17TH CENTURY FORTIFICATION AND GEOMETRY

A military and mathematical revolution

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ABSTRACT

After the French Italian wars, military architects tried to reply to the power of cannons by creating new shapes taking into account the straight lines of cannonballs trajectories: the bastions. Different models were invented by a variety of authors, but none of them gave real reasons for their constructions. The first mathematical proofs were given by Jean Errard de Bar-le-Duc, a military engineer of King Henry IV of France. His works were well received in the Low Countries, where they inspired a new Dutch "manner of fortification" (i.e. shapes with algorithms of construction). Among the authors linked to the Leiden school of engineers, Samuel Marolois was the first to explain the use of trigonometry to compute the distances and prove the shapes to be fitted for defense. This paper compares different manners of constructing bastions and different uses of geometry to be found in the original texts.

1 A short history of "modern fortification"

Fortification is a matter of geometry, as anyone would agree considering the star shaped forts, citadels or cities enclosures that are still visible around the world including Norway, with the famous Akershus fortress in Oslo. All of them are visible from the sky through Google Earth or other satellite imagery services[1]. The reasons for that geometric choice must be sought in the history of war technologies, as it is a response to the progress of artillery in the 15th century. Let's give the floor on this subject to a Dutch writer of the golden age, Matthias Dögen, whose *Architecture militaire moderne ou fortification* (Dögen, 1648) gives one of the best short histories of fortification.

According to Dögen, the oldest usual way to protect people in the cities was to build large stonewalls around them, but there was a huge drawback: the assailants could come under the cover of the walls, when the defenders had to get uncovered to try to repel them. In the Antiquity, architects invented the principle of flanking: they built outer towers to allow defenders to shoot their enemies from behind when these enemies dared to approach the walls. This principle was still in use in the medieval castles with their round towers at the edges of the main walls.

But as Geoffrey Parker showed (Parker, 1988), when the Western world discovered the use of gunpowder, it was a real revolution in attacking as well as in defending and building fortresses. The high medieval towers were ideal targets for the newly melted cannons and their easy remote destruction caused casualties inside the cities without any damage for the assailants. Height had become a tactical disadvantage; bastions were to be born. The bastions are a kind of low pentagonal towers, intended to host the defenders' canons and cannonballs, hence their names *bulwarks* (from the German *Bollwerke*). The

[1] For regular polygons, we recommend travelling by Google Earth in this order: Forte San Pedro, Cebu, (Philippines); Vardøhus, Finnmark (Norway); Fort Belgica, Banda Neira (Indonesia); the citadel of Saint Tropez (France); the Alba Carolina citadel in Alba Iulia (Romania); the cities of Neuf-Brisach (France) and Palmanova (Italy). Finally, the old city of Nicosia (Cyprus) with its eleven bastions might hold the record.

geometric shape of the bastions was generated in such a way as to force the trajectories of gunfire and to reduce damage on the walls, but as we can see on heritage monuments or original maps, there is a variety of sizes and angles according to their creators and time of their conception. The set of consistent shapes, along with their algorithms of constructions and even their justifications, were called *manners of fortification*. Early modern writers distinguish three principal manners at the turn of the 17th century: the Italian, the French and the Dutch[2]. All of them find their origin in the Italian one, but we can question the reasons of their differences and the use of mathematics for the generation and justification of shapes, especially the use of Euclidian geometry. In order to render unto Caesar the things which are Caesar's, we start with the Italian architects and follow their inventions towards North, to France and the Low Countries, where military architecture would find their (temporary) perfection through mathematics. Then we give examples of constructions described in texts designed for officers to study or practice the "new" fortification.

2 The creators of modern fortification

Modern fortification was invented in Italy around the end of the 15th century, during the Italian Wars. Architects like Michelangelo or Leonardo took part to a large movement of research about new shapes of city enclosures. The first inventor of the bastion remains unknown, if he ever existed, but this shape (see for example Figure 3.2) was widely accepted in Italy as a response to the power of cannons. Everywhere in Europe, new bastions would be built in the Italian manner of fortification, undertaken by Italian engineers themselves. Many local authorities had connections with Italy. All the Kings and Dukes, every City Council urged their recently engaged Italian counselors to secure their places with new enclosures *à l'italienne*[3] (Rogers, 1995; Parker, 1988, ch. 1).

Though universally acknowledged as taking its origins in Italy, modern fortification can't be reduced to the presence of bastions at the corners of city walls. As an architectural artefact, the bastion, and more generally the fortified enclosure would know many enhancements, practical as well as theoretical. Before the end of the 17th century, when the manner of Monsieur de Vauban becomes hegemonic, at least two major stages of successive improvement are notable as far as geometry is concerned: in France, initiated by Jean Errard with his *Fortification reduicte en art et demonstrée* (Errard, 1600) then in the Low Countries after the publication of Samuel Marolois's *Fortification* (Marolois, 1615). Let's describe in short the national contexts.

2.1 Italian authors, before 1600

Architects-artists like Leonardo and Michelangelo didn't publish their researches about the shapes of city walls. Before the mid-16th century, only a few books were published (De la Croix, 1963), but we find in Book 6 of Tartaglia's *Quesiti et invention diverse* (Tartaglia, 1546) a first attempt of reflection about the rules every architect should follow in fortifying a city. While admitting not to be a practitioner,

[2] Some authors add a fourth one, the Spanish manner, which can somehow be seen as a variant of the Italian manner.

