JORDAN'S ISOMORPHISM CONCEPT IN THE WORK "TRAITÉ DES SUBSTITUTIONS ET DES ÉQUATIONS ALGÉBRIQUES"

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ABSTRACT

This paper discusses the concept of isomorphism from the analysis of Camille Jordan's book "Traité des substitutions et des equations algébriques". Methodologically, we resort to qualitative text analysis (Kuckartz, 2014), considering three levels of analysis: reference card of the work, context and purpose of the work and of the author, and presentation and treatment of the group isomorphism concept. The results show that Jordan explicitly introduced the concepts of holoedric (total) and meriedric (partial) isomorphism in the context of substitutions groups. Thus, Jordan used the term isomorphism to designate what is currently known as bijective homomorphism (isomorphism) and surjective homomorphism (epimorphism), distinguishing them with the terms holoédrique and mériédrique, respectively. This distinction suggests the possibility of a correspondence or transmission of some properties from one group to another, recognizing the usefulness of this concept due to the similarity of the properties presented by groups that are isomorphic.

1 Introduction

Over the past four decades there has been considerable interest and growth in research in Mathematics Education towards the line of historical research (Fauvel & van Maanen, 2000; Matthews, 2014; Clark, Kjeldsen, Schorcht, Tzanakis, & Wang, 2016). Two areas of study can currently be distinguished in relation to the contributions of history in this field (differentiated themes in the International Congress on Mathematical Education [ICME]): the history of the teaching and the learning of mathematics; and the role of the history of mathematics in mathematics education. The first focuses on the history of Mathematics Education due to its development as a scientific discipline, while the second emphasizes the importance of integrating historical and epistemological issues in the teaching and learning of mathematics.

Clark et al. (2016) point out that "mathematical knowledge is determined not only by the circumstances in which it becomes a deductively structured theory, but also by the procedures that originally led or may lead to it" (p. 136). However, the teaching of mathematics usually presents the final products of years of mathematicians' work and does not consider the process of constructing those products, which would otherwise contribute to the appropriation of the meaning of mathematical concepts. In this regard, in the teaching and learning of mathematics, the integration of historical and epistemological aspects can favor the understanding of specific parts (e.g., concepts, theorems) of mathematics, and also lead to a deep awareness of the discipline itself (Clark et al., 2016).

Various researchers (e.g., Fauvel & van Maanen, 2000; Jankvist, 2009; Furinghetti, 2019) have provided arguments in favor of the use of history indicating the ways in which it can be used in the teaching and learning of mathematics. For example, Jankvist (2009) proposes two categories of arguments for the use of history, as a *tool* and as a *goal*. In the first category, history plays an important role as an auxiliary or supportive means of teaching and learning the in-issues of mathematics: mathematical concepts, theories, methods, algorithms, among others. In the second category, history as a *goal*, it is stated that although history can have a positive side effect of supporting the learning of mathematics, the main purpose of its use is not this one; it concerns the teaching of meta-issues of mathematics as a scientific discipline and its history.

We consider the use of history as a *tool* that can enhance a deep understanding of mathematical concepts, seeking to make more explicit the mathematical construction of a specific mathematical concept and other underlying concepts (in-issues) (Jankvist, 2009). Thus, we could see mathematics as something that develops and can be constructed, that is, highlighting the creative side of mathematics rather than the cultural side (Menghini, 2000). Furthermore, "to be interested in the process of knowledge construction requires a history of mathematical ideas and, therefore, an epistemological reflection" (Barbin, 1997, p. 21). Epistemology here refers to the question of the meaning of mathematical concepts and theories, what Barbin refers to as *épistémologie historique* (historical epistemology), considering that what gives meaning to concepts and theories are the problems they solve. In this respect, the historical and epistemological of concepts takes on special relevance.

On the other hand, a way in which historyhas been integrated into teaching and learning (as a tool and as a goal) of mathematics has been using primary (original) sources, that according to the categorization of Jankvist (2009) on how history may/should be used, the information from these materials is usually used as: (1) small extracts in historical epilogues (*illumination* approaches) and (2) readings of original sources or student projects (*modules* approaches). In this paper we consider the use of original sources for a historical and epistemological study in the (3) *history-based* approaches, where the main concern lies in the problems of mathematics, whose interest lies in their learning, and "do not deal with the study of the history of mathematics in a direct manner, but rather in an indirect fashion" (pp. 246-247), that is, the historical information is not directly discussed with the students.In particular, the analysis of an original source is intended to highlight the characteristics of a mathematical concept itself (e.g., its meaning and applications), and the social and cultural aspects that influenced its construction, and from the available information, rescuing episodes or enabling elements that can help students to understand mathematical concepts (Anacona, 2003).

