A CASE STUDY OF THE IMPLEMENTATION OF PRIMARY SOURCES IN UNDERGRADUATE MATHEMATICS

Matthew MAUNTEL
Florida State University, 1114 West Call Street, Tallahassee, Florida, USA
mmauntel@fsu.edu

ABSTRACT

In this paper I present a pilot case study of three different university mathematics instructors who implemented the same primary source project (PSP) as part of the Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) project. I describe four implementations (two by the author, two by non-author instructors) using a combination of student and instructor interviews, instructor implementation reports, and instructor and student surveys. Survey data revealed that most students reported perceived academic gains from their work with the PSP, with the author's students reporting some of the greatest gains. I conclude that there are differences in implementation based upon observed lines of communication and how instructors view distinctive features of the implementation, including the importance of group work and understanding language use within sources.The author stressed the importance of group work and the productive struggle associated with language, resulting in a different implementation than both non-authors.

1 Introduction

Mathematics faculty and educational researchers increasingly recognize the value of the history of mathematics as a support to student learning. The expanding body of literature in this area includes recent special issues of Science & Education (Jankvist, Fried, Katz, & Rowlands, 2014) and Problems, Resources and Issues in Undergraduate Mathematics Education (PRIMUS; Clark & Thoo, 2014), both of which include direct calls for the use of primary historical sources in teaching mathematics. For many instructors, the current lack of classroom-ready materials poses an obstacle to the incorporation of history into the mathematics classroom. As noted by Jankvist (2009), "the 'urgent task' of developing critical implements for using history in the teaching and learning of mathematics" (p. 256) is also essential to further research on the benefits and effectiveness of using history of mathematics to teach.

For decades much of the research literature in the United States on the impact of the history of mathematics on students, particularly at the secondary level or tertiary level, was focused on students' attitudes (e.g., Marshall, 2000; McBride & Rollins, 1977). There has been little focus on the use of primary sources as a classroom tool in the early work in the field of history in mathematics education. While more recent work on the use of primary sources has been done in countries such as Brazil (e.g., Bernardes & Roque, 2018), Denmark (e.g., Kjeldsen & Blomhøj, 2012), and Turkey (e.g., Alpaslan & Haser, 2015), similar research has not yet been conducted with student populations in the United States. Thus, the Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources (or, TRIUMPHS) project is committed to investigating the ways in which mathematics students respond to being taught concepts within the undergraduate curriculum via primary historical sources.

In this paper we share initial findings on the research questions we identified for this pilot study:

1. What are the differences between the author's implementation of a PSP and two non-author implementations of the PSP?
2. How do students' perceived gains in understanding content-related material differ between implementations?
3. What were the reported benefits and obstacles of each implementation of PSP from a student point of view and from an instructor point of view? How do these views align or differ?

Thus, the aim of this particular analysis was to determine whether differences exist when the instructional materials –in this case, the primary source project of interest – are implemented by the author of the project when compared to other site-testing instructors. In what follows, we explore the data sources that informed our analysis, and the ways in which we are attempting to make sense of students' experiences with PSPs. We will discuss the differences we identified from each implementation of the primary source project, *Solving a System of Linear Equations using Ancient Chinese Methods*, the ways in which multiple course populations reported similar student gains, and the ways in which students' reported benefits and obstacles for learning with primary source materials can inform future implementations in the TRIUMPHS project.

2 Brief Literature Review

Numerous examples from the literature describe outcomes of purposeful use of history of mathematics in undergraduate mathematics instruction (see, for example, Barnett, 2014; Jahnke et al., 2002; Liu, 2014; Ruch, 2014; Tamulis, 2014). However, these examples often describe the instructional approach that was undertaken (including important details of the primary sources used), or only offer anecdotal evidence observed by the authors, regarding their students' experience with the historical content or instructional materials. In other, more empirical examples, the primary focus of the intervention is on pre-service mathematics teachers (e.g., Charalambous, Panaoura, & Philippou, 2009; Clark, 2012; Huntley & Flores, 2010).An open question for the field of history in mathematics education is: What might be possible to investigate regarding the implementation of particular instructional materials (in this case, the PSPs being developed)?

Teaching from primary sources has been widespread in the other fields such as social sciences (De Guzmán, 2007; Klyve, Stemkoski, & Tou, 2013). The use of original sources enables students – as well as teachers – to enrich their understanding of subject being taught (Laubenbacher, Pengelley, & Siddoway, 1994). Due to a lack of appropriate classroom materials, the integration of history into mathematics classrooms remains a difficult approach for many.

Precisely because the use of primary sources provides students the opportunity to interpret results as they were originally presented and then reformulate them in modern terms, original (primary) source readings enable instructors to present a different view of mathematics to students. That is, instead of a classroom scenario in which "definitions and theorems are usually presented first, and then, motivation and application follow afterwards" (Jankvist, 2014, p. 879), primary source materials provide instructors with the opportunity to motivate instruction first with a problem or application, "the solution of which leads to theorems, proofs, and in the end definitions" (Jankvist, p. 879). In this way, primary historical sources have the potential to transform students' images of or views about mathematics.

3 The TRIUMPHS project

3.1 Overview

In 2015, the National Science Foundation in the United States funded a five-year, seven-institution collaborative project to design, test, and evaluate curricular materials for teaching standard topics in the university mathematics curriculum via the use of primary historical sources. The team of principal investigators (PIs) consists of six mathematicians and one mathematics education researcher. The primary source projects (PSPs) are developed by TRIUMPHS PIs, as well as external authors, and once PSPs have gone through an extensive review and testing in the author's (or authors') classroom, they are site-tested by project PIs (when appropriate) and classroom instructors (mathematicians and mathematics teacher educators) who were recruited either through workshops or through other recruitment means. There is an extensive evaluation-with-research component of the TRIUMPHS project, which addresses aspects of faculty expertise and student change. Three TRIUMPHS PIs (two mathematicians and the mathematics education researcher) and several graduate students facilitate the evaluation of classroom site testing of the PSPs.

