with fragments from (Gauss, 1899) and (Gauss, 1811), and fragments from (Cauchy 1821, 1847, 1849).

personality. Having brought them to life,we can now use them to bring to life the story of mathematical ideas – in particular the distressing square root of minus one*[slide appears]*.

$$\sqrt{(-1)}$$

GAUSS: Monsieur, in your illustrious work, you treat these symbols as mere slaves, and indeed they serve you well. But I elevate them to the status of full citizens[3]of the world of number. My paper offers a brief exposition of the principal elements of a new theory of these so-called *imaginäre Mengen*, imaginary quantities. I call them *die komplexen Zahlen*!

CAUCHY: *Des nombres complexes*! You give the impossible square roots of negatives the name of *complex numbers*–and have you thus conjured them into reality? *[shrugs and shakes his head in disbelief]*

GAUSS: *Mein Freund,* you are famous for working freely with *[quotes with mocking gesture] '*impossible numbers' …definite integrals taken between imaginary limits, and a marvellous new type of calculus analogous to the infinitesimal calculus. Why deny them equal rights with real numbers?

CAUCHY: *Bien sûr, naturellement,* I make great use of these symbols, as did our master Euler, but I hold that an imaginary equation is only a symbolic representation of two equations between real quantities. The roots of negative numbers remain impossible! *[shakes his head and pulls a face]* I confess that I harbour a horror of the square root of minus one, even while I write the symbol all over my manuscripts!

GAUSS: *Ach so*, like Euler, you saythe square root of a negative number is an impossible quantity by nature, existing merely in the imagination! It is fortunate nothing prevented him from making use of it in calculation! But between his wonders and yours we should by now have all our painful doubts removed!

CAUCHY: *[shaking his head] Non!* I am still tempted to completely repudiate that horrible symbol, abandoning it without regret, because I do not know what this alleged symbolism signifies nor what meaning to give to it. I only knowhow to make use of it.

GAUSS: *Wissen Sie*, Monsieur Cauchy, the early algebraists likewise fretted over the symbol '*minus*', and called the negative roots of equations *false* roots. And these roots are indeed false when the problem to which they relate has been stated in such a way that the quantity sought allows of no opposite.

CAUCHY: *Oui*! The old objection that there cannot be less than nothing – for how can there be less than no objects? But now we have many applications in mechanics where quantities like extension and time have opposite directions.

GAUSS: *Ganz so*! The acceptance of extensions of numberdepends on the variety of applications being forced upon us. Let us go back in time, and straight to the heart of the matter. M. Cauchy, tell me…in general arithmetic we admit fractions, although there are so many countable things where a fraction has no meaning. Not so?

CAUCHY: *Oui, oui*! And you will now hasten to point out that, just so, we ought not to deny to negative numbers the same rights, simply because innumerable things allow no opposite.

[3] *Bürgerrecht*, in (Gauss 1831, p. 171).

GAUSS: As you say – *Précisément, Monsieur*! The reality of negative numbers is sufficiently justified since in innumerable *other* cases they find an adequate substratum.

CAUCHY: But for impossible numbers there is no adequate foundation, or substratum, as you call it.

GAUSS: If imaginary quantities are to be retained in analysis – which, the versatile M. Cauchy will surely agree, seems better than to abolish them, they must be established on a sufficiently solid foundation. I have long believed it is necessary that they be considered as equally possible with real quantities.[4]

CAUCHY: Equally possible! Equally real? *Non, non, impossible*!

GAUSS: On which account I should prefer to include both real and imaginary quantities under the common designation 'possible quantities'. My recent paper gives a vindication of these names, and a fruitful exposition of the whole matter.[5]

CAUCHY: Some entities are quantities, some are numbers, some are, whatever you say, merely symbols – tools we invent in order to reach real conclusions about real possible quantities.

GAUSS: *Monsieur,* is your work really just an empty play[6] upon symbols, representing impossibilities? Consider the rich contribution which this 'play' has made to the treasure of the relations of real quantities? How can you deny them an adequate foundation?

CAUCHY: My way of treating these equations as purely formal and symbolic spares me the torture of finding out what is represented by that symbol, for which you German geometers simply substitute the letter *i*, and think thus to remove the pain.[7]

GAUSS: *Mein Freund*, I have admired your work for many years, and wondered why you never publicly worried (as the British have a habit of doing) over the question of just what you are talking about.

CAUCHY: These symbols do not have the same sort of existence as real numbers!

GAUSS: I have long considered this highly important part of mathematics from a different point of view, where imaginary quantities can be fully naturalized rather than merely tolerated, and have an objective existence.