Thus, we situate this work within the second area of study in relation to the contributions of history to mathematical education, namely, the role of the history of mathematics in mathematical education. We propose that a historical and epistemological analysis of concepts based on original sources, reveals the different approaches that mathematicians made around a concept, the difficulties that they had to overcome, the genesis of the concept and how other aspects determined the evolution of this concept until its consideration as we know it today. In this respect, some investigations justify that a historical approach to the concept may favor its understanding (e.g., Anacona, 2003; Furinghetti, 2019).

Specifically, we have carried out a historical and epistemological analysis of the group isomorphism concept that despite the importance of this concept in abstract algebra (Lajoie, 2000), research carried out around their teaching and learning in a first course of abstract algebra has shown difficulties presented by undergraduate students to understand this concept. For example, Larsen (2009) identified difficulties in determining when two groups are *essentially the same* (the same structure whose elements are not necessarily labeled differently), and in the formulating of the formal definition of isomorphism, that usually is given in terms of a bijective function, whereas the operation preservation property does not emerge easily. Leron, Hazzan, & Zazkis (1995), Lajoie (2000, 2001), Weber (2001), and Weber & Alcock (2004) showed difficulties in proving that two groups are isomorphic and in constructing an isomorphism between specific groups. In addition, Lajoie (2000, 2001) identified difficulties: in giving the interpretation experts give, to the idea that isomorphic groups are similar, in considering more than one possible isomorphism between two isomorphic groups, in seeing isomorphism both as an equivalence relation on groups and as a particular correspondence between two isomorphic groups, and in recognizing a usefulness to the isomorphism concept in algebra.

We specifically investigated the following questions:

- What kind of problems were the mathematicians of the 19th century trying to solve, which favored the emergence of the concept of group isomorphism?
- What meanings were attributed to the concept of group isomorphism by 19th century mathematicians?

This document presents the results of the analysis of the book by Camille Jordan (1838-1922): “Traité des substitutions et des equations algébriques”, published in 1870 in the sections that refer to the concept of group isomorphism. The presentation of a didactic intervention based on the results of historical and epistemological analysis of this concept in Jordan’s work is beyond the scope of this paper.

2 Method

This document studies the work of Jordan (1870), which from a literature review of secondary sources (Wussing, 1984; Kleiner, 2007), was identified as the work where the definition of isomorphism appears for the first time explicitly in the context of substitution groups (what today we would call a permutation of a finite number of letters).

For the analysis of the work we used the qualitative text analysis method (Kuckartz, 2014). Three levels of analysis (González, 2002) were considered, each one of which deepens in the information obtained in the previous one: *reference card of the work*, *context and purpose of the work and of the author*, and *presentation and treatment of the group isomorphism concept* (Table 2.1). The second and third levels have been subdivided into categories, which were constructed in a deductive-inductive way (Kuckartz, 2014).

The first level (*reference card of the work*) corresponds to the information specific to the work, which includes the name of the author, dates of birth and death, title of the work, year, publisher and place of publication analyzed and location of the work.

The second level (*context and purpose of the work and of the author*) provides an overview of the work, contextualizing it so that it can be properly analyzed in terms of the facts that influenced its writing and publication. The resulting categories were the following: (1) *Contextualization of nineteenth century mathematics*, which informs about the historical-cultural moment of the mathematics in which the work was written; (2) *Contextualization of the work,* which provides information about the historical (scientific) moment in which it was produced, the goals or intentions of the author about the work, as well as the innovations introduced in the work; (3) *Characterization of the structure of the work*, to show the extent and distribution of the content and the works considered as reference at the time of writing; (4) *Professional information of the author*, to inform about the institutions where he carried out his main studies, as well as the influences that he received from other mathematicians of the time, to highlight the most important works published by the author and to recognize his link with the mathematical object of interest.