The goal of the TRIUMPHS project is to promote students' learning and their development of a deeper interest in and appreciation of mathematical concepts by creating educational materials in the form of PSPs based on original historical sources written by mathematicians involved in the discovery and development of the topics being studied. In TRIUMPHS, PSPs contain (1) excerpts from one or several historical sources, (2) a discussion of the mathematical significance of each selection, and (3) student tasks designed to illuminate the mathematical concepts that form the focus of the sources. PSPs are designed to guide students in their explorations of these original texts in order to promote their own understanding of those ideas.

The numerous PSPs are the life force of the TRIUMPHS project. During the grant-funded effort, the PIs promised that some 50 PSPs (which span the undergraduate mathematics curriculum, from basic statistics and trigonometry, to real analysis, abstract algebra, and topology) will be developed, tested, and evaluated. Of the 50 PSPs, 20 are planned to be "full-length" and 30 are what we refer to as "mini-PSPs." Full-length PSPs are designed to typically encompass at least two to four class sessions, which represents the same amount of time that it normally takes to teach the mathematical topic of focus within the PSP. However, among the full-length PSPS there are also longer ones that could be used by instructors to comprise an entire course's content[1]. Alternatively, "mini-PSPs" can be completed in one to two class sessions and each of the mini-PSPs have been developed to teach a particular topic or concept in mathematics that would normally be addressed in a single class session, but which will be done via a primary historical source. To date, 28 full-length PSPs and 21 mini-PSPs have been developed. Though we have exceeded our commitment to develop 20 full-length PSPs, there are additional full-length PSPs in development, as well as the remaining, promised mini-PSPs.

[1] In fact, this was done recently (Spring 2018) by Janet H. Barnett, in an Abstract Algebra course.

In Fall 2015 the first PSPs were tested in two undergraduate mathematics classrooms in the United States; in Year 3 (academic year 2017–18), 46 distinct site testers tested one or more PSPs in undergraduate mathematics classrooms. In total, by the end of Year 3, 53 instructors have site tested PSPs as part of the TRIUMPHS project, with some one-third of those serving as repeat testers. In the first semester of Year 4, we have 25 site testers; again, of these, we have several repeat site testers, where 14 are new to site testing TRIUMPHS PSPs.

3.2 *Solving a System of Linear Equations using Ancient Chinese Methods*

The *Solving a System of Linear Equations using Ancient Chinese Methods* PSP (Flagg, 2017) was created to introduced students to row reduction in an introductory linear algebra course. The PSP is based upon the text *The Nine Chapters on the Mathematical Art* (Shen et al., 1999) and covers basic arithmetic using counting rods and solving systems of equations involving the Fancheng rule (see Appendix for a section of the PSP relating to the Fancheng rule). When first learning the Fancheng rule, students perform operations using columns operations in grids as opposed to row operations. Modern matrix notation is introduced in the latter half of the PSP. The PSP also introduces modern terminology, including echelon form, and requires students to complete several problems from *The Nine Chapters on the Mathematical Art* (Shen et al., 1999) using modern notation. Flagg (2017) highlights the value of the Fancheng rule in avoiding complex fractions until the very end of the row reduction process.

4 Context and setting for the study

The *Solving a System of Linear Equations using Ancient Chinese Methods* PSP (Flagg, 2017) was implemented in Fall 2017 and Spring 2018[2]. Students completed initial and final course surveys, in which they shared their beliefs about mathematics, prior experience with primary source materials in undergraduate mathematics courses, views about mathematics learning, and general demographic information. Upon completion of the PSP, students provided responses to post-PSP survey items, which captured students' perceived gains in skills relating to linear algebra content, general mathematical skills including reading and writing about mathematics, and attitudes and confidence in mathematics. Additional questions asked about the interaction of students with peers, the instructor, and the primary source material inside and outside of class. Finally, several open-ended questions asked students to reflect upon their experience with the PSP, including their perception of benefits and obstacles of learning mathematics using primary sources, and their attitude towards using primary sources in a linear algebra course. Implementation reports and surveys were collected from instructors and an instructional guide ("Notes to Instructors") was provided by the author.

4.1 Instructors and students

There were three course instructors of interest, across four implementations of the *Solving a System of Linear Equations using Ancient Chinese Methods* in 2017–18. The four student populations, as well as the survey data collected for each, are briefly described in Table 4.1 1.

[2] Flagg is a mathematician and university instructor and not a mathematics education researcher.

Table 4.1: Student populations in Linear Algebra courses (2017-18)

Instructor	Number of consenting students (estimated students enrolled)	Students completing all surveys (I = Pre-Course; P = Post-PSP; F = Post-Course)
Author Flagg (Fall 2017)	14 (15)	10 (I: 14, P: 11, F: 10)
Professor Monty (Fall 2017)	10 (25)	5 (I: 8, P: 7, F: 5)
Author Flagg (Spring 2018)	17 (21)	7 (I: 13, P: 11, F: 9)
Professor Carl (Spring 2018)	29 (32)	23[3]

In addition to the surveys and implementations reports, Professors Flagg and Monty were interviewed about their fall implementations. Three students from Professor Carl's class and two students from Professor Monty's class were also interviewed. The interviews were reviewed and relevant excerpts were transcribed for analysis and triangulation with data collected from the surveys and implementation reports.

In the following sections we describe each implementation using either the instructor's perspective, student's perspective, or both, when available, to describe the ways in which the four course populations reported similar perceived learning gains, and the ways in which students' reported benefits and obstacles for learning with primary source materials can inform future implementations in the TRIUMPHS project. We begin with our conceptualization of implementation of the PSP of interest.