[3] For a critical point of view on the expression *Italian trace*, see Bragard, 2014.

Tartaglia approaches the problem of fortification with his intellectual tools, and he determines six principles about the shape, size and areas of the cities enclosures.

Tartaglia's followers won't extend his reflection, but mostly publish their methods to draw the star shaped fortresses. The reasons for particular designs are kept secret, giving way to discussion on their concrete fulfillment on the field. Some of the treatises which are published will be translated into foreign languages, mainly in French (for example: Cataneo, 1574; Theti, 1589). In general, these books deal with architecture more than geometry. However at the very end of the century, an interesting controversy occurs between two architects in charge of building the Palmanova fortress, Giulio Savorgnano and Buonaiuto Lorini (La Penna, 1997). Having rather different conceptions about the final shape of Palmanova, both had had to advocate their own views before the Venitian Senators. Lorini published his rules as an appendix of his *Delle fortificationi* (Lorini, 1597), while Savorgnan left his unpublished. But Savorgnan's manuscript was found amongst Galileo's papers, which indicates a certain consideration.

In order to show the significant demand for knowledge on fortification, let's mention the private courses given by the same Galileo in Padova around 1590 (Valleriani, 2015). The manuscript of the course, *Trattato di fortificazione*, begins with many usual techniques of practical geometry, such as the drawing of perpendiculars, parallels, regular polygons, and so on. No doubt that there was a need for a theorization of fortification practices. Finally Jean Errard arrived.

2.2 Jean Errard and the French geometric School on fortification

At the end of the Italian Wars, the conflicts moved from Italy to the North, concentrating on the frontiers of the Spanish Habsburg Empire. Following the same path, many unemployed Italian engineers were recruited here and there in Europe to build new ramparts for fragile cities.

In France for example, when King Francis I ordered the construction of a fortified harbor at the mouth of the river Seine, this task was attributed to Girolamo Bellarmato, an Italian architect who worked in several other cities in France. Along with Castriotto and Marini, Bellarmato was one of the major military architects in France at the time. Till the end of the 16th century, it seems that no French engineers had been able to supervise the creation of important fortresses, and even more so to write something substantial about fortification. But there was a need for an elite corps of French engineers. Little by little, the Italians were replaced by local military architects, and King Henry IV ordered his favorite engineer, Jean Errard de Bar-le-Duc, to write the first French book on Fortification. The title of this book is indicative of its project: *La fortification reduicte en art et demonstrée* (Errard, 1600) meaning that the whole process will be established on a detailed analysis of the needs, then fully described, and finally justified by mathematical demonstrations. In the preface, Errard himself justifies the title (Errard, 1600, p. 1):

> *I dared undertake what every engineer so far hasn't dared or wanted, at least nothing was written about that science. Because the discourses on mechanical things do not deserve this Title, not being here a matter of strokes which for*

someone could succeed by accident, but a matter of Geometrical demonstrations that will give infallible certitude to anyone[4].

As one can read the specificity of Errard's approach lies in his attempt to turn the practices of fortification into a true science. For that purpose, the book starts with a description of all the cannons in use in the French armies, and the conversion of their power into the men's working days which would be necessary to reconstruct the collapsed walls. Four principles, called the *Maxims of fortification*, are then presented, the principal two dealing with the flanked angle, which must be right, and the line of defense, whose length must not exceed the reach of the defender's weapons (see figure 3.6 for a glossary). The algorithms of constructions are given in detail and the different steps are well illustrated. Some places in the North of France were fortified according to Errard's principles, especially the citadel in Amiens, still visible now, except for the two Eastern bastions, which were destroyed to leave space for an enlarged road.

Errard's manner was abandoned shortly after his death in 1610, but his *Fortification demonstrée* kept a reference book for many authors, especially teachers, till the end of the 17th century (Métin, 2016). His legacy was received and prospered by Dutch engineers related to the famous school of engineering at the University of Leiden.

2.3 The Leiden School

During the Eighty Years' War, many cities were besieged, even alternately by one and the other side, namely the Spanish Empire and the Dutch Republic. The young Dutch Republic had to face an experienced enemy, but Maurice of Nassau lacked time, money and trained engineers to fortify towns and protect citizen from the Spanish *furia*. Following the advice of his closest counsellor Simon Stevin, he founded in 1600 the *Duytsche Mathematique*, an engineering training course in theoretical and practical mathematics taught in Dutch (Dijksterhuis, 2017). Stevin had written a treatise on fortification (Stevin, 1594), but apparently no fortresses was built on the field according to the shapes he created.

Of course, fortification practices already existed before the foundation of the *Duytsche Matematique*. Italian engineers such as Paciotto and Marchi worked for the sovereigns of the Spanish Low Countries. But we also find non-Italian mathematics practitioners at work in Antwerp, such as Michel Coignet, a mathematics teacher and an instrument maker (Meskens, 2013) who became the equivalent of Stevin in the court of the Archdukes Albert and Isabelle of Austria, governors of the Habsburg Netherland. Several subsisting manuscripts show that Coignet gave lessons in French, Italian and Spanish. His taught *manners* are revealing of the transition between Errard and the Dutch fortifiers.