The third level (*presentation and treatment of the group isomorphism concept*) considers the context of application of the concept by the author because it unravels its meaning and scope. In addition, a concept is only one element of a conceptual field that conveys its meaning to it, so a concept will never appear isolated, which was considered in determining how the concept of study was justified and applied (Schubring, 2005). The resulting categories were the following: (1) *Definition*, which are descriptions related to the mathematical concept group isomorphism, in which they are characterized from a theoretical mathematical point of view and which will depend on the context in which it was introduced: classical algebra, number theory, geometry or analysis; (2) *Other concepts involved*, which are the concepts underlying the group isomorphism concept and which are associated with its development; (3) *Types of examples* that the author uses to present the mathematical object group isomorphism; (4) *Applications of the concept*, to identify the importance of the group isomorphism concept from the problems that are solved by this, as well as to identify the scopes and the limitations of the definition.

Table 2.1: Methodology for the analysis of historical mathematical works

Levels of analysis	Categories	Units of analysis
Reference card of the work	Name of the author Date of birth and death of the author Title of the work Year, publisher and place of publication analyzed Location of the work	
Context and purpose of the work and of the author	Contextualization of nineteenth century mathematics	Historical-cultural moment of the mathematics
	Contextualization of the work	Historical moment and place where the work was written General objectives of the work Innovations introduced in the work

	Characterization of the structure of the work	Extent and distribution of the content References
	Professional information of the author	Author’s education Other published works
Presentation and treatment of the group isomorphism concept	Definition Other concepts involved Types of examples Applications of the concept	

3 Analysis and interpretation of Jordan’s work (1870)

With respect to the presentation of the work, the proposed levels of analysis have been considered in an orderly manner: *reference card of the work*, where the data of the analyzed material are established; *context and purpose of the work and of the author*, for which in addition to the information that the analyzed material has been able to provide, it has been necessary to consider secondary sources such as Brechenmacher (2012), Kleiner (2007), Lebesgue (1926), Neumann (1999), Schlimm (2008), Timmermans (2012) and, Wussing (1984), these sources were selected according to two criteria: (1) review of mathematics history books where elements of Jordan’s life and academic work where identified, and (2) review of articles in mathematics history that provided specific information about life, Jordan’s academic work, and where specific content of the Traité was discussed; and finally, *presentation and treatment of the group isomorphism concept*. The development of the exposition of the work is organized according to the levels of analysis and is based on the categories and units of analysis previously established (Table 2.1).

Reference card of the work

Author: Marie-Ennemond- Camille Jordan.
Author’s date of birth and death: 1838-1922.
Title: Traité des substitutions et des équations algébriques.
Year, publisher and place of the edition analyzed:1870, Paris: Gauthier-Villars.
Location of work: Library of Sciences. Mathematics. University of Zaragoza. Spain.
Symbols: MAT-Ant 102; ZGG 404.

Context and purpose of the work and of the author

The French-Italian mathematician Joseph-Louis de Lagrange’s (1736-1813) work in 1770-71 had initiated the study of permutations in connection with the study of the solving of algebraic equations and had influenced, in the first third of the 19th century, the work of other mathematicians such as the Italian Paolo Ruffini (1765-1822) and the Norwegian Niels Henrik Abel (1802-1829), developing elements of the theory of groups of permutations (Kleiner, 2007). It was the French Évariste Galois (1811-1832) who first used the term *group* and distinguished between the general principles of what is now known as Galois Theory and an application of this, namely the solvability of algebraic equations by radicals. Galois’ ideas transmitted from his writings were not understood and assimilated by his contemporaries. The main works of Galois were published after his death for the first time in 1846 by Joseph Liouville (1809-1882), who like his French compatriots Charles Hermite (1822-1901), Victor Puiseux (1820-1883) and Joseph-Alfred Serret (1819-1885) studied and continued the works of Galois (Wussing, 1984); until they received a complete treatment by Jordan in his "Traité des substitutions et des equations algébriques" (Neumann, 1999).

In the first half of the nineteenth century, the French mathematician Augustin-Louis Cauchy (1789-1857) also made important contributions to the development of theory of groups of permutations, and it is because of Cauchy that this theory has been developed autonomously. In this regard, Kleiner (2007) points out that "before Cauchy, permutations were not an object of independent study but rather a useful device for the investigation of solutions of polynomial equations" (p. 24).