5 Modeling communication utilized during PSP implementation

As part of this pilot study, we analyzed the interactions (i.e., lines of communication) between students, the instructor, and the author of the PSP in question, *Solving a System of Linear Equations using Ancient Chinese Methods* (Flagg, 2017). First, we considered the role of the student and how they supplied information to the instructor and to the author. For example, each student could supply information to the instructor (who was also implementing the PSP) by completing the assigned tasks and asking questions in class. Each student also completed a survey (e.g., post-PSP survey) about the project which we also analyzed. Students could communicate with each other via group work taking place during class.

[3] Due to the implementation of the PSP at the start of the semester, students in Professor Carl's course only completed one survey, which included items from each of the three surveys, and was completed immediately after the PSP was completed.

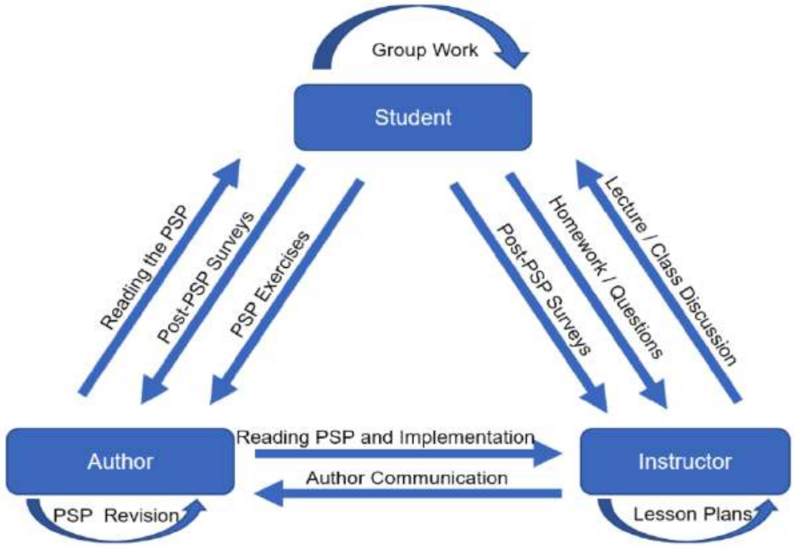

Figure 5.1: Lines of communication in a PSP implementation

Another line of communication was in the form of instructors with the students or author, through instructors lecturing or leading in-class discussions with students or via the instructor communicating directly with the author by email or phone if they had questions about implementation. The author was able to communicate both with students and the instructor through the written text of the PSP. The author also included "Notes to Instructors" with the PSP, which provided suggestions for implementation, including a timeline for implementation. Finally, it was possible for the author to further process the lines of communication by revising their PSP. The various forms of communication that we conjectured taking place are illustrated in Figure 5.1.

5.1 Case 1: The author as instructor (Flagg)

The first case we analyze is one in which the author is the instructor. According to the author's implementation report, a typical class consisted of reviewing the reading and homework from the earlier session, doing some group work in class relating to task(s) in the PSP, and having a concluding whole-class discussion. Students were expected to read the PSP outside of class as well as work on several tasks in the PSP as homework. The author was excited to share her passion for the material demonstrated by the following passage:

> I benefited from being able to share my passion for where ideas originate with my students and give them the opportunity to discover "new" old ideas. The students benefited from the struggle of trying to understand the unfamiliar language and notation. (Flagg's Implementation Report, Fall 2017)

5.2 Case 2: Non-author as "guide on the side" (Monty)

During this implementation, the instructor provided the students with the PSP and asked them to read portions of it outside of class. Class time consisted of students working together in small groups, which entailed completing assigned tasks during class. Students could ask the professor questions during class, but there was often not a concluding whole-class discussion. This is described in the instructor's implementation report in the following passage which described three days of implementation:

Students worked in groups (3-4students each) on Tasks 2-9.Students worked on a white board (rather than their own paper) so that they had to interact (as I limited it to one white board pen for each group). (Monty, Implementation Report)

This indicates that the instructor saw part of the value and struggle of the PSP as coming from the students working as a group. The instructor also noted in his commentary on the implementation that "I probably should have stepped in and brought the entire class together. I think some of them were starting to veer off track and re-orienting, them may have been helpful" (Implementation Report). This indicates the instructor's reluctance to interfere with student group work during implementation (even when, upon later reflection, the instructor indicated it might have been useful). This implementation is characterized by its focus on student interactions as opposed to instructor-student interactions (Figure 5.2).

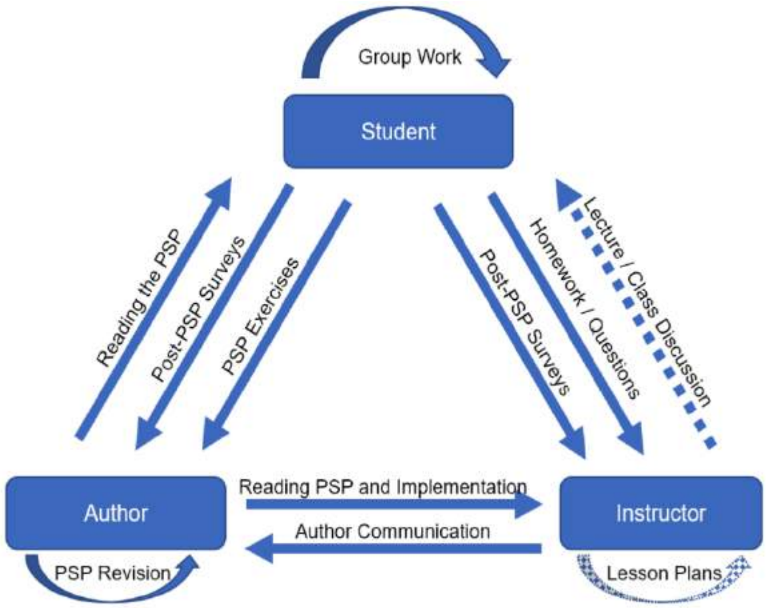

Figure 5.2: Non-author "guide on the side" approach

5.3 Case 3: Non-author without group work emphasis (Carl)

During this version of implementation, students were asked to read the material outside of class and complete some of the PSP tasks. The course consisted mostly of lecture with some in-class discussion but no group work. This implementation is depicted in Figure 5.3. During this implementation it was possible that the importance of reading was undermined by the in-class discussion as represented by the following comment offered by a student during a post-course interview:

> We would spend a couple of hours outside of class trying to go through five questions and a small section of the reading, and then we would come to class and he would answer all of our questions in like 10 minutes. (Student Interview)

Here the student noted that instructor would cover the material quickly the next day, undermining the value of struggling with the material on their own. To highlight this point, the arrow connecting author to the student is dotted to represent that the students did not struggle through the PSP.