CAUCHY: Objective existence! Nonsense!

GAUSS: That's what some of the die-hard English still say of the negatives, and they are being more consistent than you, my friend! Tell me once more, what convinces you that negative numbers exist as objective entities?

CAUCHY: Positive and negative numbers find clear application when the thing being counted – geometrical extension, or time, or velocity – has an opposite, which, when conceived of as united with it, has the effect of destroying it.

GAUSS: Exactly, and this can happen only where the things enumerated are not substances (objects thinkable in themselves), but *relations* between any two objects.

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[4] Based on (Gauss, 1899).
[5] His intention to do this was signalled 32 years earlier in his 1899 paper.
[6] This is '*inhaltleeres Zeichenspiel'*, in (Gauss, 1831, p. 175).
[7] Based on (Cauchy, 1847), quoted in (Andersen, 1999).

CAUCHY: Are you proposing to define impossible numbers as relations too?

GAUSS: For relations of a single series, plus one and minus one are sufficient to indicate the order of the transition, but for a series of series, i.e., a two-dimensional manifold, an additional pair of units denoting opposition is required, namely *plus i* and *minus i*.

CAUCHY: This is not intuitive – though you may certainly name the vertical unit anything you like.

GAUSS: These relations can be made intuitive only by a representation in space –

CAUCHY: A real number may be identified intuitively with a geometrical quantity, but they are surely not the same thing in themselves!

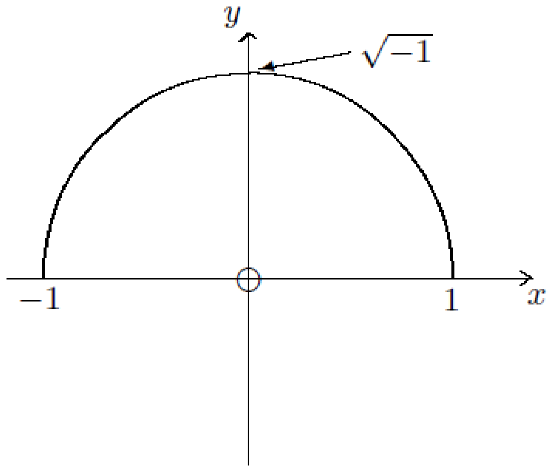
GAUSS: Just as one can think of the entire domain of all real magnitudes as an infinite straight line, so one can make the entire domain of all magnitudes, real and imaginary, meaningful as an infinite plane.[8]

CAUCHY: The directions of positive and negative unity are clearly opposite, but how do we justify assigning their square roots a real direction?

GAUSS: The directions of *plus one* and *plus i* may, in principle, be arbitrarily assigned. But if you consider that, numerically, the *square root of minus one*is a mean proportional between *plus one* and *minus one [slide appears]*

$$-1 = \left(\sqrt{-1}\right)^2 \text{gives } \frac{1}{\sqrt{-1}} = \frac{\sqrt{-1}}{-1}$$

And geometrically *[gestures with his arms throughout this speech, as another slide appears]* the rotation of the direction positive unity through a right angle to the vertical unit *plus i* is a mean proportional between the directions positive and negative unity. Then it becomes entirely natural to let *plus i* correspond to the *square root of minus one.*




CAUCHY: And you claim that an intuitive signification of your symbol *i* has now been fully justified?

GAUSS: Indeed, M. Cauchy, the arithmetic of the complex numbers is provided with *der anschaulichsten Versinnlichung,*[9]and nothing more is necessary to bring this quantity into the domain of objects of arithmetic.

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[8] From his letter to Bessel (Gauss, 1811).

[9] Concrete sensory representation. From (Gauss, 1831), in *Werke,* vol. II, 174-175

CAUCHY: Hmm … so you are claiming thatplanar direction, with rotation as the new relation, can give objectivity to the mysterious impossibles …

GAUSS: If people have considered this subject from a false point of view and thereby found a mysterious obscurity, this is largely due to an unsuitable nomenclature. If plus one, minus one and the square root of minus one had not been called *positive, negative*, and *imaginary* (or *impossible*) unity, but perhaps *direct, inverse*, and *lateral* unity, such obscurity could hardly have been suggested.

CAUCHY: It's not all in a name, surely, Herr Gauss! There are serious metaphysical objections to overcome!

GAUSS: *Nein*– by this geometrical device, the effect of the arithmetical operations on the complex quantities becomes capable of sensible representation, such that there is nothing left to be desired. In this way the true metaphysics of the imaginary quantities is placed in a bright new light.