In France, a deep and rapid advance that revealed the wide range of the concept of permutation group through mathematics was achieved by establishing a connection between the Galois and Cauchy approaches (Wussing, 1984). The work of Serret of 1866, the third edition of the"Cours d'Algèbre Supérieure" and two Comments of Jordan on Galois, the first "Commentaire sur le Mémoire de Galois" of 1865, and its continuation "Commentaire sur Galois" of 1869 are examples of this interest of unification; but it would be Jordan's work "Traité des substitutions et des equations algébriques", in which the ideas of Galois and Cauchy were finally unified, which had a great influence on the evolution of group theory. Jordan published more than 30 articles on groups in the period of 1860-1880, and "the *Treatise* embodied the substance of most of Jordan's publications on groups up to that time" (Kleiner, 2007, p. 25).

Camille Jordan was born on January 5, 1838 in Lyon, France. Jordan was admitted to the École Polytechnique in 1855. In 1873 he was appointed *examinateur* at the École and professor of Analysis, replacing Hermite in 1876. Jordan was elected member of the Académie des Sciences after the death of the French Michel Chasles (1793-1880) in 1881. Furthermore, in 1883 he was appointed professor at the Collège de France as Liouville's successor and in 1885 he assumed the position as director of the *Journal de Mathématiques Pures et Appliquées*, one of the main mathematical research journals of the time (Lebesgue, 1926).

Jordan's book "Traité des substitutions et des équations algébriques" of 667 pages was published in 1870 by Gauthier-Villars. In the preface the author states that the main goal of the work is "to develop the methods of Galois and compile them into a body of doctrine, showing how easily they solve all the major problems of equation theory" (Jordan, 1870, p. VII, our translation; see note 1). In addition to mentioning the influence of Serret's work, "Cours d'Algèbre supérieure", whose study inspired Jordan to contribute to the progress of Algebra, he recognized the contributions of Galois, for the invention of the principles of the Galois theory and the Italian Enrico Betti (1823-1892), for writing an important Memoir, where it was established for the first time the complete series of the theorems of Galois in a rigorous way. Jordan also mentioned the contributions of Germans Leopold Kronecker (1823-1891) and Abel, Hermite, and the Italian Francesco Brioschi (1824-1897) on the Galois groups of certain division problems of elliptic and Abelian functions, as well as the investigations of the German geometers Otto Hesse (1811-1874), Alfred Clebsch (1833-1872), and Ernst Eduard Kummer (1810-1893), the British Arthur Cayley (1821-1895), the Irish George Salmon (1819-1904), and the Swiss Jakob Steiner (1796-1863), who studied various geometrical problems to which Galois' methods can be applied.

The concept of substitutions group and its application not only to the theory of equations but in other areas of contemporary mathematics allowed Jordan to unify in his Traité the results of Galois, Cauchy and other mathematicians (Wussing, 1984; Kleiner, 2007). In general, Jordan's Traité contains sections dedicated to the study of substitution groups, to the Galois theory itself and to the applications of the Galois theory to equations that arise in several areas of mathematics.

Jordan's work (1870) is not a book with pedagogical intent. In fact, as Wussing (1984) indicates: "it was an expression of Jordan's deep desire to bring about a conceptual synthesis of the mathematics of his time. That he tried to achieve such a synthesis by relying on the concept of a permutation group" (p. 160). Also, Kleiner (2007) points out that "his aim was a survey of all of mathematics by areas in which the theory of permutation groups had been applied or seemed likely to be applicable" (p. 25).On the Traité of Camille Jordan one may consult Brechenmacher (2012).

The Traité consists of four books, each divided into chapters. Book I deals with the main notions related to congruences. Book II is divided into two chapters, the first dedicated to the study of substitutions in general and the second to linear substitutions. Book III is made up of four chapters. The first presents the principles of the general theory of equations and the other three contain applications of the Galois theory in algebra, geometry and problems concerning transcendental functions. Finally, in Book IV, divided into seven chapters, Jordan determines the general types of equations solvable by radicals and for each of them he obtains a complete classification system.