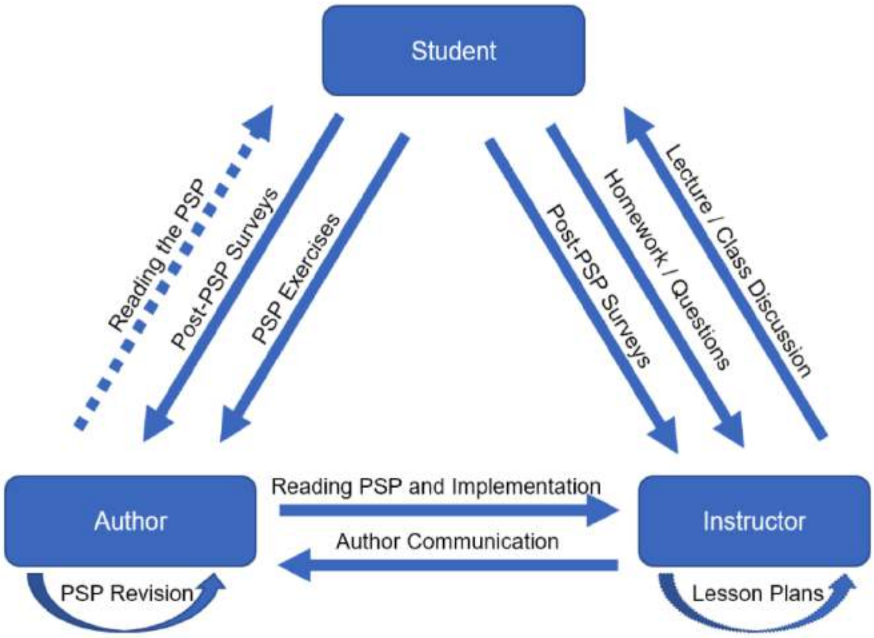

Figure 5.3: Non-author, without group work approach

6 Analysis and findings

6.1 Perceived student gains

As Figures 6.1, 6.2, and 6.3 show, students reported perceived gains in three major topics related to the PSP. We note several trends from the data collected. First, there is a trend toward students' reported perceived gains as "great gain" and "good gain," indicating that students perceived that they experienced progress in key course topics. Second, for the author's (Flagg) implementations, there was a high percentage of students reporting "good" and "great" gains, which might have resulted in the difference in implementation between the author and non-authors. Third, Figure 6.3 indicates a trend toward students' reported perceived gains as "no gain" or "small gain." A possible reason for this trend is that students in the United States are often taught back substitution prior to taking a Linear Algebra course, thus resulting in students perceiving smaller gains.

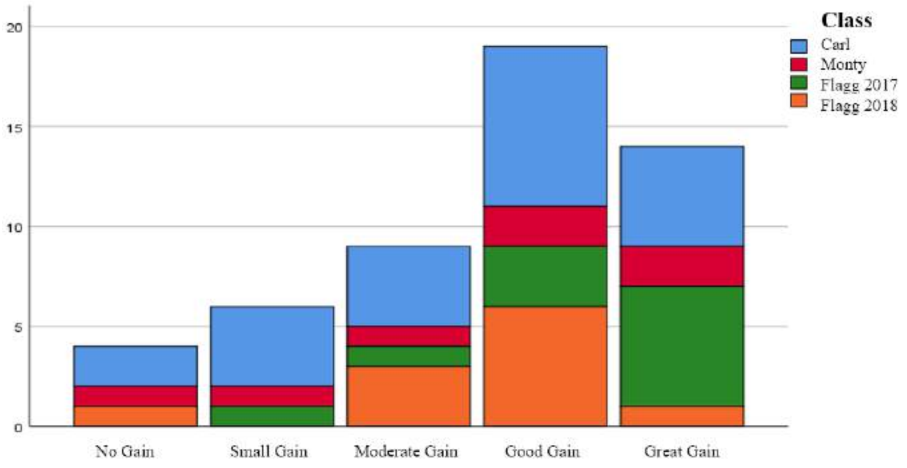

Figure 6.1: Perceived learning gains:Students' ability to set-up an augmented matrix

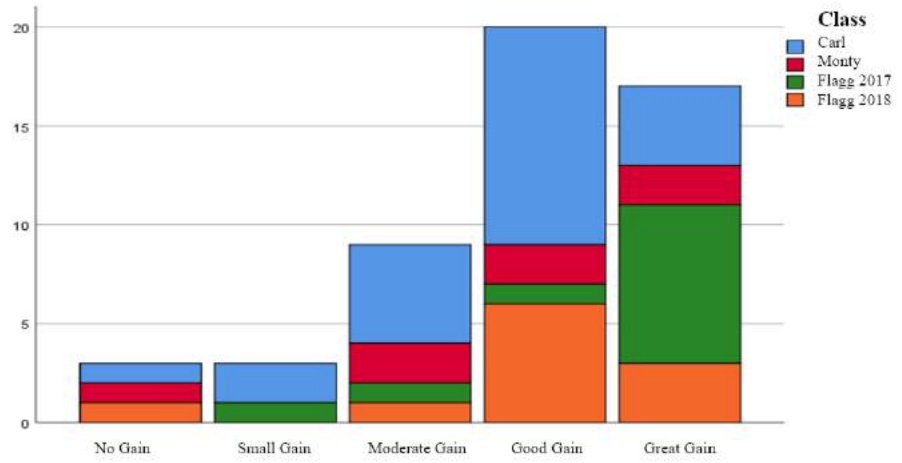

Figure 6.2: Perceived learning gains: Students' use of elementary row operation on an augmented matrix to find the unique solution of a non-singular matrix.