Michel Coignet's explanations on the practices of his time and military side are exposed in a French manuscript course on trigonometry (Coignet, 1612). The author gives a first method of fortifying polygons, which deals with lengths and no angles at all, but he is not satisfied with it. Asserting that the flanked angle needs to get a unique value of 90°, he gives a second method starting based on angles, following Errard's view. Meanwhile in the opposite camp, the disciples of Stevin free themselves from the constraint of the right

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[4] J'ai osé entreprendre ce que tous les Ingénieurs, jusqu'à présent, n'ont voulu ou osé, au moins n'en paraît-il rien par aucun écrit traitant de cette science. Car les discours des choses mécaniques ne méritent point ce Titre, n'étant ici question des traits, qui à quelqu'un pourraient réussir à l'aventure, mais de démonstrations Géométriques qui donnent à tous assurance infaillible {with modernized spelling}.

angle, thanks to trigonometry. Before 1650, publications will follow one another, more or less inspired by the first of them, Marolois' *Fortification* (Marolois, 1615).

Marolois' book was published at the time he was unsuccessful in applying to the job of professor at Leiden school of engineers. It is the third and last part of a complete mathematical textbook for the use of engineers including geometry, the use of surveying instruments, gauging, trigonometry, perspective and fortification (Marolois, 1616). The *Fortification* is composed of two parts, the first of which consists in a case study of a variety of shapes, from the square to the dodecagon, each of them being divided into several sub-cases. In his attempts to find the perfect shape is, Marolois explores the different values he can assign to the proportion between face and curtain, or flank to gorge, or even flanked angle to flanking angle (see glossary on figure 3.6). After having studied fifty or so cases, he opts for simple proportions (3 to 2 or 4 to 3) and gives the eleven maxims which will define the first Dutch *manner* of fortification. His successors will more or less follow the same rules, leaving aside explorations to focus on now universally accepted shapes.

We'll show in concrete terms in §3.3 the importance of trigonometry in Marolois' works and more generally in Dutch fortification. In Western countries, trigonometry had found his corner stone in Regiomontanus's *De Triangulis* (Regiomontanus, 1533), including the Law of Sines and the resolution of triangles. But Regiomontanus's main goal was to provide the astronomers with useful theorems (Maior, 1998, 41-46). Grattan-Guinness also points out the prominent role of trigonometry during the period 1540-1660 that he names "age of trigonometry' (Grattan-Guinness, 1998, 174-233), but he mostly mentions astronomy, surveying and navigation. According to him, one of the most important books for this period is Pitiscus's *Trigonometria*, (Pitiscus, 1600). A close examination of the different editions shows the growing range of applications of trigonometry: a first version in 1595 contained only two theoretical parts, but from 1600 on, it would be completed with the tables of sines, tangents and secants, plus an appendix on applications of trigonometry to a variety of problems in surveying, geography, gnomonic and astronomy. In the 3rd edition (Pitiscus, 1612), the appendix is extended to a new domain: architectonic, that is military architecture. The huge influence of this work amongst engineers and fortifiers can be measured by its numerous quotations and even reproductions of its contents in many books for decades in Europe.

Now we examine the three different *manners* we introduced above.

3 The *Manners* of fortification

Our paper reports a workshop, the "working part" of which consisted of studying different *manners of fortification* with ruler and compass. The main goal was to question the role of geometry in these several *manners*, beyond the simple use of instruments to draw the required shapes. We essentially describe the participant's procedures.

We have chosen to focus on the three major steps in the researches on fortification: the Italian use of diagonals in polygons, Jean Errard's hexagon "reduced into art and demonstrated", and Samuel Marolois's "trigonometricky" hexagon. Due to their minor role in history, we had to leave aside Michel Coignet's propositions (Coignet, 1612), despite their interest as transition markers between Errard and his Dutch followers.

3.1 Two examples of Italian architects

Despite Tartaglia‘s investigations on the fundamentals of fortification and his expression of what should be its principles, we find but few real justifications in the Italian books on military architecture of the 16th century. Even if the fortresses were fully described, using many diagrams, they were more detailed as stone and earth real life fortresses than as ink and paper diagrams.

In a majority of Italian books and manuscripts, the construction algorithms keep undisclosed and the reader has to make sense of the figure himself. This non-didactical aspect might be due to the identity of this targeted reader, who may have been an engineer, or at least a well-skilled person. To give a typical illustration of this specific *manner*, let’s examine first a manuscript we found in the Jagiellonian Library in Cracow (Anonymous, 16th century). Its anonymous author writes in Italian, but he favors the figures: the only comments describe lines and angles in terms of stone and earth fortresses, but don’t explain the generation of the shapes. For instance, below a diagram of a fortified hexagon we can only read: “All lines are drawn from and through the points and the intersections. Thus, measuring is not needed.”[5] (Anonymous, 16th century, fol. 12v).

Let’s try to apply this principle to reconstructing one of the numerous fortified shapes in the manuscript, for example the fortified square on fol. 24v (Fig. 3.1). Our intuitive view leads us to start with the largest square ABCD and its diagonals (AC) and (BD). The side of the square is twice the opening of the compass, that is to say the diameter of each arc drawn inside ABCD. These arcs, centered at A, B, C and D, give the midpoints of the sides, respectively E, F, G and H, so the medians (EG) and (HF) can be drawn. If we then draw the arcs centered at midpoints E, F, G and H, we obtain useful points, namely I, J and K, L, which allow us to draw further symmetrical straight lines, as do points M, N and O, P. Our initial square is now divided into sixteen smaller squares.