It is in Book II of Jordan's Traité entitled "Des substitutions", where the isomorphism concept can be found, and in which Jordan proves a part of the Jordan-Hölder theorem, "one of the fundamental theorems in the theory of groups" (Schlimm, 2008, p. 409).The first chapter deals with generalizations about substitutions and presents a synthesis of previous results from French mathematicians such as Cauchy, Serret, Joseph Bertrand (1822-1900), and Émile Mathieu (1835-1890). Jordan presents concepts (which had previously appeared in his "Commentaire sur Galois" in 1869) such as a substitution, an unit substitution, the product of two substitutions, a group (Jordan uses the term *faisceau* as a synonym for group), the derived group (le groupe dérivé), the order of a group, the degree of a group, the simple and composite groups, and alternating group. About Jordan's work on groups, he considers them as groups acting on sets and uses a generator and relations approach to groups. In Jordan's group definition, closure under multiplication is the sole property required.

> **23**. One gives the *substitution* name to the operation by which a number of things are exchanged than can be assumed to be represented by letters *a*, *b*, …. […]
>
> **27**. One says that a system of substitutions forms a *group* (or a *bundle*) if the product of any two of the substitutions of the system is again a member of that system.

> The various substitutions obtained operating successively whenever we want and, in any order, certain substitutions given A, B, C ... obviously form a group: we will call it the derived group of A, B, C, ..., and we will designate it by the symbol (A, B, C, ...). (Jordan, 1870, pp. 21-22, our translation; see note 2)

Jordan also proves the Lagrange theorem and the Cauchy theorem and introduces the isomorphism concept. In addition, he conducts an important study on transitivity and primitivity for substitution groups. In the second chapter of Book II, Jordan is devoted to the study of the properties of the General and Special linear groups.

Presentation and treatment of the group isomorphism concept

Jordan's Traité is an extensive and complex work. In this paper we do not present a general study of Jordan's group concept, rather, we consider a fragmented reading of the Traité, extracting some interesting elements about the isomorphism concept of this monumental work.

In Book II: Des substitutions, in §V.- Symmétrie des fonctions rationnelles, Jordan addresses the problem of the number of values taken on by a function of *n* variables as a result of their permutation, that it had been the subject of previous study by the same author in his 1860 thesis. Also, Jordan treats the concept of isomorphism.

Jordan imported the terms *holoédrique* (total) and *mériédrique* (partial) from Auguste Bravais' "Études cristallographiques" to substitution groups. "Similarly any two groups of operations applying to any object (movements of solid objects, substitutions of the roots of an equation, etc.) can transmit, in whole or in part, some of their properties" (Timmermans, 2012, p. 45).

Jordan used the word *isomorphism* to refer to what is now known as bijective homomorphism and surjective homomorphism, but distinguished between the two using the terms *l'isomorphisme holoédrique* and *l'isomorphisme mériédrique*, respectively. However, we must keep in mind that Jordan's isomorphism is not seen as maps.

> **67**. A group Γ is said to be *isomorphic* to a group *G* if it is possible to establish between their substitutions a correspondence such that: 1° to each substitution of *G* there corresponds a unique substitution of Γ and to each substitution of Γ one or more substitutions of *G*; 2° to the product of any two substitutions of *G* there corresponds the product of their respective corresponding substitutions.
>
> An isomorphism is said to be *meriedric* if many substitutions of *G* correspond to the same substitution of Γ, and *holoedric* in the opposite case. (Jordan, 1870, p. 56, our translation; see note 3)

Jordan does not present concrete examples of isomorphic groups. However, he explicitly shows the application of the definition of meriedric isomorphism by deducing that if a group has isomorphic groups of this type then it is composite (see note 4), as shown in the following quotation:

> Suppose, to fix the ideas, that G contains *m* substitutions $g_1,...,g_m$ corresponding to the same substitution γ of group Γ. Let γ' another substitution of Γ, g' a substitution of G

corresponding to it: $g_1^{-1}g'$ will correspond to $\gamma^{-1}\gamma'$, and consequently each of the *m* substitutions $g', g_2g_1^{-1}g', ..., g_mg_1^{-1}g'$ will correspond to $\gamma\gamma^{-1}\gamma' = \gamma'$. Each substitution of Γ having thus *m* corresponding in G, the order of Γ will be *m*-fold smaller than that of G. Group Γ contains substitution I. Let $h_1, ..., h_m$ be the corresponding substitutions of G: they form a group to which all the substitutions ofGare permutable. Because if g is one of these, γ is its corresponding: $g^{-1}h_1g$ has the corresponding $\gamma^{-1}I\gamma = I$: therefore, it belongs to the sequence $h_1, ..., h_m$.

