REVERSED PROCEDURE AND KUTTAKA METHOD

The calculation of Indian Mathematics (*ganita*) in *Aryabhatiya* and *Brahma-sphuta-siddhanta*

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ABSTRACT

Between the 4th and 5th centuries, there was remarkable mathematical activity in India related to astronomy which produced the *siddhantas*, texts that included whole chapters devoted exclusively to mathematical calculus (*ganita*). Aryabhata (476-550) and Brahamagupta (598-668) wrote their siddhantic texts, *Aryabhatiya* and *Brahma-sphuta-siddhanta*, which were a reference to later astronomers and mathematicians. In this presentation, we will focus on some of the issues related to Aryabhata and Brahamagupta calculation. High school students can study them to establish bridges between the current methods of resolution and the old Indian methods (Reversed Procedure and *Kuttaka* Method). We will present two recent didactic experiences in the classroom: the first related with the Reversed Procedure of Aryabhata, presented during a pre-service teacher course at the University of Barcelona; the second with the *Kuttaka* method, made with students from an elective course of the History of Mathematics at the Polytechnic University of Catalonia.

Keywords: Indian Mathematics, Aryabhata, Brahmagupta, *Ganita*, Reversed procedure, *Kuttaka* method.

1 Introduction

Different members of the Catalan Historians of Science group had already presented several didactic works using original historical sources1. We had begun to try introducing some aspects of ancient India mathematics. When we talk about mathematics in ancient India, we are referring to the Indian subcontinent, i.e. India, Nepal, Pakistan, Bangladesh and Sri Lanka. In the Indian subcontinent, one of the oldest civilizations in the history of humanity, the civilization of the Indus was developed. The initial basic rudiments on geometry and metrology evolved over time to achieve remarkable astronomical and mathematical knowledge. There was a long development through different stages in which the transmission and learning of knowledge was made using singular mechanisms involving aspects related to commerce and technology (civilization of the Indus 2500-1800 BC), ritualistic religious and linguistic structures (Vedic age 1500-500 BC) or non-theist cosmological ideas (Jainism 500-200 BC) (Puig-Pla& Guevara, 2017).It was during the so-called classical era (400-1200) when such prominent personalities as Aryabhata (5th and 6th centuries), Brahmagupta (7th century) or Bhaskara II (12th century) appeared.
In this presentation, we will focus on some of the issues related to Aryabhata and

[1]These works are related with Greeks, medieval Muslim or ancient Chinese mathematics. See Romero *et al.*, 2015 (last ESU-7 in Copenhagen) or Massa *et al.*, 2011 (ESU-6 in Vienna).

Brahamagupta calculation (implemented in the 2017-2018academic year) in the classroom: the Reversed Procedure and the *Kuttaka* Method.

The Reversed Procedure from Aryabhata was presented during a pre-service teacher courseduring an Interuniversity Master in Teacher Training on Didactics of Mathematics - Universitat de Barcelona, Universitat Autònoma de Barcelona, Universitat Politècnica de Catalunya, Universitat Pompeu Fabra- (UB, UAB, UPC, UPF).

The *Kuttaka* Method from Aryabhata and Brahamagupta was offered in an elective History of Mathematics course for students of the Degree in Mathematics – Universitat Politècnica de Catalunya (UPC).

In general, if we want to implement a mathematical activity based on historical texts in the classroom it seems advisable, first, to locate the country and the historical time, then to identify the context in which it was developed and, finally, to raise the problem that it wanted to solve.

2 Mathematics in siddhanta texts

In the 4th and 5th centuries, there was remarkable mathematical activity in India related to astronomy which produces the s*iddhantas*, treatises on mathematical astronomy. They were texts with instructions to calculate positions of celestial bodies and solve questions related to the calendar, geography or astrology (Joseph, 1996:360).

The authors of the *siddhantas* wrote their works in Sanskrit verses following a structure established by the different astronomy schools (or *paksas).*In general, *siddhantas* pursued three goals: the first one consisted of predictive astronomy, the computation of times, locations and appearances of celestial phenomena –future or past– as seen from any given terrestrial location; the second was computational astronomy, explanations on computational procedures in terms of the geometry of the spherical models (presented in a separate section called *gola* (sphere)); and the third objective was instruction in general mathematical knowledge, basic arithmetic operations, calculation of interest on loans, rules for finding areas, volumes, sum of series, etc.

The *siddhanta* texts include one or some chapters dedicated exclusively to calculation in the proper mathematical sense(*ganita*).

2.1 Aryabhata & Brahamagupta

Two Indian astronomers and mathematicians, Aryabhata (476-550) and Brahamagupta (598-668), wrote two relevant *siddhanta* texts, *Aryabhatiya* and *Brahma-sphuta-siddhanta*, respectively. They were references for later astronomers and mathematicians. In Indian mathematics, *Aryabhatiya*(499) played, in some manner, the role of the *Elements* of Euclid in Greek mathematics (Moreno, 2011:46). *Brahma-sphuta-siddantha* (or *Corrected Teatrise of Brahma*), a work written in 628 when Brahmagupta was 30 years old, was in some way, a response to *Aryabhatiya*. He criticized Aryabhata, for example, for considering in a *kalpa* (the period between the creation and recreation of a world) 1008 *mahayugas* (a period of time) instead of 1000 according to Hinduism.

The last work of Brahmagupta was *Khanda-khadyaka*, written at the age of 67. It is a *karana* or manual of mathematical astronomy with simplified calculations.

