AN INTERNATIONAL COMPARATIVE STUDY ON HOW MATHEMATICAL CULTURE IS IMPLEMENTED IN THE TEXTBOOKS

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ABSTRACT

It is claimed in this study that "mathematics in culture" (MiC, a macro-view of the development of mathematical knowledge) and "culture of mathematics" (CoM, a micro-view of the development of mathematical knowledge) are the two main dimensions of mathematical culture. This study has two purposes. First, I attempted to identify the constituents of mathematical culture in terms of the interrelationship between mathematics and culture, particularly the interaction of mathematics with Eastern and Western culture. Second, I selected three sets of high school mathematics textbooks from Taiwan, China, and the United States respectively to investigate how, if any, mathematical culture is implemented. A quantitative analysis revealed that: (1) Taiwanese and Chinese textbooks have many features in common. Both sets of textbooks stress applications of math in daily life and provide several opportunities for exploring, but ignore ethnic features, mathematical dialogue, and the evolution of mathematical concepts. In contrast, the US textbook employs a lot more problems of applications of math in daily life and in nature, with nearly half of the examples and exercises devoted to promoting students' problem-solving abilities to resolve authentic problems. However, as with the other two sets of textbooks, a treatment of ethnic features is absent.

Keywords: International comparison, textbook analysis, mathematical culture

1 Introduction

The United Nations Educational, Scientific and Cultural Organization (UNESCO) proposedthe years of 2003~2012 as the United Nations Literacy Decade (UNLD) (UNESCO, 2003). Its slogan "Literacy as Freedom" aims to increase literacy levels and to empower all people everywhere. The original meaning of literacy is being able to read and write. But in the modern sense, "Literacy is about more than reading and writing – it is about how we communicate in society. It is about social practices and relationships, about knowledge, language and culture." (UNESCO, 2003, p.1).School is not a place for cultivating all kinds of literacy, but it is well-recognized that one of the best approaches of empowering students' literacy, particularly for those that are knowledge-based, is school education. Tarr et al. (2008) indicated that though teachers' teaching practices may be influenced by their beliefs, knowledge, and students' responses, the textbook is still one of key factors determining their instructional decisions. Since mathematical culture is an essential component of mathematical literacy, this study aims to investigate and compare how mathematical culture is implemented in the high school mathematics textbooks of Taiwan, China, and the United States.

2 Constituents of Mathematical Culture

Historically, mathematics education in various countries initially focused on elite education and vocational education, and rarely considered the relationship among mathematics, society, history and philosophy. However, due to the increasing popularity of higher education in the late twentieth century, the advocacy of "mathematics for all" has arisen. Mathematics is no longer viewed just a practical instrument, but also as a discipline for lifelong learning. The cultural facet of mathematics thereafter begins to receive increasing attention as well. However, the scope of mathematics culture is broad, and it is necessary to clarify the content of literacy for mathematical culture.

Based on their review of 164 definitions of culture, Kroeber and Kluckhohn (1952) proposed the following definition of culture:

> Culture consists of patterns, explicit and implicit, of and for behaviour acquired and transmitted by symbols, constituting the distinctive achievement of human groups, including their embodiments in artifacts. (p. 181)

According to Kroeber and Kluckhohn, the essential core of culture consists of traditional ideas and attached values, and culture systems may, on the one hand, be considered as the products of human action, on the other as decisive factors of further action. Following Kroeber and Kluckhohn's conception, this study defines mathematical culture as follows:

> Mathematical culture consists of patterns, explicit and implicit, inductive and deductive, logical and illogical, of and for problem solving behavior acquired and transmitted by symbols, constituting the distinctive achievement of human groups in general, the mathematician in particular.

This definition stresses the inductive and deductive features of mathematical knowledge in the making, and the logical and illogical development of mathematics. While searching for the cultural basis of mathematics, Wilder (1950) reminded us that mathematics is a part of, and is influenced by, the culture in which it is found. In this manner, the culture dominates its elements, and in particular its mathematics. For instance, a Chinese mathematician living about the year 1200 C.E. would mainly focus on computing with numbers and solving equations without paying attention to geometry as the ancient Greeks understood it. In contrast, a Greek mathematician of 200 B.C.E. would focus more on geometrical proofs than on algebra and numerical computation as the Chinese practiced it. This depicts the mathematics in culture. On the other hand, mathematicians in different cultures share some common methodological views and paradigms for working on mathematics that ensure their creations can be recognized by other mathematicians. These common methodological views and paradigms shape a working academic culture, which is the culture of mathematics. This study therefore proposes two major constituents of mathematical culture: mathematics in culture and culture of mathematics.