If $m >$ I, the group $(h_1,...,h_m)$ cannot be reduced to the only substitution I: it will be less than G, if we assume that Γ is not reduced to the only substitution I; therefore G will be composite. Hence this conclusion: *The composite groups have only meriedric isomorphs* (not formed exclusivelyby substitution I). (Jordan, 1870, p. 56, our translation; see note 5)

Jordan then poses as a problem the determination of isomorphic groups to a given G group. In the development, Jordan treats the problem by constructing a new group through the values taken by a function of *n* variables as a result of their permutations, which turns out to be transitive (see note 6) and isomorphic to G:

68. PROBLEM. — *Determine the isomorphic groups to a given group* G.

The problem is reduced to determining those groups that are transitive. […]

69. So let's search isomorphic groups at G and transitive. We will see that their determination is reduced to that of the various groups contained in G.

Let $x, x_1, ...$ be the letters that G exchanges between them; H=$(h_1,...,h_n)$ any group contained in G. Substitutions of G can all be put in the form of $h_\alpha g_\beta, g_1, ..., g_\beta, ..., g_m$ which are appropriately chosen substitutions, the first of which is reduced to unit and whose number *m* equals the proportion of the orders of G and H.

Now let F_1 be any rational function of $x, x_1, ...$, invariable by substitutions H; F_s becomes substitution *s*. The *m* functions $F_1, ..., F_{g_m}$ will be transformed into each other by any substitution of G. Indeed, the substitution $h_{\alpha'}g_{\beta'}$ transforms F_{g_β}, for example, into $F_{g_\beta h_{\alpha'} g_{\beta'}}$ .Moreover $g_\beta h_{\alpha'} g_{\beta'}$, belonging to G, can be put in the form $h_{\alpha''}g_{\beta''}$ and $h_{\alpha''}$ does not alter the function F_1 : so F_{g_β} will be transformed into $F_{h_{\alpha''}g_{\beta''}} = F_{g_{\beta''}}$.

Each substitution of G, performed in the functions $F_1, \ldots$, is thus equivalent to a certain substitution performed *between* these functions. The latter substitutions obviously form a group Γ, isomorphic to G. This new group will be transitive, the substitutions $g_1, ..., g_m$ make it possible to transform F_1 into any of the functions $F_1, ..., F_{g_m}$.

70. We will show reciprocally that any transitive group, isomorphic to G, is identical to one of those we have just formed. […]

Our proposition is thus established. (Jordan, 1870, pp. 56-59, our translation; see note 7)

Finally, in Jordan's treatment of the concepts of isomorphism and isomorphic groups, some theorems involving them can be identified, for example, in section §XI. Groupes isomorphes aux groups linéaires of Book II; as well as in Book III: Des irrationnelles, in Chapter IV: Applications a la théorie des transcendantes in the section § III. Fonctions hyperelliptiques; and in Book IV: De la résolution par radicaux, in the first chapter, Conditions de résolubilité. Below is a theorem taken from Book III.

THEOREM. — *Any group of degree q isisomorphicnot meriedricto a group of the degree* $2^{2k}-1$, *with abelian linear substitutions, where k is the largest integer contained in* $\frac{q-1}{2}$.(Jordan, 1870, pp. 364-365, our translation; see note 8)

4 Discussion

Implicitly, the groups became the object of study in algebra when mathematicians like Lagrange and Ruffini used permutations when dealing with one of the main problems of the eighteenth and nineteenth century in this mathematical domain, the solvabilityof algebraic equations of degree higher than the fourth by radicals (Kleiner, 2007).Abelproved that it was impossible solving the general equation of the fifth degree by radicals and posed the following problems: (1) Constructing all algebraic equations of a given degree that are solvable by radicals. (2) Given an equation, recognizing whether it is soluble by radicals and make this resolution when possible (Lebesgue, 1926).

On the other hand, the term group was usedfor the first time, without being defined, by Galois, who created a theory (Galois theory) considered one of the great achievements of the nineteenth century, and its application to the solvability of equations by radicals. After the publications of Galois' writings by Liouville in 1846, the interaction between the theory of equations and the theory of permutations favored the emergence and consolidation of the concept of a permutation group (Wussing, 1984). Thus, initially, group theory was considered under the aspect of finite group theory of permutations.