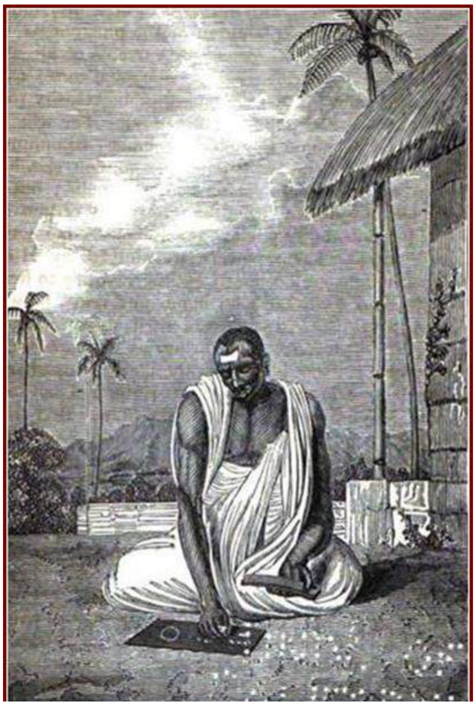


Figure 2.1: Imaginary representation of Brahmagupta (598-670) according to a 19th century engraving

2.2 *Aryabhatiya* and *Brahma-sphuta-siddantha*

The *Aryabhatiya*, written in 499 by Aryabhata, is the earliest completely preserved *siddhanta*. Chapter 2 is devoted to mathematics (*ganita*) and it consists of 33 verses[2] in a Sanskrit metric including methods for resolution of first degree, quadratic and first-degree indeterminate equations (Plofker, 2009, pp. 122-136). Bhaskara I wrote *Aryabhatiyabhasya* (629), a comment in Sanskrit prose about these 33 verses of chapter 2 of the *Aryabhatiya* (Keller, 2005, pp. 279-280).

From the *Aryabhatiya* of Aryabhata with commentary by Bhaskara I, it is known what Aryabhata said about how to teach the Reversed Procedure (Chapter 2, 28): "*In a reversed* [operation], *multipliers become divisors and divisors, multipliers, and an additive* [quantity], *is a subtractive* [quantity], *a subtractive* [quantity] *an additive* [quantity]".

In relation to the *Brahma-sphuṭa-siddhanta* (or BSS from now on), the book has 24 chapters. Chapters 1 to 10 deal with basic topics of astronomy: average lengths of planets; true lengths of the planets; the problems of daytime rotation; lunar eclipses; solar eclipses; rising and setting of the sun; phases of the moon; the shadow of the moon; conjunctions of the planets and conjunctions of the planets with the fixed stars. Chapter 11 is a criticism of *Aryabhatiya*. Chapter 12 deals calculation with numbers (*ganita*) and chapter 18 with calculation with unknown quantities and contents of *kuttaka* method. The construction of the sine appears in chapter 21. Chapters 13 to 17, 19 to 20 and 22 to 24 are dedicated to the topics related to astronomy.

[2]Verses 11 through 17 of *Aryabhatiya* contain calculations of half strings (which will evolve and become the sine) (Puig-Pla*et al*., 2011: 53).

3 The Reversed Procedure and the*Kuttaka*Method

Reversed Procedure and *Kuttaka* Method are both procedures of calculation to solve equations. In the first case it is about finding a number if we know the final result of several consecutive operations with the first number. The problem informs about the operations and the final result; the question is which is the initial number. We have to reverse the operations described in the problem, starting with the last one and finishing with the first one.

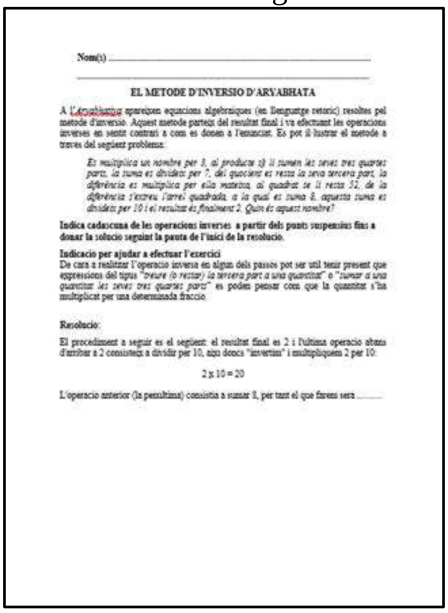
In the second case, we take successive steps to transform an equation with two unknowns into a simpler one repeating the process until reaching an equation that has one of the coefficients equal to 1, so that from thence the solutions for the original equation can be obtained by going backwards. This method can be considered as an application of the reversed procedure. Indeed, when the original equation $ax + c = by$ is transformed into $a'x' + c = y'$, clearly $x' = 0$ and $y' = c$ constitute one solution, from whence a solution (indeed all solutions) of the original equation $ax + c = by$ will be obtained, and the procedure agrees exactly with the algorithm described in the *Kuttaka* Method.

On the one hand, this explains the term *Kuttaka* ("Pulverizer" or reducing to powder) in that the coefficients are stepwise made smaller through the Euclidean algorithm (equivalent to the ancient Chinese algorithm in the *Geng Xiang Jian Sun* method) as well as tying up the two topics under discussion in this paper.

3. 1 The activity using the Reversed Procedure of Aryabhata

We presented this activity in a *pre-service teacher course* in an Inter-university Master in Teacher Training. After locating the country and the historical time (Ancient India at the end of 5th century), we gave a short explanation about the context (mathematics in *siddhanta* texts) and the procedure in the *Aryabhatiya* of Aryabhata. We used the*Aryabhatiya*commentary of Bhaskara I. After students were asked to solve an activity.