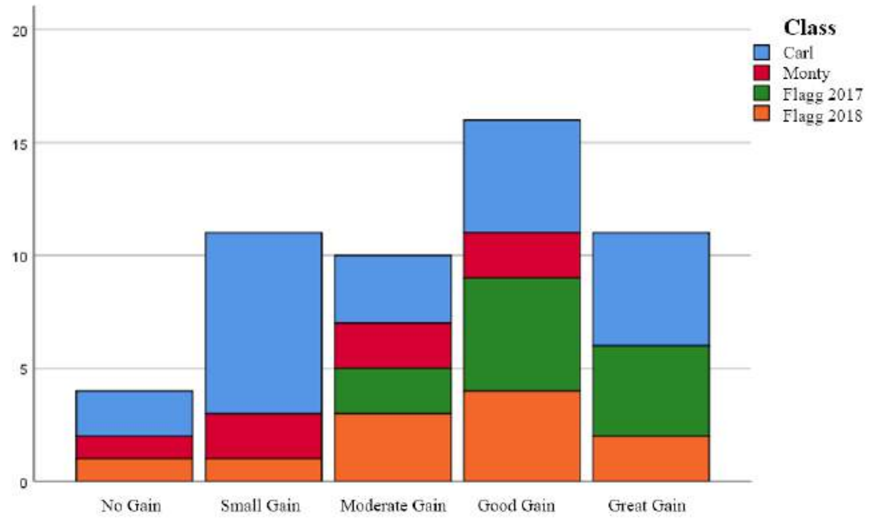

Figure 6.3: Perceived learning gains: Students' use of back substitution method for finding the solution of a system of linear equations represented by an upper triangular matrix.

6.2 Students' reported benefits and obstacles to using PSPs

On the post-PSP survey, students were asked what they believed the benefits and drawbacks were from learning mathematics by reading primary historical sources used in the *Solving a System of Linear Equations using Ancient Chinese Methods* project. Students identified multiple benefits, but one of the most prevalent was that students noted that seeing the development of mathematics or mathematical context was important and could instill confidence in the material. This is exemplified in the following student response:

> I think the benefit is knowing where methods originated, and know that what is studied and used today is completely valid and has history. It creates confidence in the subject because it has backing. (Flagg Student, Spring 2018)

A second benefit the students noted was that they experienced a change in perspective during PSP implementation. This is exemplified by the following student's response:

> One of the benefits of learning mathematics by reading historical sources is for students to have a perspective [on] how mathematics developed throughout history. This also gives [an] example how different societies can have multiple ways to solve a mathematical problem that differs from contemporary mathematics. (Flagg Student, Fall 2017)

Students also reported obstacles they experienced during this PSP implementation. Two general themes that arose were difficulty with language and general confusion. The first obstacle noted was a difficulty with the text itself, in which students described the language as archaic or that translations did not provide the clarity they needed. This is exemplified by the following comment:

> The main drawback was the archaic language used to describe the methods. I didn't have a clue what the author was asking me to do until the professor explained in more fluid language. (Monty Student, Fall 2017)

It is possible that students used the term "language" to represent general frustration with the material. There is some indication that the order in which topics were presented in the first implementation was less than optimal. This was acknowledged by students (in surveys) and in the instructor's implementation reports in Fall 2017. The PSP was revised for the 2018 implementation by the author (Flagg). Despite these revisions, there were still objections to the difficulty of understanding the language contained within the PSP, indicating that students were challenged with interpreting some "foreign" and unfamiliar terms.

The second obstacle that students reported was that the material was confusing or frustrating. This was often mentioned with reference to difficulty in understanding language, but it was not exclusively paired with this difficulty. For example, one student reported the following:

> It's more confusing than learning in a more traditional manner and I struggled with even understanding some of the primary source material. (CarlStudent1, Spring 2018)

Here it is important to note that the student was comparing the material to traditional material. This was another common theme among reported obstacles. This might indicate that students had problems learning from what was not a traditional, didactical text and the shift from such a format to something quite different presented a possible difficulty. In the United States, it is entirely possible that students might not have encountered a non-didactical mathematical text prior to this experience; thus, students may not know how to approach the text. As well, this is indicated by several students reporting that they had to read more than in a standard mathematics class.

Students were also asked what they would tell a friend about their PSP experience. Here, a number of students reported that potential peers should expect confusion and frustration, but the confusion and frustration had a payoff in terms of knowledge gains:

> They should expect to have to put a lot of time and effort into the project, but it will be worth it in the end. It will also be frustrating, but then interesting. (Carl Student 2, Spring 2018)

6.3 Instructors' consideration of language

The fact that students experienced difficulty interpreting language was also reported by instructors in their implementation reports. In general, there were two views relating to this obstacle reported by instructors. This first viewpoint was the perspective of Monty. He indicated that language was an unnecessary bug in the PSP, and perhaps the PSP could be modified to provide a clearer explanation in terms of modern terminology. In this instance, Monty wanted to "make the unfamiliar familiar" for the students. That is, Monty proposed to do this cognitive work for the students, rather than students potentially engaging in productive struggle to do this work.