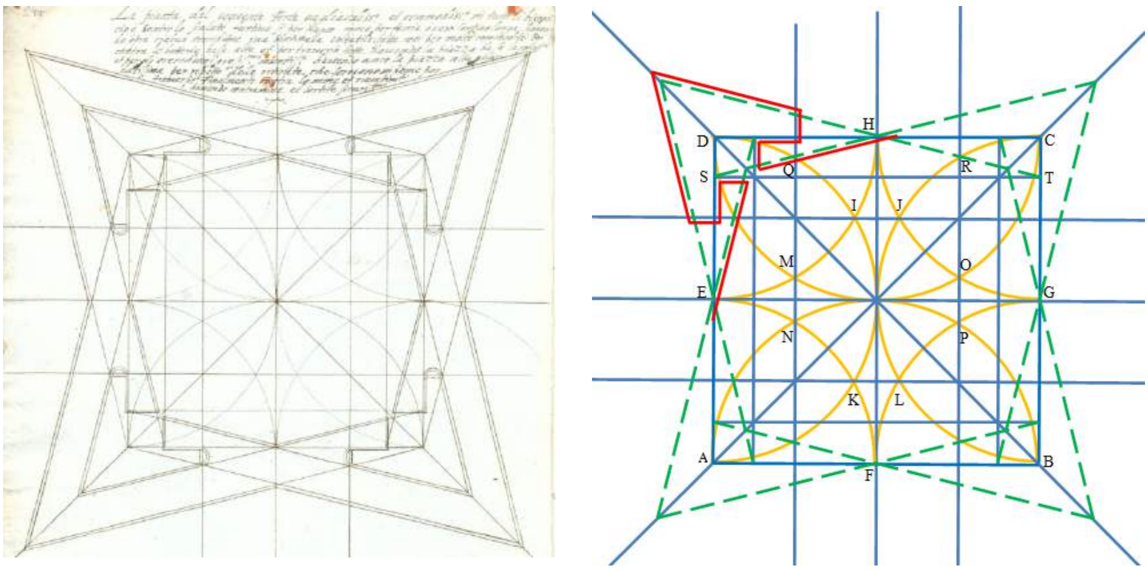

Figure 3.1: Original drawing and attempted reconstruction (Anon., 16th century, fol. 24v)

[5] *Tutte le line sono tirate per i punti et intersegationei regolari. Et peró non c’é di bisogna la misure* (All translations into English were made by the author of this paper).

But the arcs centered at E and G intersect vertical lines (MN) and (OP) at Q and R respectively. Lines (HQ) and (HR) meet two opposite sides of the square at S and T respectively. It looks like (ST) bisects the formerly drawn squares. Indeed S and T could have been found as midpoints of segments, but no indication is clearly given on the original figure. To complete the construction, we just have to produce rays [TH) and [SH) to their intersection with the diagonals of the initial square, and repeat this operation three times to get the four vertices of the bastions. Finally, we pick the appropriate segment lines to obtain the desired shape, as shown on the top left of the reconstruction in Figure 3.1.

We take our second example from a very rare book (Scala, 1598) in which the constructions are based both on unmeasured lines and on lines measured by a scale. It has been written by Giovanni Scala, a Roman mathematician and engineer, otherwise known as the geometer who completed the edition of Pomodoro's *Geometria prattica* (Pomodoro & Scala, 1599). It is likely that Scala gave lessons on fortification to several German and Polish officers in Roma before 1600. His book on fortification was first published as a portfolio of plates, and we can find copies of it with hand-written comments in Poland and in Paris. A frontispiece and a preface to Henry IV King of France were added in 1598 in order to make it a real book. It contains more than 50 various shapes of bastions, the constructions of which are more or less explained. As far as we know, none of them were actually built on the field.

In Figure 3.2, we show the bastion on plate 6.

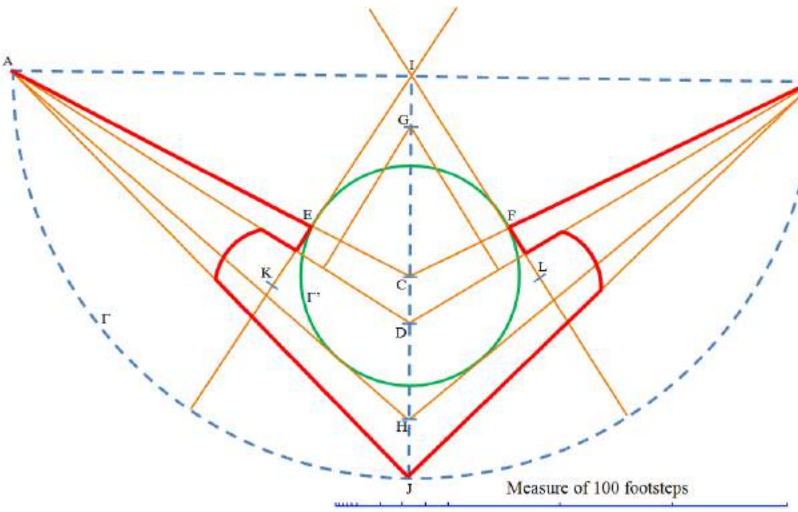

Figure 3.2: Giovanni Scala's bastion (reconstructed from Scala, 1596, plate 6)

Here is our reconstruction of the figure: take segment [AB] of 180 steps[6]; on [AB] draw the half-circle Γ and the perpendicular bisector (IJ), which intersect at J. Join the midpoint C of [IJ] to A and B; from C as a center and with radius 25 steps, draw the circle Γ', which cuts (CA) and (CB) at E and F respectively. Put point D on [JC] at 10 steps from C, and G on [IC] at 30 steps from C; join both points to A and B; segments [CA] and [CB] cut circle Γ' at respectively; produce lines [IE] and [IF] to the circumference of Γ and draw segments [DA] and [DB]; from A and B, draw the tangents to circle Γ', which meet at H;