We prepared an initial exercise for students. It was based on verse 50 of the work *Lilavati* by Bhaskara II, a problem that had to be solved by the Aryabhata method (Plofker, 2007:454). The exercise had the following information and questions:



Nom(s) ....

EL METODE D'INVERSIO D'ARYABHATA

A L'Aryabhatiya apareixen equacions algebraiques (en llenguatge retoric) resoltes pel metode d'inversio. Aquest metode parteix del resultat final i va efectuant les operacions inverses en sentit contrari a com es donen a l'enunciat. Es pot il·lustrar el metode a través del següent problema:

> *Es multiplica un nombre per 3, al producte se li sumen les seves tres quartes parts, la suma es divideix per 7, del quocient es resta la seva tercera part, la diferència es multiplica per ella mateixa, al quadrat se li resta 52, de la diferència s'extreu l'arrel quadrada, a la qual es suma 8, aquesta suma es divideix per 10 i el resultat és finalment 2. Quin és aquest nombre?*

Indica cadascuna de les operacions inverses a partir dels punts suspensius fins a donar la solucio seguint la pauta de l'inici de la resolucio.

Indicacio per ajudar a efectuar l'exercici
De cara a realitzar l'operacio inversa en algun dels passos pot ser util tenir present que expressions del tipus "*treure (o restar) la tercera part a una quantitat*" o "*sumar a una quantitat les seves tres quartes parts*" es poden pensar com que la quantitat s'ha multiplicat per una determinada fraccio.

Resolucio:

El procediment a seguir es el segient: el resultat final es 2 i l'ultima operacio abans d'arribar a 2 consisteix a dividir per 10, aixi doncs "invertim" i multipliquem 2 per 10:

2 x 10 = 20

L'operacio anterior (la penultima) consistia a sumar 8, per tant el que farem sera ....

Figure 3.1: The Reversed Procedure Exercise

“In *Aryabhatiya* algebraic equations appear (in rhetorical language) solved by the Reversed procedure. This method starts from the final result and performs the reversed operations in the opposite direction as given in the statement. The procedure can be illustrated through the following problem:

A number is multiplied by 3, the product is added to its three quarters, the sum is divided by 7, the ratio is subtracted from its third part, the difference is multiplied by itself, the square is reduced to 52, from the difference is extracted from the square root, which is added 8, that sum is divided by 10 and the result is finally 2. What is this number?

Indicate each of the inverse operations from the ellipsis in order to give the solution following the guideline of the beginning of the resolution.

Guidelines to solve the exercise.

In order to perform the reversed operation in one of the steps, it may be useful to bear in mind that expressions of the type "remove (or subtract) the third part from an amount" or "add up to three quarters" can be thought of as the amount that has been multiplied by a certain fraction.”

Resolution

The Reversed Procedure is as follows: the final result is 2 and the last operation before reaching 2 is to divide by 10, so we do the "reversed operation" by multiplying 2 by 10:

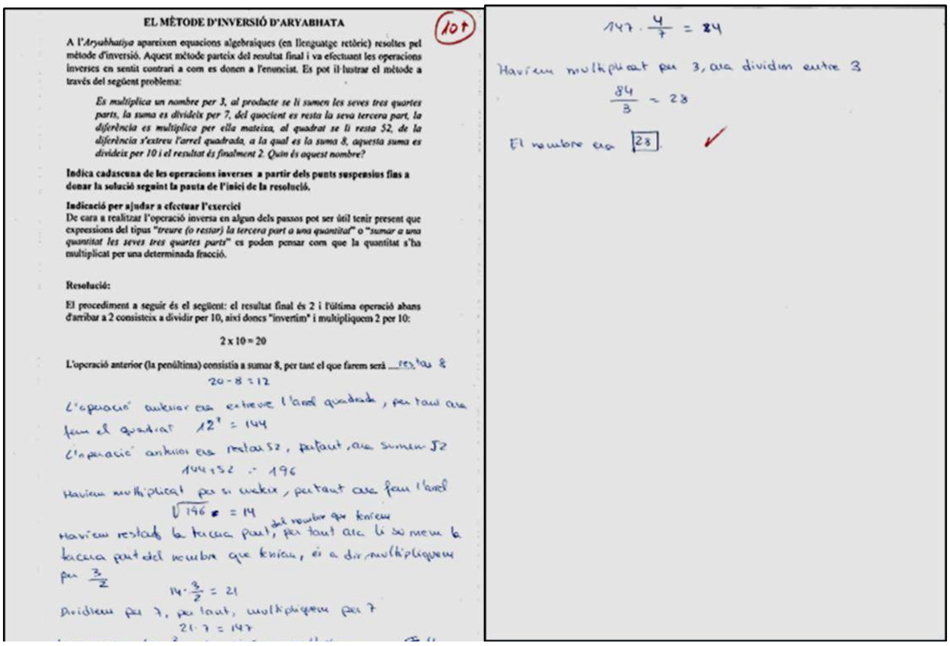
$$2 \times 10 = 20$$

The previous operation (the penultimate one) consisted of adding 8, therefore what we will do will be ....”

3.2 Students’ production susing the Reversed Procedure of Aryabhata

26 students carried out the activity. They formed freely 7 groups that had to be from 3 to 4 students (in fact there was also 1 group of 2 students and 1 group of 5).The group code is indicated thus, e.g. G2 (3) means group 2 with 3 students. In the statement the students had the pattern to follow (at the beginning of the resolution). They were asked to write a rhetorical explanation about the previous operation to carry out the reversed operation.