The other perspective was that the difficulty interpreting language provided a great opportunity to discuss the use of language in communicating mathematical ideas for example the need for clarity and precision. This is best exemplified by Carl describing his process of "turning it [language] into a feature not a bug." This viewpoint was endorsed by the author, who stated that

> …some struggle is important to understand the power and challenge of mathematical language as [students] go deeper into the subject. *A little exercise in having trouble reading the PSP can be leveraged into a lesson on why it is important to be clear.* (Flagg, Fall 2017, emphasis added)

For both Carl and Flagg, "making the familiar unfamiliar" through the PSP was a meaningful and significant aspect of the use of the PSP with regard to overall mathematics instruction, and was a shared cognitive activity among students and instructors alike.

7 Conclusion and Discussion

7.1 Limitations and future research

This study was a pilot used as a preliminary analysis of the implementation of the PSP. The researcher's viewpoints into the implementation are a result of self-reported data from the students and instructors. These viewpoints may have omitted vital information regarding classroom interactions. Any results regarding how the students and instructors recalled these interactions are limited. Also, the number of participants included in this study is small; therefore, any quantitative results should be considered limited in scope (in particular the learning gains).Future research should include data collection (and analysis) of implementation. Observing student teacher interactions may result in increasing or decreasing the cognitive demand of the tasks contained within the PSP. This might also have an impact on how students perceive the PSP and their own learning gains, and could be grounds for further research.

It is also unclear if the student remarks about the language in the PSP resulted from frustration with the mathematics of the problem (which were communicated as language issues), issues with translating particular words within the text, the didactical style of the text which was different than most traditional text books in the United States, or something else. A careful analysis of each of these conditions in future PSP implementations might shed some light on what students mean by the claim that the language (in the PSP) is frustrating.

7.2 Discussion

In this pilot study, the data indicated that there were discernible differences among different implementations of the *Solving a System of Linear Equations using Ancient Chinese Methods* PSP. First, there were different lines of communication that were open among students, the instructor, and the author. The author's implementations contained all possible lines of communication while some lines of communication were not present in non-author implementations of the PSP. The presence of these lines of communication might be a contributing reason for why students in the author's implementations of the PSP reported high perceived learning gains.

While many students viewed language as a difficulty in the PSP, there was a significant difference in how instructors viewed language. Carl and Flagg both viewed language as an obstacle for students, but one that students should embrace as part of the learning experience, as opposed to a bug that needs to be corrected in future implementations. This is in line with the conceptions of Barnett, Lodder, and Pengelley (2014) who stated that

> In short, the primary source is now being used not just to introduce the mathematics in an authentically motivated context, but also as a text which the student is explicitly challenged to actively "interpret" as part of their personal process of making modern mathematics their own. In alignment with this shift, the tasks we now write for students increasingly adopt a more active "read, reflect, respond" approach to these sources. (p. 10)

Monty and Flagg viewed group work as productive struggle for students and prioritized it in their implementation. Furthermore, Flagg noted in her implementation that

> I learned a great deal from writing and implementing my project. I think that I learned the most from incorporating the readers' suggestions in framing tasks as more open-ended questions. I will continue to use that lens as I create course material for all my classes. The implementation of the project also takes me one step closer to creating a more interactive classroom. (Flagg, Implementation Report).

7.3 Implications for Instruction

Many participants viewed using the PSP as positive experience even if it was frustrating at times. The results indicate that some students had a frustrating time reading and interpreting excerpts from a primary source. Some students were able to overcome this difficulty by the time they submitted their post-PSP surveys, others were not. This indicates that it is important for instructors to carefully monitor student progress through the PSP to ensure that while some student struggle is productive, excessive student struggle might make for an unpleasant and unproductive experience. In particular, it is important that students receive feedback early and often at the beginning of the PSP when they are first encountering a new didactical style, new mathematics, and/or issues with translation that might introduce complexity to the problem.

7.4 Conclusion

In conclusion, we found that instructors can implement PSPs in vastly different ways, even when they are provided with supporting "Notes to Instructors" that are included as part of the PSP. Monty viewed the PSP as a chance to change the social dynamic of his classroom into a more active learning environment. Carl viewed the PSP as a chance to change the academic material presented to engage students in a cognitively different way, especially through overcoming the challenge of reading and interpreting the PSP. And, the author considered both of these struggles as productive and sought to incorporate them into her classroom which resulted in some of the greatest perceived learning gains on the part of students. This is only a preliminary report and this pilot study provides some evidence that the author might be better positioned to implement their own PSP (in terms of perceived student gains). This relationship could be due to the fact that authorship and preparation of "Notes to Instructors" require the author to think deeply about not only the material, but its implementation in the classroom, which includes the consideration of student engagement and learning. We believe that the relationship between authorship and quality of implementation is grounds for interesting future research, especially with regard to consideration of professional learning experiences for non-author instructors prior to PSP implementation.

REFERENCES

Alpaslan, M., &Haser, Ç. (2015). Selecting and preparing original sources for pre-service mathematics teacher education in Turkey: The preliminary report of a dissertation. In E. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds), *History and Epistemology in Mathematics Education: Proceedings of the Seventh European Summer University*. Copenhagen, Denmark: Danish School of Education, Aarhus University.

Barnett, J. H. (2014). Learning mathematics via primary historical sources: Straight from the source's mouth, *PRIMUS, 24*(8), 722–736.

Barnett, J. H., Lodder, J., &Pengelley, D. (2014). The pedagogy of primary historical sources in mathematics: Classroom practice meets theoretical frameworks. *Science & Education, 23*(1), 7–27.

Bernardes, A., & Roque, T. (2018). History of matrices: Promoting commognitive conflicts and encouraging reflection on meta-discursive rules in prospective teachers. In K. M. Clark, T. H. Kjeldsen, S. Schorcht, & C. Tzanakis (Eds.), *Mathematics, Education and History: Towards a harmonious partnership*, ICME-13 Monographs(pp. 209–227). Cham, Switzerland: Springer International Publishing.