Mathematics in culture (MiC) is a macro-view description about how mathematical knowledge as a whole has expanded in various cultures, and includes three components:

- *Historical development*: mathematics developed throughout the history to establish its distinguished features.
- *Social needs*: mathematics grew along with the society to meet various demands.
- *Ethnic features*: mathematics evolved over time with the influence of its host culture and gradually established distinct ethnic-characteristics.

On the other hand, the culture of mathematics (CoM) refers to the emergence and construction of mathematical concepts, which is a micro-view description about the dialectical methodology by which a mathematical idea is created and validated through conversation among mathematicians. This methodology includes the following three components:

- *Inductive conjecture*: This component is the very beginning of how a mathematical idea is revealed and created, which can be seen as a thought experiment (Polya, 1954).
- *Deductive validation*: An isolated fact may not be seen as a generalized truth without a logical verification. Deduction is a significant feature of mathematics.
- *Social construction*: It has been generally held that mathematical knowledge not only is a product of self-construction, but also a polished outcome of public dialectic (Ernest, 1998).

Note that a clear distinction between MiC and CoMis impossible and both are intertwined with each other. This suggests the framework for the constituents of mathematical culture shown in Figure 2.1.

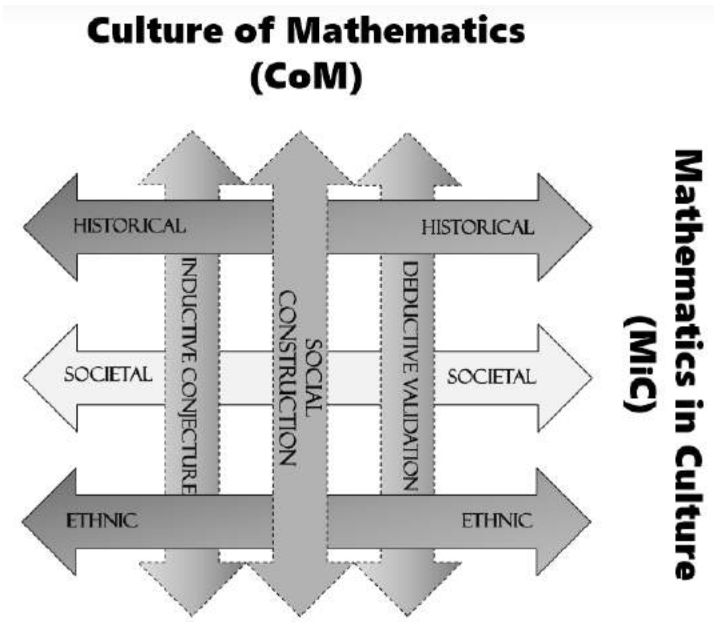


Figure 2.1: The constituents of mathematical culture

Furthermore, based on theoretical consideration, several indices for MiC and CoM were created for coding the components of mathematical culture in the textbooks. There are 13 indices for Mic (Table 2.1) and 8 for CoM (Table 2.2). The validity and appropriateness of each index has been checked and validated by a historian of mathematics and a HPM researcher.

Table 2.1: Indices for MiC

MiC	Index	Definition
History	Concept (H-C)	The origin of concepts
	Method (H-Me)	Different methods in history
	Problem (H-P)	Famous problems in history
	Episode (H-E)	Significant events in history
	Mathematician (H-I)	Distinguished mathematicians in history
	Nature (S-N)	Applications of math in nature
	Living (S-L)	Applications of math in daily life

Society	Economy (S-E)	Applications of math in economy
	Politics (S-P)	Applications of math in politics
	Arts (S-A)	Applications of math in arts
Ethnic	Context (E-C)	Concepts different cultural context
	Difference (E-D)	Approaches in different cultures
	Philosophy (E-P)	Concepts and their philosophical background