Although all the groups arrived at the correct solution, only one of them, G3(4), followed "exactly" this pattern. They wrote rhetorical explanations such as “[…] the previous operation was to extract the square root, therefore we will now square, $12^2 = 144$ […]”.



EL MÈTODE D'INVERSIÓ D'ARYABHATA

A l'*Aryabhatiya* apareixen equacions algebraiques (en llenguatge retòric) resoltes pel mètode d'inversió. Aquest mètode parteix del resultat final i va efectuant les operacions inverses en sentit contrari a com es donen a l'enunciat. Es pot il·lustrar el mètode a través del següent problema:

Es multiplica un nombre per 3, al producte se li sumen les seves tres quartes parts, la suma es divideix per 7, del quocient es resta la seva tercera part, la diferència es multiplica per ella mateixa, al quadrat se li resta 52, de la diferència s'extreu l'arrel quadrada, a la qual es la suma 8, aquesta suma es divideix per 10 i el resultat és finalment 2. Quin és aquest nombre?

Indica cadascuna de les operacions inverses a partir dels punts suspensius fins a donar la solució seguint la pauta de l'inici de la resolució.

Indicació per ajudar a efectuar l'exercici
De cara a realitzar l'operació inversa en algun dels passos pot ser útil tenir present que expressions del tipus "*treure (o restar) la tercera part a una quantitat*" o "*sumar a una quantitat les seves tres quartes parts*" es poden pensar com que la quantitat s'ha multiplicat per una determinada fracció.

Resolució:

El procediment a seguir és el següent: el resultat final és 2 i l'última operació abans d'arribar a 2 consisteix a dividir per 10, així doncs "invertim" i multipliquem 2 per 10:

2 x 10 = 20

L'operació anterior (la penúltima) consistia a sumar 8, per tant el que farem serà ....restar 8

20 − 8 = 12

$12^2 = 144$

144 + 52 = 196

$\sqrt{196} = 14$

$14 \cdot \frac{3}{2} = 21$

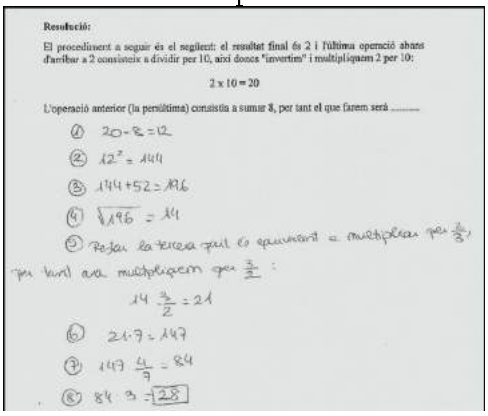
21 · 7 = 147

$147 \cdot \frac{4}{7} = 84$

$\frac{84}{3} = 28$

Figure 3.2: A student production from group G3(4) of the Reversed Procedure

All students correctly indicated the operations to be performed in each of the eight steps but "without" or "with few" rhetorical explanations.



Resolució:

El procediment a seguir és el següent: el resultat final és 2 i l'última operació abans d'arribar a 2 consisteix a dividir per 10, així doncs "invertim" i multipliquem 2 per 10:

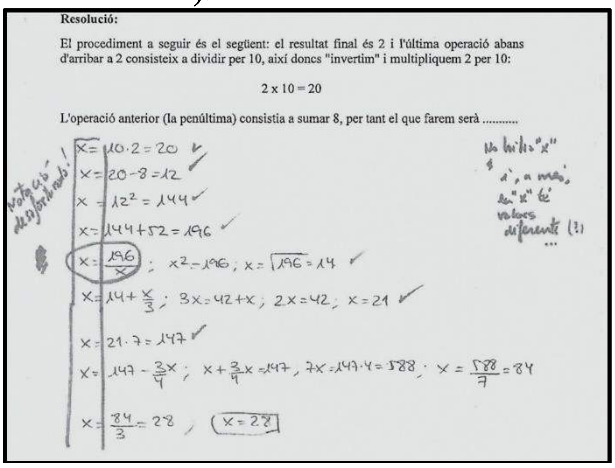
2 x 10 = 20

L'operació anterior (la penúltima) consistia a sumar 8, per tant el que farem serà ....

① 20 − 8 = 12
② $12^2 = 144$
③ 144 + 52 = 196
④ $\sqrt{196} = 14$
⑤ $14 \cdot \frac{3}{2} = 21$
⑥ 21 · 7 = 147
⑦ $147 \cdot \frac{4}{7} = 84$
⑧ 84 : 3 = 28

Figure 3.3: A student production from group G4(4) of the Reversed procedure

Two groups G5(4) and G6(2) introduced modern mathematical formalism by writing equations (using "x" for the unknown).



Resolució:

El procediment a seguir és el següent: el resultat final és 2 i l'última operació abans d'arribar a 2 consisteix a dividir per 10, així doncs "invertim" i multipliquem 2 per 10:

2 x 10 = 20

L'operació anterior (la penúltima) consistia a sumar 8, per tant el que farem serà ....

x = 10 · 2 = 20
x = 20 − 8 = 12
$x = 12^2 = 144$
x = 144 + 52 = 196
$x = \frac{196}{x}$; $x^2 = 196$; $x = \sqrt{196} = 14$
$x = 14 + \frac{x}{3}$; 3x = 42 + x ; 2x = 42 ; x = 21
x = 21 · 7 = 147
$x = 147 - \frac{3x}{4}$; $x + \frac{3}{4}x = 147$, 7x = 147 · 4 = 588 ; $x = \frac{588}{7} = 84$
$x = \frac{84}{3} = 28$, x = 28

Figure 3.4: A student production from group G5(4) of the Reversed procedure

In general, we can conclude thatthe students of the pre-service teacher course easily understood the Reversed Procedure, although not all of them were able to solve the problem “in the manner of Aryabhata”, that is, in a rhetorical way and without formal symbolism (23% of students used the modern formalism of algebra).