Charalambous, C. Y., Panaoura, A., &Philippou, G. (2009). Using the history of mathematics to induce changes in preservice teachers' beliefs and attitudes: insights from evaluating a teacher education program. *Educational Studies in Mathematics*, *71*(2), 161–180.

Clark, K. M. (2012). History of mathematics: illuminating understanding of school mathematics concepts for prospective mathematics teachers. *Educational Studies in Mathematics*, *81*(1), 67–84.

Clark, K., &Thoo, J. B. (Eds.). (2014). Special issue on the use of history of mathematics to enhance undergraduate mathematics instruction. *PRIMUS, 24*(8), 663–773.

De Guzmán, M. (2007). Enseñanza de las Ciencias y la Matemática. *RevistaIberoamericana de Educación*, *43*, 19–58.

Flagg, M. (2017). *Solving a system of linear equations using ancient Chinese methods.* Retrieved from https://digitalcommons.ursinus.edu/triumphs_linear/1/

Huntley, M. A., & Flores, A. (2010). A history of mathematics course to develop prospective secondary mathematics teachers' knowledge for teaching, *PRIMUS, 20*(7), 603–616.

Jahnke, H. N., Arcavi, A., Barbin, E., Bekken, O., Furinghetti, F., El Idrissi, A., da Silva, C. M. S., & Weeks, C. (2002). The use of original sources in the mathematics classroom. In J. Fauvel& J. van Maanen (Eds.), *History in mathematics education: The ICMI Study* (pp. 291–328). Dordrecht, The Netherlands: Kluwer.

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235–261.

Jankvist, U. T. (2014). On the use of primary sources in the teaching and learning of mathematics. In M. R. Matthews (Ed.), *International handbook of research in history, philosophy and science teaching* (pp. 873–907). Dordrecht, The Netherlands: Springer Science+Business Media.

Katz, V., Jankvist, U. T., Fried, M. N., & Rowlands, S. (Eds.). (2014). Thematic issue: History, philosophy and mathematics education. *Science & Education, 23*(1), 1–250.

Kjeldsen, T. H., &Blomhøj, M. (2012). Beyond motivation: history as a method for learning meta-discursive rules in mathematics. *Educational Studies in Mathematics, 80*(3), 327–349.

Klyve, D., Stemkoski, L., &Tou, E. (2013). *Teaching and research with original sources from the Euler Archive*. Retrieved from http://www.maa.org/press/periodicals/convergence/teaching-and-research-with-original-sources-from-the-euler-archive

Laubenbacher, R., Pengelley, D. &Siddoway, M. (1994). Recovering motivation in mathematics: Teaching with original sources. *Undergraduate Mathematics Education Trends.6*, 1–7.

Liu, P.-H. (2014). When Liu Hui meets Archimedes: Students' epistemological and cultural interpretations of mathematics. *PRIMUS, 24*(8), 710–721.

Marshall, G. (2000). *Using history of mathematics to improve secondary students' attitudes towards mathematics.* Unpublished doctoral dissertation, Illinois State University, Bloomington-Normal, IL.

McBride, C. C., & Rollins, J. H. (1977). The effects of history of mathematics on attitudes toward mathematics of college algebra students. *Journal for Research in Mathematics Education*, *8*(1), 57–61.

Ruch, D. (2014). Creating a project on difference equations with primary sources: Challenges and opportunities. *PRIMUS, 24*(8), 764–773.

Tamulis, A. (2014). Using Albrecht Dürer's*Instruction on Measurement* for projects in the mathematics classroom. *PRIMUS, 24*(8), 753–763.

Appendix

Excerpt from *Solving a System of Equations by Ancient Chinese Methods* (Flagg, 2017)

3 Solving by the Fangcheng Rule

The Fangcheng Rule gives step-by-step instructions for solving problems on a counting board that are equivalent to systems of linear equations. The Rule can be broken into two separate steps. The first step is sometimes called **forward elimination** in modern mathematics, since the procedure uses an equation, starting with the first, to remove variables from the equations that follow. As we will see, the ancient Chinese version of forward elimination will result in the array resembling the shape of a triangle, and this form of the array will be referred to as **triangular form** or **lower triangular form**.

The algorithm for solving the triangular array will be referred to as **substitution** since the simplest method of solution is to start by solving the equation with only one variable, and then substitute the known values of variables into the remaining equation to solve for one variable at a time.

The ancient Chinese method of forward elimination is equivalent to the modern method, yet the procedure for substitution presented in the Fangcheng Rule is different from modern methods. To highlight the similarities and differences in the procedure and the resulting arithmetic, we will separate the elimination and substitution steps.

3.1 The Rule

Before we begin a careful explanation of each step of the Chinese method, read the English translation of the Fangcheng Rule. The term shi in the Fangcheng Rule is the yield of grain or fruit. It refers to the seeds of rice as it comes off the plant and before it is husked. It also means the constant in the equation [Shen et al., 1999, p. 400].

The Fangcheng Rule: [Let Problem 1 serve as example.] Lay down in the right column 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 dou of grain. Similarly for the middle and left column. Use [the number of bundles of] top grade paddy in the right column to multiply the middle column then merge. Again multiply the next [and] follow the pivoting [11]. Then use the remainder of the medium grade paddy in the middle column to multiply the left column and pivot. The remainder of the low grade paddy in the left column is the divisor, the entry below is the dividend. The quotient is the yield of low grade paddy. To solve for the medium grade paddy, use the divisor [of the left column] to multiply the shi in the middle column then subtract the value of the low grade paddy. To solve for the top grade paddy also take the divisor to multiply the shi of the right column then subtract the values of the low grade and the medium grade paddy. Divide by the number of bundles of top grade paddy. This is the number of bundles of top grade paddy. The constants are divided by the divisors. Each gives the dou of yield [of one bundle].