Table 2.2: Indices for CoM

CoM	Index	Definition
I & G*	Survey (I & G-O)	Providing opportunities forobserving examples or data
	Pattern (I & G-P)	Leading students to look for patterns
	Conjecture (I & G-C)	Encouraging students to make conjectures
D&P**	Intuition (D & P-I)	Explaining properties by intuitive observation
	Example (D & P-E)	Explaining properties by particular examples
	Logic (D & P-L)	Proving properties by logical deduction
C&D***	Community (C & D-C)	Mathematical dialogue among mathematicians
	Evolution (C & D-E)	Evolution of mathematical concepts

*I & G: Induction and Guessing, **D & P: Deduction and Prove, ***C & D: Community and Dialectic

3 Target Textbooks

Three high school mathematics textbook series produced inTaiwan (T-textbook), China (C-textbook), and the United States (US-textbook) were selected for review. T-textbook and C-textbook are the most popular textbooks in Taiwan and China respectively, and US-textbook is purchased by the National Academy for Educational Research in Taiwan, and is therefore accessible by the researcher. T-textbook has 6 volumes and C-textbook has 5 volumes. US-textbook consists of 2 volumes of algebra and 1 volume of geometry. All examples and exercises in these textbooks were coded.

4 Inter-Rater Reliability

To insure the reliability of the analysis, in the pilot stage, two raters with mathematics and mathematics education background were trained to do the categorization of each index. Their analyses were not regarded as reliable until the agreement rate was above 90%.During the subsequent analysis stage, the third rater (a high school mathematics teacher with a master’s degree in education) and the researcher made final decision if there was any inconsistency between the two raters.

5 Results

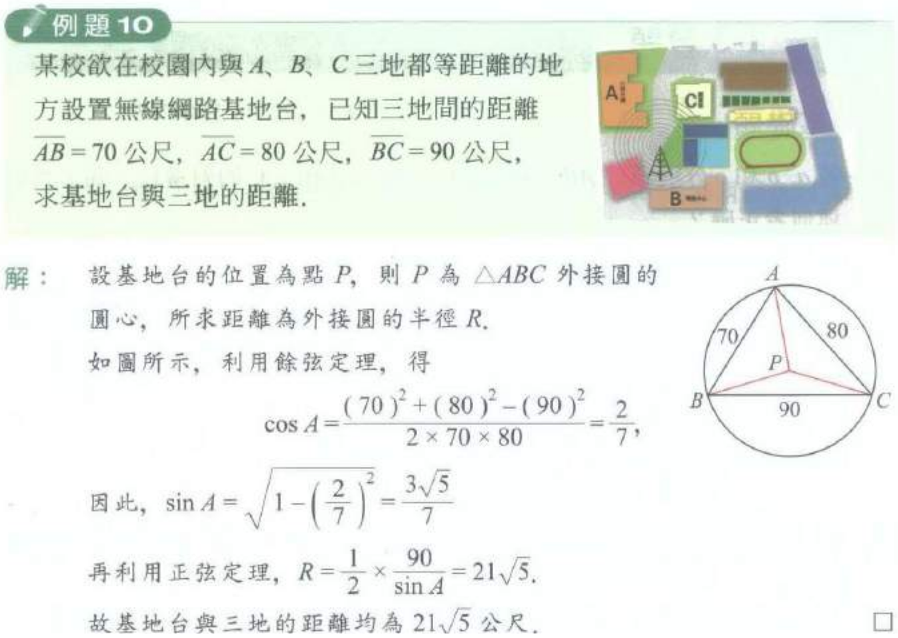
5.1 T-textbook

Table 5.1 shows the frequency and percentage for each index in the Taiwanese T-textbook textbook. It appears that, for MiC dimension, S-L (applications of math in daily life) is the most common index (26.7%) among all. For instance, following the introduction of Law of sine and Law of cosine, students are asked to determine the location of a cellular base station by applying the two laws such that the cellular base station has equal distance from three campus buildings A, B, and C. In another example, lengths of the football (i.e., soccer) field and football net are given, and a player on the point P that is 35 meters away from the bottom edge of the diagram is set to kick the ball. The student is then asked to find the tangent value of the kicking angle APB. Though the two examples can be solved by applying trigonometric identities, they are not realistic because it is impossible to build a cellular base station in the campus and the football player’s main concern is the kicking angle itself, but not the tangent value of that angle. As for CoM dimension, I&G-O (providing opportunities for observing examples and data) is most widely used (23.9%).