3.3 The *Kuttaka* Method

We presented this activity in an *elective course of the History of Mathematics* of the Degree in Mathematics (Universitat Politècnica de Catalunya). After locating the country and the historical time (Ancient India at the end of 5th century), we gave a short explanation about the context (mathematics in *siddhanta* texts) and the problem that Indian astronomers wanted to solve (the conjunction of the planets).

We continued with a short explanation of the method of Brahmagupta in *Brahma-Sphuta-Siddhanta* (chapter 18, 3-6) and finally students were asked to solve a problem, to assess the method and to answer a questionnaire about the history of math.

3.4 Chapter 18: Calculation with Unknowns

In this chapter Brahmagupta explains a calculation procedure with unknowns, first and second degree. But the innovation is the*Kuttaka*Method or “pulverizer" to treat equations or systems of indeterminate equations.The establishment of the calendar and astronomical calculations led to this type of procedureto solve equations or systems of indeterminate equations.

About the subjects that a master must know, he stated: “*A master* [acarya] *among those who know treatises* [is characterized] *by knowing the pulverizer, zero, negative* [and] *positive* [quantities], *unknowns, elimination of the middle* [term, that is, solution of quadratics], *single-color* [equations, or equations in one unknown], *and products of unknowns, as well as square nature* [problems, that is, second-degree indeterminate equations]” [BSS, Ch. 18, 2].

Brahmagupta gave the first known explanation of Indian mathematics about “the rules of signs and the arithmetic of zero”. Almost all the rules are identical to our modern algebra: “[The sum] *of two positives is positive, of two negatives negative; of a positive and a negative* [the sum] *is their difference; if they are equal it is zero. The sum of a negative and zero is negative,* [that] *of a positive and zero positive,* [and that] *of two zeros is zero* [. . .]. *The product of a negative and a positive is negative, of two negatives positive, and of positives positive; the product of zero and a negative, of zero and a positive, or of two zeros is zero.* [BSS, Ch. 18, 30 and 33, respectively]”.

Verses 43 through 59 refer to techniques and examples for solving equations with an unknown, both first and second degree. In the case of second degree, when a multiple “b” of the unknown added to a multiple “a” of the square of the unknown is equal to a number c. The calculation algorithm is focused on “removing the middle term” given the equation in the form: $ax^2 + bx = c$.

3.5 The *Kuttaka* or “pulverizer” method

Pulverizer (*Kuttaka*) is the successive steps to transform an equation with two unknowns into a simpler one. Given an equation $ax + c = by$ where a and b do not have common divisors, by means of a change of variable, the initial equation is transformed into another equivalent, until reaching an equation that has one of the coefficients equal to 1.From here the solutions are reconstructed until they reach the initial equation. But, for which part of astronomy were these calculations necessary? What was the problem to be solved?

We have to consider that we have a geocentric point of view, as the ancient Indians had. Then, it is a question of counting the days that have elapsed since the last time that two planets, A and B, were in conjunction at a certain point with respect to the fixed stars (that is since they had the same ecliptic longitudeλ_0).

For example, let us say we know - after long systematic observations - the following data:

1) Planet A takes an average time of 90 solar days to return to the same point on the stellar background (through the Zodiac) while planet B takes 33 days (see figure 3.5).

2) 19 days have elapsed since planet A completed an entire number of turns following the Zodiac (from λ_0).

3) 28 days have elapsed since planet B completed an entire number of turns following the Zodiac (from λ_0) in the same direction (see figure 3.6).

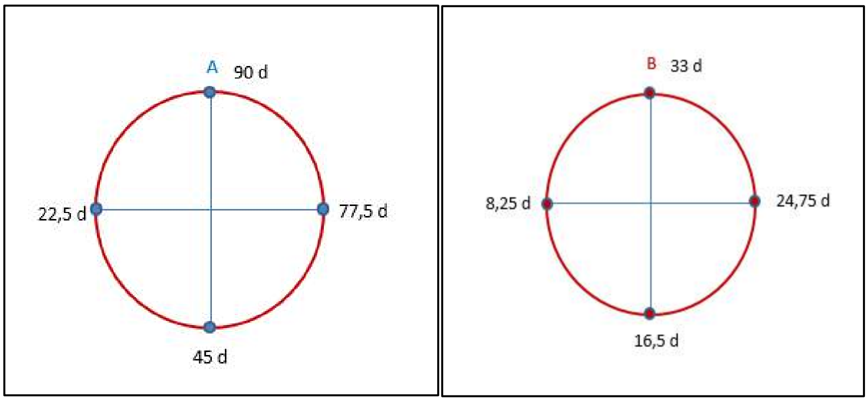


Figure 3.5: Planet A (to the left) takes 90 days to return to the same point. Planet B (on the right) takes 33 days