[11] The word 'pivot' used here as a modern term the translator chose to use to describe the process, it is not the original Chinese word.

Task 10 Try solving your array for Problem 1 of the *Nine Chapters* using the Fangcheng Rule before reading further. At what step were the instructions unclear?

Did you follow the whole rule? Probably not! Don't be discouraged. The fact that Liu Hui gave detailed commentary to the Fangcheng Rule in his edition from 263 CE indicates that other ancient Chinese mathematicians needed help as well! The original students would have likely read the rule in conjunction with a visual demonstration of the procedure on a counting board. Since we no longer have any record of the visual part of the lesson, it will take a little more effort to translate the verbal description.

In this section we will use Liu Hui's commentary to help us understand how to perform the Fangcheng Rule. We will use modern numerals to aid in understanding, but the layout will correspond to the ancient Chinese format in columns. Read Liu's introduction to the Fangcheng Rule.

The character *cheng* means comparing quantities. Given several different kinds of item, display [the number for] each as a number in an array with the sums (shi) [at the bottom]. Consider [the entries in] each column as rates, 2 items corresponds to a quantity twice, 3 items corresponds to a quantity 3 times, so the number of items is equal to the corresponding [number]. They are laid out in columns [from right to left], [and] therefore called a rectangular array (*fangcheng*). [Entries in each] column are distinct from one another and [these entries] are based on practical examples.

Task 11 What is the significance of Liu's statement that 'Entries in each column are distinct from one another'? Why is it important that the numbers are based on practical examples?

3.2 Forward Elimination in the *Nine Chapters*

We will now work through the Fangcheng Rule one step at a time, using Liu's commentary to help us understand the procedure. Our goal in this section is to reduce the array to triangular form.[12]

Rule: [Let Problem 1 serve as example.] Lay down in the right column 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 dou of grain. Similarly for the middle and left column.

Liu's Commentary: This is the general rule [for arrays]. It is difficult to comprehend in mere words, so we simply use paddy to clarify. Lay down the middle and left column like the right column.

[12] In this section the source text will be labeled as part of the original (Fangcheng) Rule or from Liu's commentary to make the distinction clear.

Task 12 Why is the Fangcheng Rule given using the numbers in a specific example instead of as a general procedure?

The array for Problem 1 is the following:

1	2	3
2	3	2
3	1	1
26	34	39

(9)

The first step of the solution procedure is the following:

Rule: Use [the number of bundles of] top grade paddy in the right column to multiply the middle column then merge.

Liu's Commentary: The meaning of this rule is: subtract the column with smallest [top entry] repeatedly from the columns with larger [top entries], then the top entry must vanish. With the top entry gone, the column has one item absent. However, if the rates in one column are subtracted [from another column], this does not affect the proportions of the remainders. Eliminating the top entry means omitting one item from the sum (shi). In this way, subtract adjacent columns from one another. Determine whether [the sum is] positive or negative. Then one can obtain the answer. First take top grade paddy in the right column to multiply the middle column. This means homogenizing and uniformizing. To homogenize and uniformize means top grade paddy in the middle column also multiplies the right column. For the sake of simplicity, one omits saying homogenize and uniformize. From the point of view of homogenizing and uniformizing this reasoning is natural.

Lui first uses the terms 'homogenize and uniformize' in Chapter 1 of the *Nine Chapters* after Problem 9 when he is explaining the rules for adding fractions. The term refers to the process of multiplying the numerator and denominator of each fraction by a specific factor in order to create equivalent fractions over a common denominator [Shen et al., 1999, pp. 70-72]. In the case of the Fangcheng Rule, 'homogenizing and uniformizing' refers to multiplying a column by the specified number in order for an entry to cancel when one column is subtracted by another.

We are instructed to first multiply the middle column by 3, the number for the top grade paddy in the right column. To multiply a column by a number means to multiply each entry in that column by the given number. After multiplying the middle column by 3, the middle column now has a larger top number than the right column. Liu tells us to subtract the right column repeatedly from the middle column until the top number vanishes (is zero). Subtracting the right column from the middle column means to replace each entry in the middle column by the difference between that number and the number on the same row of the right column. Modern mathematicians would record a zero if that was the result of subtraction. However, the ancient Chinese did not use a symbol for zero in counting board arithmetic, so we will follow the traditional procedure and leave the space blank.

Task 13 Use the instructions in this piece of the Fangcheng Rule to eliminate the number for the top grade paddy in the middle column of the array for Problem 1.

Liu next explains how to continue the process of elimination.

Liu's Commentary: Again eliminate the first entry in the left column. Again, use the two adjacent columns to eliminate the medium grade paddy.

Task 14 Follow the same procedure Liu outlines.

(a.) Eliminate the 1 in the top row of the left column.

(b.) Follow the elimination procedure for the middle and left column with the medium grade paddy.

At this point your array should resemble a triangle. Today this is called *lower triangular form* or simply *triangular form*.

The array for Problem 1 should now be in lower triangular form. Practice this procedure with the other Problems from the *Nine Chapters* and the modern problem by completing the following tasks.

Task 15 Reduce the array you created in Task 6 for Problem 3 from the *Nine Chapters* to lower triangular form using the elimination procedure the Fangcheng Rule. Note that this problem involves using negative numbers in the elimination process.

Task 16 Reduce the array you created in Task 7 for Problem 7 from the *Nine Chapters* to lower triangular form using the elimination procedure the Fangcheng Rule.

Task 17 Reduce the array for the modern problem created in Task 9 to lower triangular form using the Fangcheng Rule.

The Fangcheng Rule, together with the Sign Rule, were used to solve problems that involved excess and deficit in the form of both positive and negative numbers. However, problems arose from practical examples, so the final solutions were always positive. In triangular form, the column containing only one unknown and a yield should be positive. Consider again Problem 8 in the *Nine Chapters*.