THE TEACHING OF LOGARITHMS IN UPPER SECONDARY SCHOOL FROM A HISTORICAL PERSPECTIVE

Eleni LAPPA[1], Kostas NIKOLANTONAKIS[2]

University of Western Macedonia, Florina School of Education,
Department of Primary Education, P.O. Box 21, Florina, 53 100, Greece
[1]elelappa@gmail.com, [2]knikolantonakis @uowm.gr

ABSTRACT

Secondary school students (16 – 17 years old) face difficulties concerning the creation of a structured mental scheme about the concepts of logarithm and logarithmic function. This paper focuses on a didactic intervention following the Jankvist's modules approach and utilizing the History of Mathematics (HM) as a tool for the conceptual understanding of mathematical notions and more broadly as a goal for the perception of the human evolving nature of Mathematics. The use of HM, through studying historical texts and the awareness of the concept historical evolution, combined with the collaborative teaching method and the Geogebra software. This case study aimed to address the following research questions: a) Does the use of HM as a tool affect students' conceptual understanding of the logarithmic concepts? b) Does the HM also be promoted as a goal for students even merely to approach Mathematics as an evolving cultural product?

Keywords: History of Mathematics, Semiotic Instrumental Discursive dimensions, Geogebra, Collaborative teaching model, logarithm, logarithmic function.

1 Introduction

The abstract nature of mathematical concepts is considered as a central source of difficulties for students in the process of mathematics' conceptual understanding. Mathematical concepts are the main subjects of instructional approaches mainly through their representations, which cannot often be perceived by the senses (Sfard, 1991). What is required in mathematical education is the kind of comprehensive understanding, which focuses on the knowledge not only of 'what I have to do' but also of 'why should I do it' (Skemp, 1976). In every attempt to observe the learning path of a mathematical concept, the separation of mathematical knowledge into procedural and conceptual divisions lies in the forefront. The execution of routine algorithms, as well as the usual manipulations performed on mathematical objects, refers to the procedural knowledge which is necessary for the accomplishment of a mathematical work but is not capable of acquiring a fully integrated cognitive structure relating to the particular mathematical concept. At a higher level, conceptual knowledge is required to fill in the answer to 'why'. Conceptual knowledge is not an isolated part of knowledge but includes the necessary links to create a structured and flexible cognitive scheme (Gray & Tall, 1994). The adequately structured communication between the epistemological and cognitive content of the mathematical concept through the development of semiotic, instrumental and discursive dimensions, is achieved by developing two qualitatively different and at the same time complementary approaches of this concept: the operational and the structural ones (Sfard, 1991). Operational approach perceives the concept as a process, giving a dynamic character and is expressed by performing actions, whereas structural considers the concept as an object and it is the product of self reflection processes.

The concepts of logarithm and logarithmic function are basic parts of the curriculum content of the educational system. Logarithm is a notion with a long evolutionary course. Being at first an auxiliary tool for the calculation of difficult and time-consuming operations, has become, in its functional form, an instrument for explaining various kinds of phenomena applicable in many and varied scientific fields. However, students usually face difficulties in constructing a concept image that incorporates the exponential process into the mathematical object - the numbers called logarithms (Thomaidis, 1986; Vagliardo, 2006; Panagiotou, 2014).

In parallel, students are also called to deal with the concept of the logarithmic function. Function is considered to be a fundamental concept of mathematics while at the same time the effort of its conceptual approach faces a variety of challenges (Sfard, 1991; Sierpinska, 1992; Hitt, 1998; Artigue, 1999; Gagatsis & al., 2006). Difficulties are located in points such as the idea of functional dependence as a co-variation, the understanding of the role of the independent and dependent variables, the manipulation of functional symbolism, the use of different semiotic registers and various representations, the transition from algebraic to functional thinking, as well as the dual nature of the concept as a process and as an object (Artigue, 1992; Gagatsis & al., 2006; Kieran, 2007; Lagrange & Psycharis, 2014; Minh & Lagrange, 2016). Therefore, the teaching of logarithmic function faces two major obstacles: the first concerns the difficulty of understanding the logarithmic concept and the second is the one that emerges from the concept of function itself.

The reasons justifying the utilization of History of Mathematics (HM) in instructional approaches are varied and widely discussed (Fauvel, 1991; Furinghetti, 1997; Tzanakis & al., 2002; Fried, 2008; Jankvist, 2009; Tzanakis & Thomaidis, 2012). Characteristically, it could be said that mathematical discipline is not exhausted in its final results, as its development path involves years of efforts, doubts and revisions (Tzanakis & Thomaidis, 2000). The knowledge of the evolution trajectory of a mathematical concept can contribute positively to overcoming epistemological and didactic obstacles so as to achieve conceptual understanding. Teaching the concept in its final refined form, according to the current Curriculum, is an example of what Freudental calls ‘anti – didactical inversion’ (Barbin, 2015).