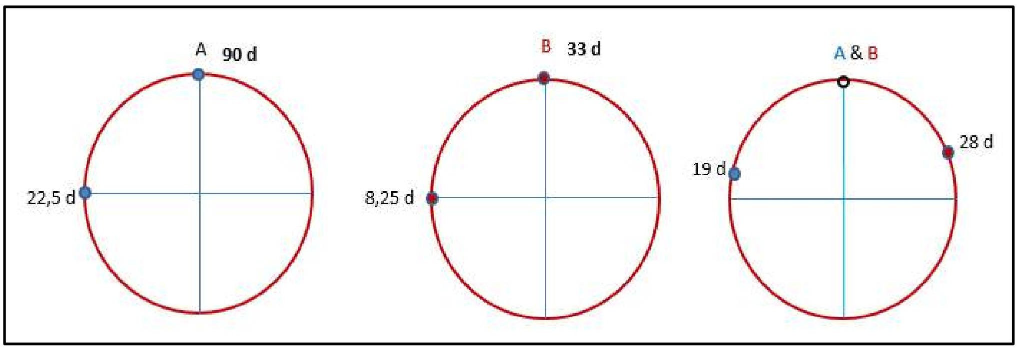


Figure 3.6: Planet A (on the left), Planet B (in the middle) and Planets A & B (on the right) after completed a different integer number of revolutions each one of them

The problem to solve is “how many days (N) have passed since the last conjunction?” Using the current notation, if x is the integer number of revolutions performed by A from the conjunction,

$$N = 19 + 90x$$

If y is the integer number of revolutions performed by B from the conjunction,

$$N = 28 + 33y$$

then, we have $N = 19 + 90x = 28 + 33y$

so, $90x - 33y = 28 - 19$

in other words that is,

$$90x - 33y = 9$$

a linear equation with two unknowns (with infinite integer solutions). We look for the positive integers such as $90x - 33y = 9$.

3.6 Brahmagupta in *Brahma-Sphuta-Siddhanta* (Chapter 18, 3-6)

Brahmagupta explains in verses 3-6 how to proceed with *Kuttaka* method:

1. *Divide the divisor having the greatest remainder* [*agra*] *by the divisor having the least remainder*

[Indication: Euclid's algorithm].

2. *Once mutually divided, the last remainder will be multiplied by an arbitrary number* [integer] *such that if the product that we obtain is added* [if the number of quotients in the process is odd] *or we take it* [if it is even] *the difference of remainders* [the additive], *which results is divisible by the penultimate remainder.*

3. *Place the quotients of the mutual divisions one below the other in columns, until you reach the optional divisor and then the quotient you have obtained.*

4. *Multiply the penultimate by the previous one and the one that follows is added to it. Repeat the process* [the result is saved in the next column and occupies the penultimate position and then the penultimate number of the previous column].

5. *Divide the last number obtained* [*agranta*] *by the divisor having the least remainder. Then multiply the remainder by the divisor having the greatest remainder and add the largest remainder. The result will be the remainder of the product of the divisors* [to obtain a smaller solution].

In our activities, we explain the *Kuttaka* method Brahmagupta has created and we ask students to do the same. We also give them the translation of the method from Nolla (2006: 196-203).

3.7 The activity

As Brahmagupta explained in verses 3-6 of Chapter 18 in *Brahma-Sphuta-Siddhanta,* we try to solve the following problemof the *Aryabhatiya* by Aryabhata with comments from Bhaskara I (Chapter 2, 33):

"[A quantity when divided] *by twelve has a remainder which is five, and furthermore, it is seen by me* [having] *a remainder which is seven, when divided by thirty-one. What should one such quantity be?*"

$$\left.\begin{array}{l} 12y + 5 = N \\ 31x + 7 = N \end{array}\right\} \qquad 31x + 2 = 12y$$

3.8 Students' solutions of the activity

Although the answers of the students are not in English, we have collected their answers step by step and the connection they made with the Bézout method. We follow the assignments of two students, A and B.

Student A, translates the problem to modern notation

N ? tal que $N \equiv 5 \pmod{12}$ i $N \equiv 7 \pmod{31}$

x, y ? tal que $N = 5 + 12y = 7 + 31x \Rightarrow -12y + 31x = -2$

$\mu = 1$
$\lambda = 4$

Figure 3.7: Student translation of the problem

1. Student A makes the first step, obtains the divisors and the residues and orders them in columns as established by the method

$-12y + 31x = -2$

1) Mètode Brahmagupta: kuttaka

	q_1	q_2	q_3	q_4		
	2	1	1	2	2	$n=4$
$a = 31$	$b = 12$	7	5	2	1	
7	5	2	1	0		
r_1	r_2	r_3	r_4			

Figure 3.8: Student A's solution, step 1

2. Student A makes the second step, to find numbers λ and λ that meet certain conditions

Buscar λ, μ tq $r_n \cdot \lambda + (-1)^n c = r_{n-1} \cdot \mu$: $\lambda - 2 = 2\mu \Rightarrow \begin{cases} \mu = 1 \\ \lambda = 4 \end{cases}$

Figure 3.9: Student A's solution, step 2

3. Student A places the quotients of the mutual divisions one below the other in columns, until the optional divisor is reached and then the quotient is obtained.