[6] The *passo*, or geometric footstep, was the thousandth part of the Roman mile, i.e. around 1.5 meters. For our reconstruction, we took the numerical values from the figure, using the scale engraved on it. Our starting point was the value of 180 footsteps, which is the explicit basis of several other Scala's constructions.

from G, draw two segments parallel to (IE) and (IF) respectively, ending on [AD] and [BD] respectively; K is the midpoint of the segments cut on line [IE) by lines [AD] and [AI]; draw an arc with center K and radius 12.5 between these two lines, and do the same symmetrically with L. The bastion is obtained by selecting the appropriate arcs and line segments among the various ones that have been drawn (in bold red on Figure 3.2).

Of course, we can't be sure of our reconstruction, especially when they deal with tangents or measures, as the engravings in the portfolio are sometimes imprecise, or show hesitations. Nevertheless, we can assume that the engraver, and consequently the inventor of the shape (they may be the same person), were aware of construction programs. These programs may have had to keep illegible for non-specialists, which would explain the lack of instructions in the texts. Anyway, the shapes do not look like they had been created responding to scientific principles, but rather in an aesthetic aim. The way of generating them indicates a global idea of the bastion profile, but no mathematics is used, except the usual geometrical concepts and drawing procedure. This will be changed in 1600 by French engineer Jean Errard de Bar-le-Duc.

3.2 Jean Errard, and the Euclidean proof

Errard's fundamental example is the regular hexagon, explained on one of the six equilateral triangles it is composed of (cf. Fig. 3.3). In fact, the author considers the triangle, the square and the pentagon as unfit for receiving right-angled bastions, what can be justified nowadays by the impossibility of applying Errard's general algorithm of construction to these particular shapes.

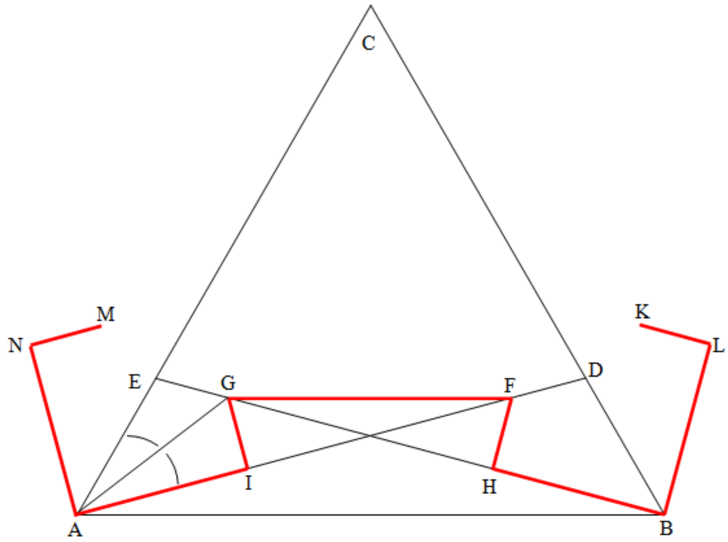

Figure 3.3: Errard's hexagonal construction (after Errard, 1620, p. 40)

Here is the text given in the posthumous edition made by Alexis Errard, Jean's nephew and also an engineer, supposedly according to his uncle's will. We give this version preferably to the very first one, as it clearly separates the construction and the proof (Errard, 1620, p. 40):

> *Let be proposed to fortify a hexagon, as far as the hexagon can be divided into six equilateral triangles. On AB describe the equilateral triangle ABC, and angle CAD of 45 degrees. Draw line AE equal to line BD, then drawn line BE. Divide angle EAD into two equal parts by AG, & let DF be taken equal to EG. Draw the curtain*

wall GF, as well as FH perpendicular to BE. Let AI be taken equal to BH, and GI be drawn perpendicularly as FH. So are described the two half-bastions AIG & FHB[7].

Since Errard based his construction algorithms on scientific principles, he had to prove that the results met his requirements, the most important being about the line of defense (i.e. AF on Fig. 3.3 or BF on Fig. 3.4), whose length must not exceed 100 toises[8]. Here follows a modernized version of Errard's demonstration (after Errard, 1600, p. 24; Errard, 1620, p. 41-42; see Fig. 3.4).