Note that the History index (H-C) receives less attention, with only 15.7% of the examples and exercises related to it. For instance, an example related to the application of Apollonius Circle is given to find the running trajectory of two hunting dogs. Furthermore, both the Ethnic features and the C&D index (mathematical dialogue and evolution of mathematical concepts) are totally absent in the text.

Table 5.1: Frequencies and percentages for each index in T-textbook

Index	**S-L**	**I&G-O**	**D&P-E**	**H-I**	**D&P-I**	**I&G-P**	**H-C**	**S-N**
Frequency	56	48	34	17	16	12	8	7
Percentage	26.7%	23.9%	16.2%	8%	7.6%	5.7%	3.8%	3.3%
Index	**I&G-C**	**H-P**	**H-E**	**H-M**	**S-A**			
Frequency	3	3	3	2	1			
Percentage	1.4%	1.4%	1.4%	0.95%	0.47%			


例題 10

某校欲在校園內與 A、B、C 三地都等距離的地方設置無線網路基地台，已知三地間的距離 $\overline{AB}=70$ 公尺，$\overline{AC}=80$ 公尺，$\overline{BC}=90$ 公尺，求基地台與三地的距離.

解：設基地台的位置為點 P，則 P 為 $\triangle ABC$ 外接圓的圓心，所求距離為外接圓的半徑 R.

如圖所示，利用餘弦定理，得

$$\cos A=\frac{(70)^2+(80)^2-(90)^2}{2\times 70\times 80}=\frac{2}{7},$$

因此，$\sin A=\sqrt{1-\left(\frac{2}{7}\right)^2}=\frac{3\sqrt{5}}{7}$

再利用正弦定理，$R=\frac{1}{2}\times\frac{90}{\sin A}=21\sqrt{5}$.

故基地台與三地的距離均為 $21\sqrt{5}$ 公尺. □

Figure 5.1: Determine the location of cellular base station

例題 6

如下圖，有一足球場寬 63 公尺，球門寬 7 公尺，某足球員沿邊界帶球突破，在距底線 35 公尺 P 處起腳射門．設此時 P 對球門所張的角為 θ，求 $\tan\theta$ 的值．

Figure 5.2: Find the tangent of kicking angle

5.2 C-textbook

Table 5.2 indicates the frequency and percentage for each index in the Chinese C-textbook. Similar to the T-textbook, S-L (applications of math in daily life) is the most common index (25.7%) among all MiC indices. Of the CoM indices, I&G-O (providing opportunities for observing examples or data) and D&P-E (explaining properties by particular examples) are two generally adopted strategies. Even though the percentage of S-L in the C-textbook is almost the same as that in the T-textbook, S-L problems in the C-textbook are more realistic in nature. This appears to be due to a greater use of mathematical modeling as a means for increasing students' problem solving ability. For instance, in the beginning of 'The Concept of Functions', an example is given below:

If you plan to do investment and there are three different proposals.
1. Proposal A gets $40 reward every day.
2. Proposal B gets $10 reward on the first day, $20 on the second day, $30 on the third day, and so on.
3. Proposal C gets $0.4 reward on the first day and double reward thereafter.