$q_1 = 2$
$q_2 = 1$
$q_3 = 1$
$q_4 = 2$
$\lambda = 4$
$\mu = 1$

Figure 3.10: Student A's solution, step 3

4. Student A obtains a solution

$q_1 = 2$	$q_1 = 2$	$q_1 = 2$
$q_2 = 1$	$q_2 = 1$	$q_2 = 1$
$q_3 = 1$	$q_3 = 1$	$\alpha_2 = 1 \cdot 9 + 4 = 13$
$q_4 = 2$	$q_4 \cdot \lambda + \mu = \alpha_1 = 9$	$\alpha_1 = 9$
$\lambda = 4$	$\lambda = 4$	
$\mu = 1$		

$q_1 = 2$ $\qquad \alpha_4 = 2 \cdot 22 + 13 = 57 = y$
$\alpha_3 = 1 \cdot 13 + 9 = 22$ $\qquad \alpha_3 = 22 = x$
$\alpha_2 = 13$

Comprovem-ho: $-12y + 31x = -12 \cdot 57 + 31 \cdot 22 = -684 + 682 = -2$

Figure 3.11: Student A's solution, step 4

5. Student A obtains a smaller solution

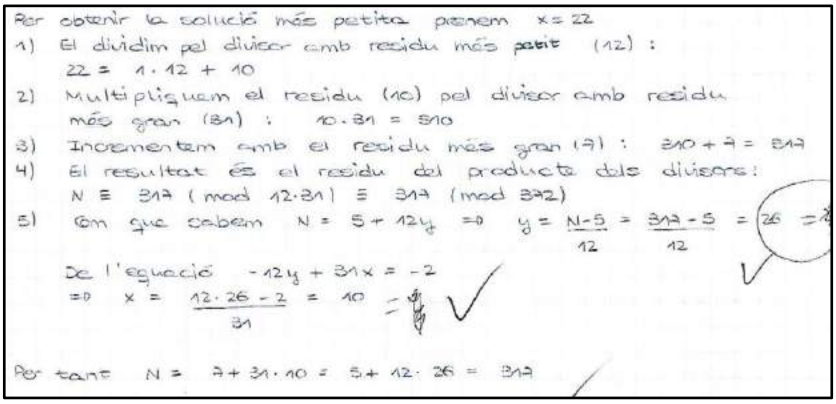


Figure 3.12: Student A's solution, step 5

We can see another student's solution (Student B) in figure 3.13. This student also used Bézout's identity to solve the problem.

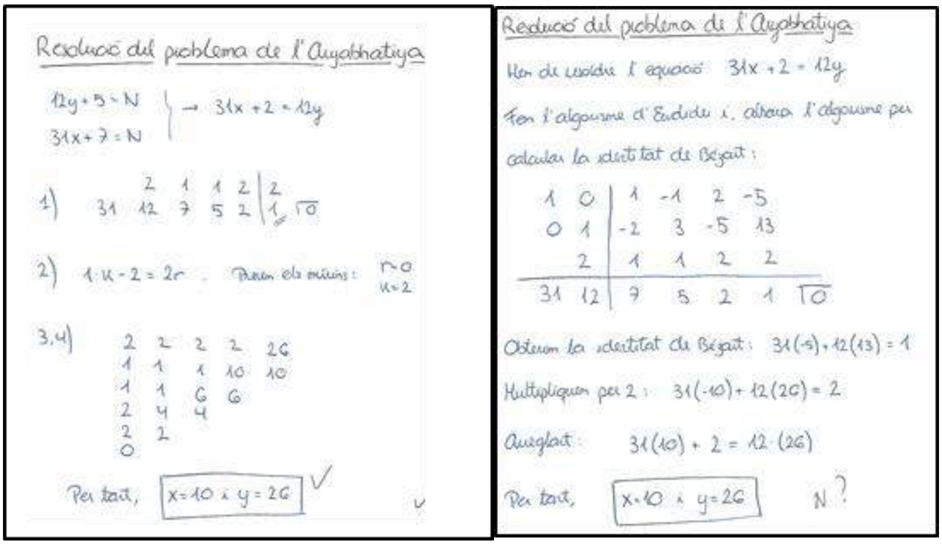


Figure 3.13: Student B's solution of Aryabhata's problem using *Kuttaka* method (left) and Bézout's identity (right)

We did not explain Bézout's identity in this course but some students connected the *Kuttaka* method with Bézout's identity: *Let a and b be integers with greatest common divisor d. Then, there exist integers x and y such that ax + by = d*, and they solve the same problem with both methods, as we can see in the two pages of figure 3.13 done by student B. These students of the Degree in Mathematics are in the fourth year and had studied Bézout's identity in the first year.

4 Questionnaire for Students

After the *Kuttaka* activity we asked the students some questions in order to assess the method but also some questions about their knowledge of math history. Again, we write the question followed by some students' answers. The group had 8 students.

Question1. Did you have any knowledge about mathematics in ancient India?

1.1.- If yes, how did you get acquainted with it?
Explain briefly what you knew about it

Students' answers

Student P: positional notation and decimal base of numbers.
Student Q: research project in High School about Indian Math (only early civilizations). She remembered that the actual system of numbers came from Indian numbers.

Question 2. Before studying the *Kuttaka* method, did you know any other way to solve this kind of equations (linear with two unknowns)?

2.1. If yes, where did you learn this other method? Which one?
2.2. Evaluate the advantages and disadvantages of both methods.

Students' answers

All of them knew another method for solving this kind of equation. They learned it in the subject ofFoundations of Mathematics (1st year).

Only two of them (students A and B) knew the name: Bézout Identity and only one of them (student B) said the full name: Étienne Bézout.

In general they preferred Bézout's method.

Students A & B like Bézout Identity to solve these equations because they think it is a proof but they thought "both methods" were similar.