The independence of the theory of permutations resulted in the application of the concept of a permutation group outside the theory of algebraic equations, for example, in geometry, analysis, number theory, and mechanics in the works of mathematicians such as Jordan, Serret and Hermite, just to mention a few (Wussing, 1984).In fact, Schlimm (2008) points out that with the publication of Jordan (1870), "the theory of substitution groups was established as an independent tool for the study of algebraic equations" (p. 410).

Jordan (1870) explicitly introduced the concepts of *isomorphism holoédrique* (total) and *mériédrique* (partial), terms which he imported from crystallography, particularly from Études de Auguste Bravais, to his "Traité des substitutions et des equations algébriques" in the context of substitution groups. The distinction between meriedric and holoedric isomorphism presented by Jordan proves that isomorphism was not always bijective (holoedric). In his

Traité, Jordan uses the term *similitude* (similarity) to refer to the properties of isomorphic groups *identiques* (identical), which he acknowledges as a utility of isomorphism. The approach to the definition of Jordan's meriedric isomorphism in 1870 suggests the possibility of a correspondence or transmission of *some* properties from one group to another.

Two groups are isomorphic (similar), if it is possible to establish a biunivocal correspondence among their elements in such a way that the product of the *corresponds* is equal to the *corresponding* product; that is, the operation is preserved. The correspondence Jordan refers to is what is called today group isomorphism.

On the other hand, the usefulness of the notion of isomorphism of substitution groups, according to Jordan (1870), concerns "the similarity of properties that isomorphic groups present between each other. […] therefore in many cases replace the direct consideration of a group by that of any of its isomorphs" (p. 60). Today, the importance of isomorphism is difficult for students to recognize; for example, Lajoie (2000) showed that the ideas, images and conceptions that students assign to group isomorphism do not allow them to understand the importance of this concept in mathematics. Students tend to refer to isomorphic groups with expressions such as *similar* or *equivalent*, and in practice, to determine whether two groups are isomorphic, students tend to rely on the literal interpretation of these words, so they try to find similarities between the operations or elements of the groups.

Finally, an important idea for the design of tasks has been extracted from the analysis of the work; this idea includes asking the students to construct a group of permutations isomorphic to a given group G.

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NOTES

[1] de développer les méthodes de Galois et de les constituer en corps de doctrine, en montrant avec quelle facilité elles permettent de résoudre tous les principaux problèmes de la théorie des équations. (Jordan, 1870, p. VII)

[2] **23**. On donne le nom de *substitution* à l'opération par laquelle on intervertit un certain nombre de choses que l'on peut supposer représentées par des lettres *a*, *b*, …. […]
27. On dira qu'un système de substitutions forme un *groupe* (ou un *faisceau*) si le produit de deux substitutions quelconques du système appartient lui-même au système.
Les diverses substitutions obtenues en opérant successivement tant qu'on voudra et dans un ordre quelconque certaines substitutions données A, B, C,… forment évidemment un groupe : nous l'appellerons le *groupe dérivé de* A, B, C,…, et nous le désignerons par le symbole (A, B, C,…). (Jordan, 1870, pp. 21-22)

[3] **67**. Un groupe Γ est dit *isomorphe* à un autre groupe*G*, si l'on peut établir entre leurs substitutions une correspondance telle : 1° que chaque substitution de *G* corresponde à une seule substitution de Γ, et chaque substitution de Γ à une ou plusieurs substitutions de*G* ; 2° que le produit de deux substitutions quelconques de *G* corresponde au produit de leurs correspondantes respectives.

L'isomorphisme sera dit *mériédrique*, si plusieurs substitutions de *G* correspondent à une même substitution de Γ, *holoédrique* dans le cas contraire. (Jordan, 1870, p. 56)

[4] Jordan (1870) points out that: "un groupe G est *simple*, s'il ne contient aucun autre groupe auquel ses substitutions soient permutables, *composé* dans le cas contraire" (p. 41).