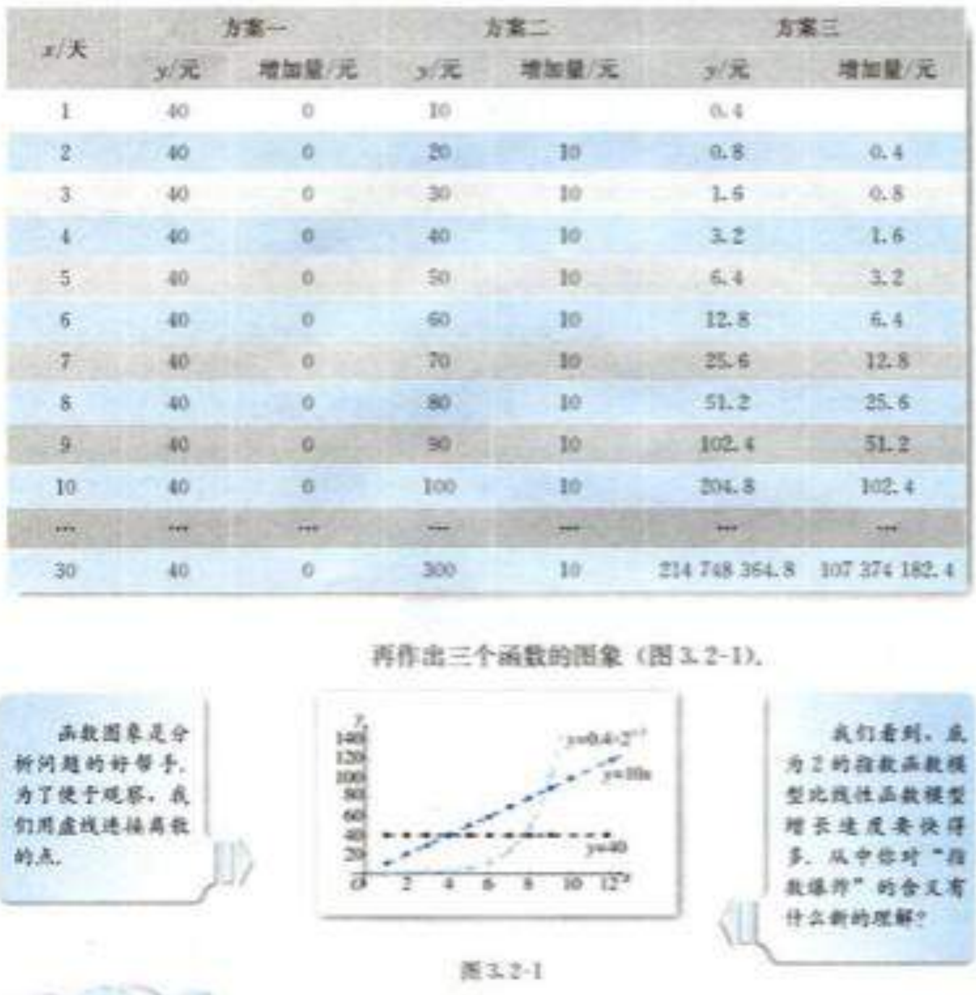
Which one do you prefer?

This example is followed by a table indicating the reward for each proposal in 30 days and a graph showing the tendency for growth of each proposal. Students are then led to an algebraic representation for each proposal, thereby fully demonstrating the rule of four (verbal, numerical, graphical, and algebraic representation) for introducing the concept of functions.

Analogous to the data for the C-textbook, it was found that only 14.3% of the examples and exercises are related to the History index, and that the Ethnic features and the C&D indices are also lacking.

Table 5.2: Frequencies and percentages for each index in P-version

Index	S-L	I&G-O	D&P-E	S-N	D&P-I	H-I	H-C	H-P
Frequency	54	36	34	28	16	10	7	6
Percentage	25.7%	17.1%	16.2%	13.3%	7.6%	4.8%	3.3%	2.9%
Index	**H-M**	**S-E**	**I&G-P**	**I&G-C**	**H-E**	**D&P-L**		
Frequency	5	5	4	2	2	1		
Percentage	2.4%	2.4%	1.9%	0.95%	0.95%	0.47%		



x/天	方案一		方案二		方案三	
	y/元	增加量/元	y/元	增加量/元	y/元	增加量/元
1	40	0	10		0.4	
2	40	0	20	10	0.8	0.4
3	40	0	30	10	1.6	0.8
4	40	0	40	10	3.2	1.6
5	40	0	50	10	6.4	3.2
6	40	0	60	10	12.8	6.4
7	40	0	70	10	25.6	12.8
8	40	0	80	10	51.2	25.6
9	40	0	90	10	102.4	51.2
10	40	0	100	10	204.8	102.4
…	…	…	…	…	…	…
30	40	0	300	10	214 748 364.8	107 374 182.4

再作出三个函数的图象（图 3.2-1）.