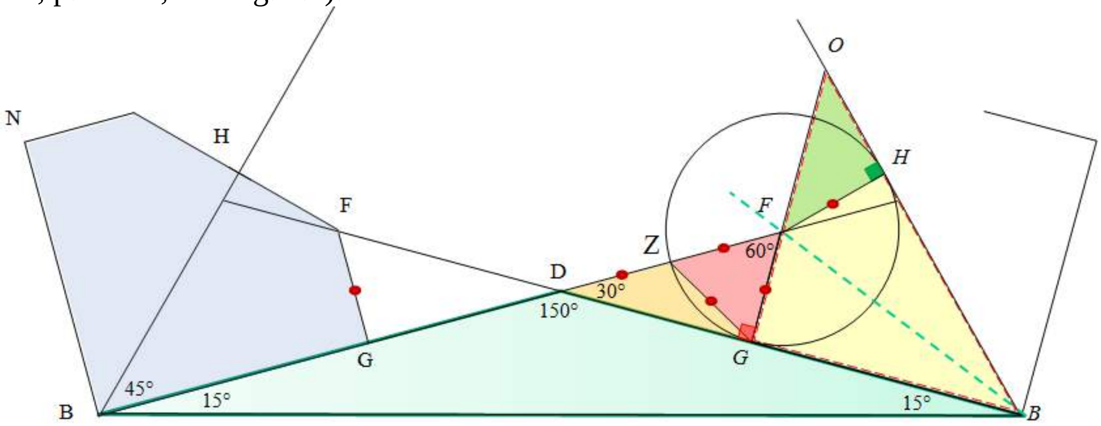

Figure 3.4 : Errard's demonstration (after Errard, 1600, p. 25 & Errard, 1620, p. 42)

Remembering that F is on the bisector of angle *O*B*G*, let's draw a circle centered at F and tangent to the sides [*BO*) and [*BG*) of angle *OBG* at *H* and *G* respectively. The circle cuts [*F*D] à Z. We first consider triangle DB*B*: since ∠HB*B* = 60° and ∠HBG = 45°, then ∠DB*B* = 15°, same for ∠D*B*B, by symmetry. Thus ∠BD*B* = 150° and consequently ∠*F*D*G* = 30°. Now let's examine triangle *F*D*G*: right-angled at G, it has an angle of 30° at D, so ∠D*FG* = 60°. But Z*F* = Z*G*, so *F*Z*G* is equilateral and *F*Z = Z*G* = *FG*; moreover, ∠*F*Z*G* = ∠*FGZ*, so ∠Z*G*D = 30°, ΔZ*G*D is isosceles and Z*G* = ZD (= FH).

Having linked these lengths together, Errard takes FG = *FG* = 16 toises as a common unit for all of them. Using the Pythagorean theorem in right triangle *FG*D, he obtains $\mathrm{D}G = \sqrt{32^2 - 16^2} \approx 27.713$ (he takes $27\frac{3}{4}$). in the isosceles right triangle *OHF*, $OF = 16\sqrt{2} \approx 22.63$ (Errard gives $22\frac{3}{5}$), and finally, in the isosceles right triangle *BGO*, $BG = GO = GF + FO = 38\frac{3}{5}$. The line of defense F*B* (or *F*B) can now be evaluated: $\mathrm{F}B = \mathrm{FD} + \mathrm{D}G + GB = F\mathrm{D} + \mathrm{D}G + GB = 32 + 27\frac{3}{4} + 38\frac{3}{5} \approx 98\frac{1}{2}$, which is less than 100.

Thanks to his choice of appropriate angles, Errard needed only basic Euclidean propositions to systematically find what was missing. No trigonometric lines there but essentially the Pythagorean Theorem. We would call this demonstration Euclidean, no doubt it would have pleased Jean Errard. But this pleasant aspect of the right angle has major drawbacks on the field: the defenders on the flanks are turned towards the walls instead of the counterscarp, and the faces of the bastions are too large to resist a long time

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[7] *Soit proposé à fortifier un Hexagone, d'autant que l'Hexagone se divise en six triangles équilatéraux. Soit sur AB décrit le triangle équilatéral ABC, puis soit fait l'angle CAD de quarante-cinq degrés. Soit faite la ligne AE égale à la ligne BD, en après soit tirée BE. Soit divisé l'Angle EAD en deux également par la ligne AG, & soit prise DF égale à EG, & tirée la Courtine GF : comme aussi FH perpendiculaire sur la ligne BE. Soit prise AI égale à BH, & soit tirée la ligne GI perpendiculairement comme FH. Ainsi seront décrits les deux demi Bastions AIG, & FHB* {modernized spelling}.

[8] An old unit, roughly corresponding to the human height (approximately 195 meters).

to the pounding of artillery. Errard's heirs in Holland would use the newly invented methods of trigonometry to get rid of the right flanked angle and set the generation of bastions free of it.

3.3 Samuel Marolois's "trigonometricky" hexagon

Unlike Errard, Marolois doesn't establish his constructions on necessity and principles drawn from the field practices. The first few pages of his *Fortification* state the values of angles for any regular polygon from the square to the dodecagon. The flanked angle is not right; on the contrary its value depends on the angle at the center of the polygon. The different values given in a table (Marolois 1638, 5) correspond to a simple algorithm, which is guessable at first glance as in a modern spreadsheet, but not formally expressed. In our time, it would be: flanked angle = half-angle at the center + 15°. Knowing that the angle between the flank and the curtain is always right, all the other angles are determined. It is only after a variety of case studies according to diverse proportions of lengths that Marolois gives "the manner how to describe succinctly the designs or Plots of some regular Fortifications" (*Idem*, 27) and finally the *Maxims* of regular fortification (*Ibid*., 43).