[5] Supposons, pour fixer les idées, que G contienne *m* substitutions $g_1,\ldots,g_m$ correspondantes à une même substitution γ du groupe Γ. Soient γ' une autre substitution quelconque de Γ, g' une substitution de G qui

lui corresponde : $g_1^{-1}g'$ correspondra à $\gamma^{-1}\gamma'$, et par suite chacune des m substitutions $g', g_2g_1^{-1}g', ..., g_mg_1^{-1}g'$ correspondra à $\gamma\gamma^{-1}\gamma' = \gamma'$. Chaque substitution de Γ ayant ainsi m correspondantes dans G, l'ordre de Γ sera m fois moindre que celui de G.
Le groupe Γ contient la substitution I. Soient $h_1, ..., h_m$ les substitutions correspondantes de G : elles forment un groupe auquel toutes les substitutions de G sont permutables. Car soient g l'une de ces dernières, γ sa correspondante : $g^{-1}h_1g$ a pour correspondante $\gamma^{-1}I\gamma = I$: elle appartient donc à la suite $h_1, ..., h_m$.
Si $m > \text{I}$, le groupe $(h_1, ..., h_m)$ ne peut se réduire à la seule substitution I : il sera d'ailleurs moindre que G, si l'on suppose que Γ ne se réduise pas à la seule substitution I ; donc G sera composé. D'où cette conclusion : *Les groupes composés ont seuls des isomorphes mériédriques* (non exclusivement formés de la substitution I). (Jordan, 1870, p. 56)

[6] According to Jordan (1870),"un groupe est *transitif* lorsque, en opérant successivement toutes ses substitutions, on parvient à faire passer une des lettres à la place de l'une quelconque des autres; plus généralement, il sera *n fois transitif* si ses substitutions permettent d'amener simultanément *n* lettres données *a, b, c, ...* aux places primitivement occupées par *n* autres lettres quelconques *a', b', c'*, …" (p. 29).
[7] **68**. PROBLÈME. — *Déterminer les groupes isomorphes à un groupe donné* G.
Le problème se réduit à déterminer ceux de ces groupes qui sont transitifs. […]
69. Cherchons donc les groupes isomorphes à G et transitifs. Nous allons voir que leur détermination se ramène à celle des divers groupes contenus dans G.
Soient en effet $x, x_1, ...$ les lettres que G permute entre elles ; $\text{H}=(h_1, ..., h_n)$ un groupe quelconque contenu dans G. Les substitutions de G peuvent toutes être mises sous la forme $h_\alpha g_\beta, g_1, ..., g_\beta, ..., g_m$ étant des substitutions convenablement choisies, dont la première se réduit à l'unité et dont le nombre m est égal au rapport des ordres de G et de H.
Soient maintenant F_1 une fonction rationnelle quelconque de $x, x_1, ...$, invariable par les substitutions H ; F_s ce qu'elle devient par la substitution *s*. Les *m* fonctions $\text{F}_1, ..., \text{F}_{g_m}$ seront transformées les unes dans les autres par toute substitution de G. En effet, la substitution $h_{\alpha'}g_{\beta'}$ transforme F_{g_β}, par exemple, en $\text{F}_{g_\beta h_{\alpha'} g_{\beta'}}$. D'ailleurs $g_\beta h_{\alpha'} g_{\beta'}$, appartenant à G, peut être mise sous la forme $h_{\alpha''}g_{\beta''}$ et $h_{\alpha''}$ n'altère pas la fonction F_1 : donc F_{g_β} sera transformée en $\text{F}_{h_{\alpha''}g_{\beta''}} = \text{F}_{g_{\beta''}}$.
Chaque substitution de G, effectuée dans les fonctions $\text{F}_1, ...$, équivaut ainsi à une certaine substitution effectuée *entre* ces fonctions. Ces dernières substitutions forment évidemment un groupe Γ, isomorphe à G. Ce nouveau groupe sera transitif, les substitutions $g_1, ..., g_m$ permettant de transformer F_1 en l'une quelconque des fonctions $\text{F}_1, ..., \text{F}_{g_m}$.
70. Nous allons montrer réciproquement que tout groupe transitif, isomorphe à G, est identique à l'un de ceux que nous venons de former. […]
Notre proposition se trouve ainsi établie. (Jordan, 1870, pp. 56-59)

[8] THÉORÈME. — *Un groupe quelconque de degré q est isomorphe sans mériédrie à un groupe de degré* $2^{2k} - 1$*, à substitutions linéaires abéliennes, k étant le plus grand entier contenu dans* $\frac{q-1}{2}$.(Jordan, 1870, pp. 364-365)

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