图 3.2-1

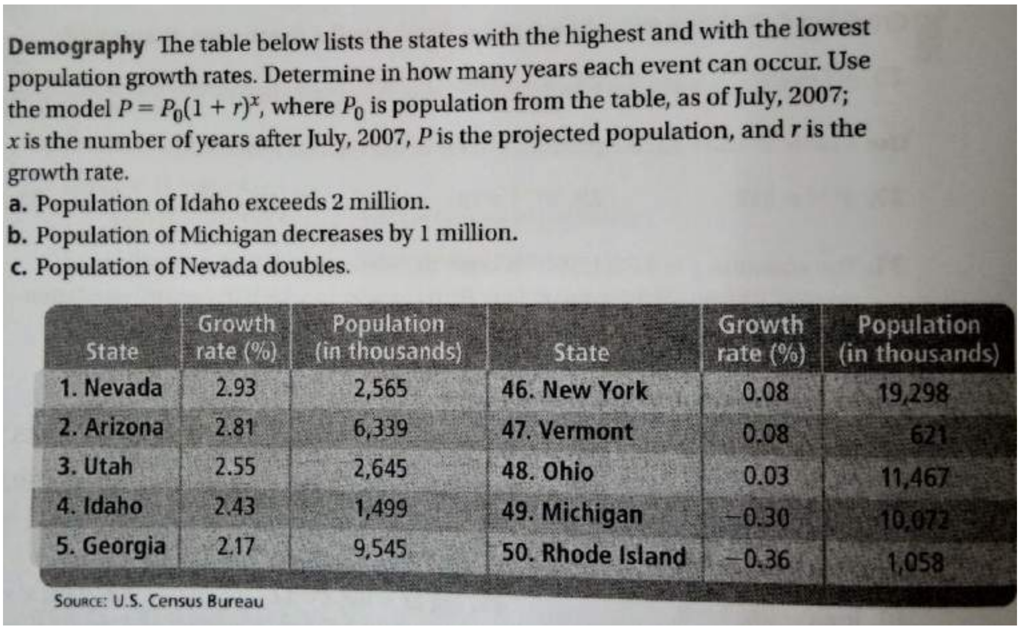
Figure 5.3: Table and graph of the reward for each proposal

5.3 US-textbook

Similar to the cases of the C-textbook and T-textbook, the US-textbook employs many S-L (applications of math in daily life) problems (35.4% as shown in the Table 5.3), intended as a meansto promote students' interest and increase their understanding. This percentage is significantly greater than that of either the C-textbookor the T-textbook. It was further noted that the S-L problems in the US-textbook are more realistic in nature. For instance, in a demography problem (Figure 5.4), the population in the year 2007 and growth rate of several states are given, and students are asked to determine in how many years it will take for each state to reach a specified population.

Table 5.3: Frequencies and percentages for each index in US-textbook

Index	**S-L**	**D&P-I**	**I&G-O**	**S-N**	**D&P-E**	**I&G-P**	**D&P-L**	**H-P**
Frequency	258	143	116	84	68	12	12	12
Percentage	35.4%	19.6%	15.9%	11.5%	9.3%	1.6%	1.6%	1.6%
Index	**I&G-C**	**S-P**	**S-A**	**S-E**	**H-E**	**C&D-E**		
Frequency	6	6	4	3	2	2		
Percentage	0.8%	0.8%	0.55%	0.41%	0.27%	0.27%		


Demography The table below lists the states with the highest and with the lowest population growth rates. Determine in how many years each event can occur. Use the model $P = P_0(1 + r)^x$, where P_0 is population from the table, as of July, 2007; x is the number of years after July, 2007, P is the projected population, and r is the growth rate.

a. Population of Idaho exceeds 2 million.

b. Population of Michigan decreases by 1 million.

c. Population of Nevada doubles.