CAUCHY: Hmm … it is not impossible that I may come to grant some sort of existence to these *quantités imaginaires* – pardon, Herr Gauss, I should say, *die komplexen Zahlen*, complex numbers! [*laughs*] I perhaps owe them some dignity for I have made such profitable use of them!

GAUSS: I predict that the architect of the new calculus will, after mature reflection, adopt not only the geometric representation of so-called impossible numbers, but also begin to call them by my far more respectful name.And I prophesy that your new calculus will oneday be called 'complex function theory'. Come M. Cauchy, let us proceed to my house and discuss more mathematics over some good *Bier und Wurst*…

*[BOTH RISE]*

CAUCHY: Complex numbers! Complex function theory! Fine names, Herr Gauss! But our fears of alien concepts die hard – it reminds me of these scandalous 'non-Euclidean geometries' that some of our colleagues are whispering about –

GAUSS: *Ah, ja*, now there's an interesting conversation we might have…

*[EXIT DEEP IN CONVERSATION]*

## 5 Conclusion

And so we end with a hint of the wider context in which strange new algebras and incredible new geometries were breaking through the official boundaries of mathematical authenticity. The battle to grasp the true nature of abstract mathematics raged throughout the nineteenth century. Cauchy himself, as late as 1847, would refer to the torture of finding out what is represented by the symbol for square root of minus one.

*[CAUCHY appears again, about 15 years older, shrugging and looking irritated]*

CAUCHY: We completely repudiate the symbol, abandoning it without regret because we do not know what this alleged symbolism signifies nor what meaning to give to it.[10]

*[EXIT]*

Shortly afterwards, still driven by his lifelong crusade for rigour, he describes in more nuanced form his embracing of a geometrical representation. He admits that he has arrived

[10] From (Cauchy, 1847).

at this new position after 'mature reflections'. He will have owed much to the influence of Gauss, whom we heard deriding Cauchy's earlier style as an empty play upon symbols.

CAUCHY: In my *Analyse Algebrique*, back in 1821, 1 was content to show that the theory of imaginary expressions and equations could be rendered rigorous by considering these expressions and equations symbolic. But after new and mature reflections the better side to take seems to be to abandon entirely the use of the sign[for] $\sqrt{-1}$ and to replace the theory of imaginary expressions by the theory of quantities which I shall call geometric. Of course I shall have to define very carefully the term 'geometric quantity' and further define the different functions of these quantities, especially their sums, their products and their integral powers, by choosing such definitions as agree with those admitted when we are dealing with algebraic quantities alone.[11] *[EXIT]*

Thus, at last the primary architect of complex analysis comes around to a geometric grounding of the complex numbers he has been using so fruitfully for over thirty years! Today, we keep both the symbolic and the geometric in fruitful union, much as Gauss did.Curiously, Gauss did express some doubts, both privately and publicly, in 1834 and 1849, about whether the geometric representation could really capture the true essence of complex numbers.In 1834 he wrote in a letter:

*[cameo appearance of GAUSS]*

GAUSS: I admit that this *Darstellung* – this geometrical representation, is not really *der Wesen*–the essence – of their being, which is to be grasped by higher faculties in a more general way. But it may be the only completely pure and convincing example of their application.[12]

*[EXIT]*

Our aim in this paper has been to illustrate, by example, how the devices of theatre, based on extracts from primary sources, can bring out the excitement, the struggle, the sheer achievement, of historical advances in mathematics. Concepts we expect students to swallow unquestioningly are seen to have been fiercely contested by the great mathematicians who gave them birth in a community dialogue. Re-constructing and re-living with the pioneers such debates may help us understand what conditioned prejudices at the time, and what fuelled the journey to new thought paradigms. Use of theatre can bring vividly to life both the people and events behind the abstract concepts of the mathematics curriculum. Such an activity may be mounted with very little preparation, and few theatrical props, as indeed this dialogue was at ESU8 in Oslo. From my own experience introducing complex numbers at university level, motivating concepts in this way, using historical characters in dialogue, is well worth the time taken, as the receptivity of learners is enhanced.

I pose the concluding challenge: Can similar plays add value at various levels of mathematical instruction, for simpler ideas or even more complex ideas? How can we make use of theatre to ease the pain and enhance the joy in learners' conceptual development?

[11] Based on (Cauchy, 1849).
[12] From (Gauss, 1834).

## Acknowledgements

I would like to express my gratitude to the three people who helped to bring life and authenticity to the dialogue by their portrayal of Wallis, Cauchy and Gauss: Peter Ransom, Fred Metin, and Gero Stoffels. In addition, Peter Ransom kindly contributed items of period dress and equipment to enhance the staging.

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