Here is a slightly modernized version of the English translation of the first example, the design of a hexagonal fortress (*Ibid*., 29-30, to be followed on figure 3.5, using the glossary on figure 3.6 for specific terms[9]):

> *Let there be given a Hexagonal Fortress to be fortified, whereof the face AC makes 24 rods[10], and the flanked angle 80 degrees, according to which the interior flanking angle will make 20 degrees, and the exterior 140 degrees. Let the curtain be 30 [i.e. 32] rods, which gives the reason of the face to the curtain as 3 to 4. To do this, we shall draw the infinite covered line AB, by the help of a graduated instrument, the other angle CAD of 20 degrees (of 20, because the interior flanking angle, which is always equal to it, makes here 20 degrees) by means of the indefinite line, upon which you make the length of the face 24 rods, as from A to C; from which point C, the perpendicular CD being drawn upon the line AB, shall be placed from D the length of the curtain, which is here 32 rods as from D to E. Finally the distance AD from E to B, and the perpendicular EF the distance of CD as from E to F. Drawing the line FB, you have the other face, so that all the part of the given reason are described; […] we make the angle HKA only of 35 degrees, according to which the gorge in the flank will be almost as 4 to 3, or somewhat more by reason of the line HK, cutting the diagonal line AG at H, from which point H the line HN being drawn parallel to AB, you shall have the interior Polygon, upon which the lines CL and FN being drawn in length, the lines DC to L and EF to M. In doing so, all the essential parts of the said fortress will be described.*

This excerpt needs some comments. The first four lines remind us of the previously calculated angles, of the chosen length of 24 rods for the face, and of the proportion of the face to the curtain, that is 3 to 4. This being established, the construction program is described step by step: on line AB as a basis, with an angle of 20 degrees and a length of 24 rods, we draw the face AC of the left bastion; we put on line AB the following points:

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[9] According to Marolois's notations, the figure shows lowercase letters, while the text in written in capitals. Moreover, the letters in figure 3.6 are not consistent with those in figure 3.5.

[10] The unit of measurement is the Rhineland rod (a 12-foot rod, approximately 3.8 meters), used during the reign of Maurice de Nassau. The contemporary English rod is equivalent to 5.5 yards (around 5 meters).

D, by orthogonal projection of C on AB, then E and B such as DE = 32 (length of the curtain wall), and EB = AD. Symmetry is at work, even if not mentioned; point K is determined on line AB by ∠AKC = 35°, then point H as the intersection point of (KC) and the side AG of equilateral triangle AGB (G is not visible on figure 3.5); (HN) is drawn parallel to (AB), then C is orthogonally projected on (HN) to get L, and the complete shape ACLMFB is obtained by symmetry.

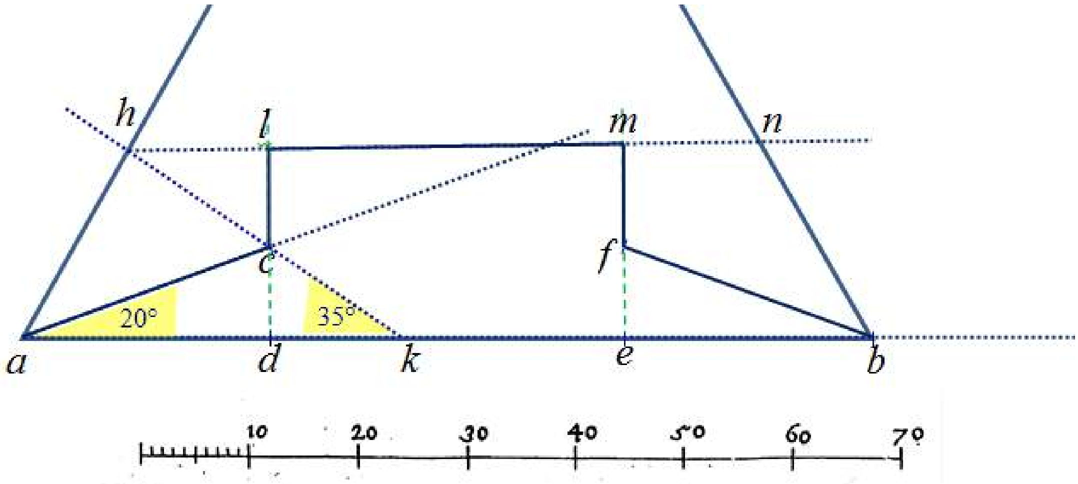

Figure 3.5 : Marolois's construction (after Marolois, 1615, plate 13, fig. 72)