State	Growth rate (%)	Population (in thousands)	State	Growth rate (%)	Population (in thousands)
1. Nevada	2.93	2,565	46. New York	0.08	19,298
2. Arizona	2.81	6,339	47. Vermont	0.08	621
3. Utah	2.55	2,645	48. Ohio	0.03	11,467
4. Idaho	2.43	1,499	49. Michigan	−0.30	10,072
5. Georgia	2.17	9,545	50. Rhode Island	−0.36	1,058

SOURCE: U.S. Census Bureau

Figure 5.4: The population and growth rate of several states in the US

Furthermore, the US-textbook apparently stresses the role of observation while working on the problems since D&P-I (explaining properties by intuitive survey) and I&G-O (providing opportunities for observing examples and data) percentages are19.6% and 15.9% respectively. In one "looking for a pattern" exercise, the first five rows of Pascal Triangle are given (Figure 5.5), and students are asked (*a*) to predict the numbers in the seventh row and (*b*) to find the sum of the numbers in each of the first five rows and predict the sum of the numbers in the seventh row. Another particular feature of this textbook series is that, as compared to the other two textbooks, the US-textbook tends to emphasize the S-N index (applications of math in nature). A sample problem about Exponential Functions that is related to archaeology is shown below:

Archaeologists use carbon-14, which has a half-life of 5730 years, to determine the age of artifacts in carbon dating. Write the exponential decay function for a 24-mg sample. How much carbon-14 remains after 30millennia?

However, the percentage of History index of the US-textbook is extremely low at only1.87% total.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Figure 5.5: Pascal triangle

6 Conclusions and Discussions

This study aimed to investigate how mathematical culture is implemented in high school mathematics textbooks from Taiwan, China, and the United States. We defined mathematical culture and created a framework consisting of mathematics in culture (MiC) and culture of mathematics (CoM) to serve as a guideline for the analysis. Results show that the T-textbook from Taiwan and the C-textbook from China have many features in common. Both sets of the textbook stress S-L (applications of math in daily life), use I&G-O (providing opportunities for observing examples and data) widely, and totally ignore Ethnic features and the C&D index (mathematical dialogue and evolution of mathematical concepts). Additionally, it was found that the C-textbook is likely to use mathematical modeling as a means for increasing students' problem solving ability. In contrast, S-L problems in the T-textbook are more unrealistic. Compared to the T-textbook and the C-textbook, the US textbook employs more problems of Society index (35.4% of S-L and 11.5% of S-N). Nearly half of the examples and exercises in the US-textbook are devoted to promoting students' problem-solving abilities to resolve authentic problems. However, as in the other two versions, the index Ethnic features is absent. Introducing in what ways an identical mathematical concept was implemented in different cultures may trigger learners' critical thinking and mathematical understanding. Though ethnic features are also related to History index, examples and exercises would not be counted as the Ethnic index if a cultural comparison was not made.

Owing to the ways in which mathematical culture reflects the true nature of mathematics, it should receive consistent attention from teachers and students, and the mathematics textbook is an appropriate agent for achieving the purpose. However, the present study found that certain components of mathematical culture are not employed very widely or profoundly in the high school mathematics textbooks from Taiwan, China, and the United States that were analyzed. The three sets of textbooks overwhelmingly emphasize S-L index, which is expected, but overlook the evolution of mathematical concepts, ethnic features, and the social construction of mathematical knowledge, which are all related to the history of mathematics. In this manner, mathematics is treated more like an instrument than a particular kind of cultural wisdom. As a preliminary stage for entering college, high school students are further supposed to realize that mathematics not only can be applied to resolve daily practical problems, but may be advanced to an abstract level for its own sake. For instance, a historical and inductive approach may connect the Fibonacci sequence with the golden ratio. Further, while introducing the concept of infinite series, the problem of determining the length of coastline can be associated with geometrical fractal. Unfortunately, the three sets of textbooks in the present study also fail to meet that purpose. The role of history of mathematics in teaching has been advocated for decades (Jankvist, 2009, 2011; Liu & Niess, 2006; Liu, 2009; Radford, 1997). Only history can address the ways in which concepts were created and polished through the ages, reveal the distinctive mathematical characteristics of different cultures, and demonstrate the dialectical nature of mathematical knowledge. When studying the textbook, students usually spend more time on doing examples and exercises than reading the text. To show the full essence of mathematical culture, a mathematics textbook should therefore include examples and exercises that refer to history of mathematics as a means to shed more light on the nature of mathematical thinking and knowledge construction.

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