Question 3.The *Kuttaka* method is a calculation procedure to solve indeterminate equations. Can you cite applications or contexts (current or historical) in which indeterminate equations are used?

Students' answers

Five of them relate this procedure to astronomy, the context used in the classroom to introduce *Kuttaka* method.

The student who knew Étienne Bézout (student B) said that linear equations with two unknowns are also used to calculate the intersection of varieties.

Student C referred to Fermat's theorem and Pythagorean Triples.

Question 4. Throughout the Degree in Mathematics or in high school, have you studied the solutions of a linear equation with two variables or unknowns?

Students' answers

All students said they had studied how to solve this equation with integer solutions in Foundations of Mathematics (1st year).

One of them studied Diophantus's method in high school and did a research project in high school about Indian math.

Question 5.Is there any subject in the Degree in Mathematics in which the history of the involved conceptswas introduced?

Students' answers

Five students said: **no**

Three students said:

Student A: **yes** a little, sometimes teacher introduced the historical context and the biography of the mathematician who discovered a theorem.

Student B: **yes** in many subjects, the biography of the mathematician who discovered a theorem, but never in context.

Student C: **yes** but it depended on the teacher of the subject.

Question 6. Do you remember if any historical development of any mathematical concept was introduced in high school?

Students' answers

Six students said: **no**. One of them said: I would have liked to know something.

Two students said: **no**, only about Pythagoras and his theorem. One of them said: the goal of high school is for students to understand the concepts in order to pass the university entrance exam.

Question 7. Give your opinion about if History of Mathematics can help to understand better the topics studied in the Degree.

Students' answers

All students agree on the importance of knowing the History of Mathematics. In fact, they have chosen this elective course.

There are two points of view:

a) In order to understand modern concept sbetter.

b) Although it is not useful for a better understanding of modern mathematics, it could be interesting for understanding ancient mathematics.

Four students take position a) and four in b).

Question 8.For which mathematical topics would you recommend introducing the history of mathematics?

Students' answers

Some students associate history with specific topics: geometry, root of a polynomial, limit, calculus, and so on.

Others said the history of math must be taught at the beginning, in order to understand why math developed as well as in high school or in the first year of the degree.

Yet others said, in general in all the subjects and in both ways:

a) Related to concepts, to understand why other civilizations use other methods.

b) Related to problems, to compare motivations to solve problems and to have a broader view of the problem.

5 Final Remarks

We have been discussing throughout the paper various questions related to each topic exposed. As final remarks we can say that the activities based on the analysis of historical texts connected to the curriculum contribute to improving the students' integral training, giving them additional knowledge of the social and scientific context of the periods involved. They also get students to achieve a vision of mathematics, not as a final product but as a science that has been developed on the basis of trying to answer the questions that mankind has been asking throughout time about the world around us.

NOTE:

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REFERENCES

Guevara, I., & Puig, C.(2017). *Brahmagupta. El àlgebra de las estrelles*. Barcelona: RBA.

Joseph, G. G., (1996). *La cresta del pavo real. Las matemáticas y susraíces no europees*. Madrid: Editorial Pirámide.

Keller, A. (2005). Making diagram sspeak, in BhaskaraI'scommentary on the *Aryabhatiya*. *Historia Mathematica 32*, 275-302.

Massa-Esteve, M. R., Guevara, I., Romero, F., & Puig-Pla, C. (2011). Understanding Mathematics using original sources. Criteria and conditions. In E. Barbin, M. Kronfeller, & C. Tzanakis (Eds.) *History and Epistemology in Mathematics Education. Proceeding of the 6th European Summer University* (pp.415-427). Wien: Holzhausen Verlag.

Moreno, R. (2011). *Aryabhata, Brahmagupta y Bhaskara. Tres matemáticos de la India*. Madrid: Nivola.

Nolla, R. (2006). *Estudis i activitats sobre problemes clau de la història de la matemàtica*. Barcelona: Publicacions de la SCM, IEC.

Plofker, K., (2007). Mathematics in India. In V. Katz (ed.) *TheMathematics of Egypt, Mesopotamia, China, India, and Islam: a sourcebook* (pp. 385- 514). Princeton and Oxford: Princeton University Press.

Plofker, K., (2009). *Mathematics in India*. Princeton: Princeton and Oxford University Press.

Puig-Pla, C., Guevara, I., Romero, F., & Massa, M. R. (2011). La trigonometria a la matemàtica de l'Antiga Índia. Algunes idees per treballar a l'aula. In P. Grapí, & M. R. MASSA (Eds.). *Actes de la VI Jornada sobre la Història de la Ciència i l'Ensenyament "Antoni Quintana Marí"*(pp. 53-60). Barcelona: Societat Catalana d'Història de la Ciència i de la Tècnica – Institut d'Estudis Catalans,

Puig-Pla, C.,& Guevara, I. (2017). Maons, altars i versos: els singulars mètodes de transmissió dels coneixements matemàtics i astronòmics a l'antiga Índia. In P. Grapí, & M. R. Massa (Ed.), *Actes de la XV Jornada sobre la Història de la Ciència i l'Ensenyament* (pp.135-142). Barcelona: SCHCT-IEC. DOI: 10.2436/10.2006.03.17.

Romero, F., Massa, M.R., Guevara, I., & Puig-Pla, C. (2015). Teachertraining in theHistoryofMathematics. In É. Barbin, U. T. Jankvist, & T. H. Kjeldsen (eds.) *History and Epistemology in Mathematics Education. Proceedings of the Seventh European Summer University* (pp. 113-128). Copenhagen: Aarhus University.