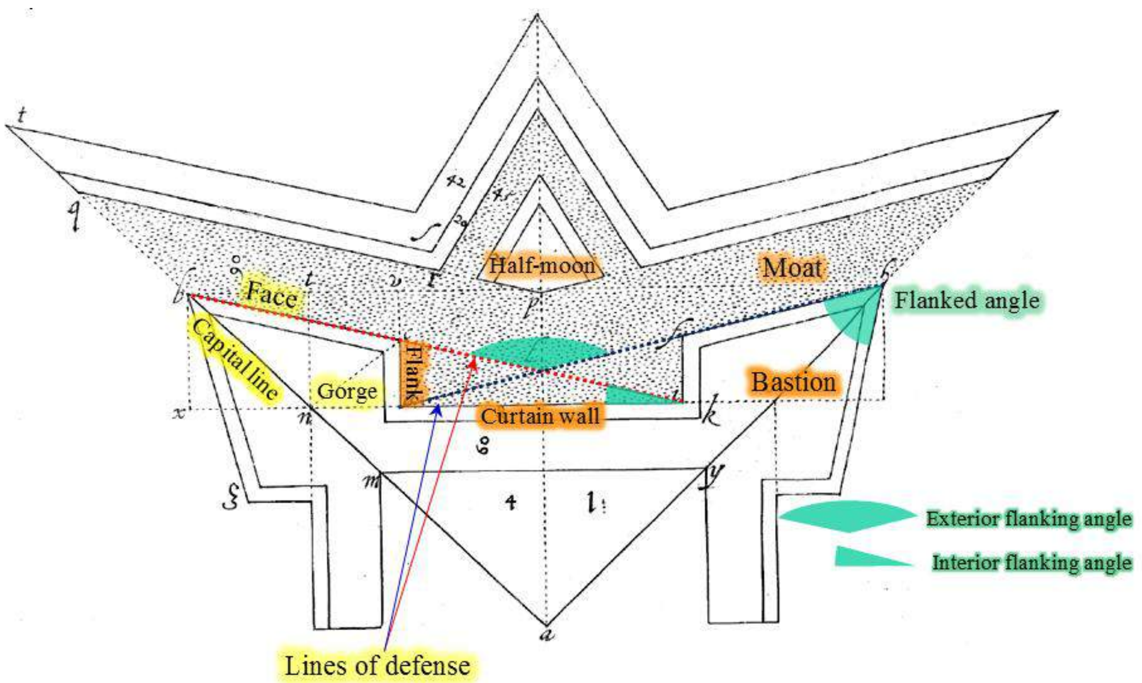

Figure 3.6: Glossary (after Marolois, 1615, plate 1, fig. 1)

The profile being constructed, Marolois doesn't give any calculation or demonstration, because all of them have been detailed before. His translator Henry Hexham is even terser, writing for example (*Ibid.*, 20):

Here is nothing but that which is ill calculated by the Author, or rather by his disciples, as from the beginning (without all doubt) seeking to help themselves with *the figures put here under, which was needless, supposing that they are skilled in Trigonometrie.*

In fact, trigonometry is excessively used by Marolois to calculate every length and distances in every case study in the first part of his book. The hexagonal fortress we showed the construction above had already been the subject of calculations in the 44th example (Marolois, 1615, fol. Tv). For example, DL is take, as the sine of $\angle$DAC, providing that the face AC (of 24 rods) corresponds to the *sinus totus*[11] of 100 000 parts. Then AD is computed using the sine rule in triangle ACD. Unfortunately, the flanked angle (i.e. twice $\angle$HAC), chosen of 80° according to the rectified table of angles of part 1, is in fact taken of 75° according to the unrectified table. The demonstration is thus entirely in conformity with another case, what turns the readers into confusion. This may be the reason why it doesn't belong to the English version.

4 Conclusion: towards an European military architecture

From the Italians to the Dutch we have shown changes in the way of using geometry. In a certain way we could infer that Italian architects were driven by the images of what should be the final shapes, together with a bright idea of symmetry and even beauty. Following this supposition, Errard may be seen as a direct heir of their way of thinking. It is quite clear that the final shape of the bastion led his thinking throughout its establishment process. Indeed Errard's bastions having three right angles are truncated squares. Demonstrations and calculations are based on the existence of these right angles, which allow the use of classical Euclidean theorems without need for trigonometric tables. But this makes the difference between Errard and his Italian predecessors: the rigorous approach of Errard promotes scientific discussion and anticipation for future adaptation to the improvements of attacking practices. The Dutch, thanks to their virtuoso practice of trigonometry, do not seem to be stopped by a closed vision of the final shapes, but create them with more liberty. The use of the sine rule and trigonometric methods allow them to master the distances, lengths and angles, whatever the proportions they choose. It was a necessity for them to keep their fortresses suitable for new material conditions of sieges, especially the use of explosives and the trench approaches.

Even off the field, the question of adapting the shapes of fortresses was taken seriously. For instance mathematics teachers of that time had links to the milieu of military architects. The numerous courses on fortification of the 17th century, printed as well as manuscripts, that we found in France echo the discussion of engineers about the qualities of one or another angle. Teachers expose and compare the constructions, generally drawing their examples from Errard, Marolois and the next generation of French military engineers, especially Antoine de Ville and Blaise-François Pagan. The manners of fortification were not unified yet, but it would be the case at the end of the century, when everywhere in Europe engineers would fortify places according to the *manière* of Monsieur de Vauban (or Menno van Coehoorn, his counterpart in the Low Countries). Unfortunately for us, Vauban didn't promote a mathematical approach of fortification. Quite the contrary, he claimed that geometry was useless for that purpose.

[11] That is: the radius of the trigonometric circle.

For the reader to realize how simple and non-mathematical it was, here is a summary of Vauban's manner of bastionning a line AB of 180 toises[12] (Muller, 1746): AB being perpendicularly bisected at C, set point D on the perpendicular 30 toises off from C; then put E on [AD] 50 toises from A and H on [BD] 50 toises from B; G and F are the symmetrical points of H and E with respect to D. The profile of the two half-bastions is given by AEFGHB (readers are invited to drawn this profile themselves; as an easy exercise). As one can see, there are no angles to consider here, but only distances, which are to be taken with the compass on a scale of 180 toises.

In fact, the former period is much more interesting for nowadays mathematic educators. The military revolution process is more attractive than his final results about fortification, especially when you consider the role played by geometry. At the beginning of the 17th century, things were not entirely determined and many controversies happened, on the field as well as in the offices, in the mansions or even at the Royal Courts. Nobles, whether officers or not, had to know the terms, concepts and practices of fortification perfectly. Many noble families needed mathematics skills to be taught to their teenagers, because geometry was the language of fortification. For our present math classes, studying fortification in the original texts can bring the students a good example of a concrete use of geometry and justify the learning of this vanishing discipline.

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[12] Approximately 350 meters.

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