2 Methodology

This intervention was designed in the context of Jankvist’s modules approach, using a PowerPoint file, an introductory worksheet and 3 main worksheets. It was applied to a 16-student class (17 years old) in the upper secondary school. The role of the HM in this intervention is twofold. On the one hand, from a macroscopic point of view, through the negotiation of the historical evolution of logarithms, history was used as a goal to highlight as far as possible the dynamic and evolving nature of the content of mathematical discipline, its links with the social and cultural environments in which it develops, as well as its connection to other fields of science. On the other hand, history was used as a tool to achieve a more wholistic approach to mathematical concepts, focusing on key components of the historical evolution that are necessary in constructing an integrated image of these concepts. The most important stages in the historical evolution of the logarithms were presented emphasizing the necessity for estimating difficult numerical operations in the historical context of 17th century, as though the reasons that led to the invention of the concept, the importance of logarithms in the modern reality, as well as the individuals who determined with their contribution the development of this historical route. The method of Prosthaphaeresis in the arithmetic-geometric progressions' correspondence was also studied in the Napier kinematic model. The logarithm was defined as the 'number of ratios' in a continuous ratio within the bounds of a historical context in which the notion of power had not yet been clarified. The way that Napier constructed his logarithmic tables and the potential of estimating the logarithm of any positive number by the hyperbolic areas were also discussed. The historical review was completed by presenting and discussing logarithmic applications in various fields. The students then worked on translated historical texts by Leonard Euler from his book "Introductio in analysin infinitorum" (1748) which contained the definition of the logarithm and the logarithmic rules as well as the definition of the function as analytical expression with reference to the reverse function.

A combination of various teaching tools was used in this instructional approach for the creation of a structural mental scheme by the students concerning the specific concepts. The collaborative teaching method favored the exchange of views, the development of arguments and the active participation of students. At the same time, the digital environment of the educational software Geogebra, in combination with the historical texts provided a proper environment for experimentation and exploration, driving the development of the instrumental dimension of work for conceiving the logarithmic function as a co-variation reverses to the exponential. Through the observation of the development of the groups' work along the axes of semiotic, instrumental and discursive dimensions and the analysis of students' individual answers in two questionnaires completed after the end of the intervention, the following questions were addressed: a) Does the use of HM as a tool affect students' conceptual understanding of the logarithmic concepts? b) Does the HM could be also promoted as a goal for students to even merely approach mathematics as an evolving cultural product?

2.1 Worksheets

2.1.1. Introductory, first and second worksheets - logarithms

The introductory worksheet contained difficult and time-consuming numerical calculations (multiplications, divisions, cubic root extractions) which students were asked to estimate. After their initial failed attempts and in the course of the intervention they managed to make the same calculations using logarithms.

The first worksheet contained Leonard Euler's text (1748), concerning the symbolism of logarithms and the definition as a power exponent and as a function value, approaching the logarithmic concept as a process and at the same time as a result:

> **"***Just as, given a number a, for any value of z, we can find the value of y [α^z] so in turn, given a positive value for y, we would like to give a value for z, such that α^z = y. This value of z, insofar as it is viewed as a function of y, it is called the LOGARITHM of y"* (Down & al., 2004).

The knowledge of the text was intended to enrich the theoretical frame of reference in the workspace of the groups. The following tasks in this worksheet aimed at the acquirement of a comprehensive understanding of the definition and the symbolism through the clarification of the exponential and logarithmic relation of the variables involved and the application of the definition in order to deal with relative activities. On the other hand, the first worksheet aimed at the investigation of students' knowledge about basic components concerning the concept of function such as the set of values and the domain of the function. The responses of the groups were used to design the third worksheet, which focused on the concept of the logarithmic function in order to examine in detail the nature of function as a co-variation, the notion of reversibility and the alternating roles of dependent - independent variables. The groups were asked to apply the exponential process, included in the definition, in order to activate the operational nature of the logarithmic concept. Through these actions, it was expected that students would make a first step towards the internalization of the process and the identification of the loga symbol as an autonomous mathematical object.

On the same wavelength as the first worksheet, the second one contains Euler's text, focusing on the logarithmic rules which proof is indirectly given to the text as a result of the definition:

"In like manner if log y = z, then log y^2 = 2z, log y^3 = 3z, etc., and in general log y^n = nz or log y^n = n log y, since z = log y. If follows that the logarithm of any power of y is equal to the product of the exponent and the logarithm of y… If we already know the logarithms of two numbers, for example log y = z and log v = x, since y = α^z and v = α^x, it follows that log vy = x + z = log v + log y. Hence, the logarithm of the product of two numbers is equal to the sum of the logarithms of the factors" (Down & al., 2004).

The usefulness of rules for estimating logs of many numbers using logarithms of few ones is also emphasized without providing clear reference on the ease of numerical operations that is achieved through the rules. Students were expected to delve deeper into the internalization of the logarithm concept by using the definition in constructing proof of logarithmic rules and developing the required justification on answering the questions were posed.

1.1.2. Third worksheet – logarithmic function

The third worksheet contained two phases. In the first one, a historical text including the Euler's definition of the function as analytical expression constituted the core of the work. There was also reference to the uniqueness of the dependent variable for each value of the independent one and the existence of the inverse function, switching in fact the roles of dependent and independent variables.

"*a function of a variable quantity ...[as] an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities ... a single-valued function is one for which, no matter what value is assigned to the variable z , a single value of the function is determined ... If y is any kind of function of z, then likewise z will be a function of y"* (Down & al., 2004) .

The tasks following the text were structured so as to lead the focus of the students' work on the concepts of co-variation, dependent and independent variables and inverse function. Through a real problem in the seismology field, students were also expected to deal with the notions mentioned above, focusing on the exponential and logarithmic functions.

The second phase of the worksheet was characterized by the involvement of the Geogebra mathematical software for studying the logarithmic function. Students had to swift independent with dependent variables finding out the mathematical formulas of inverse functions, to get to the proper formula from the graph of the logarithmic function, to create graphical and algebraic resolutions of the logarithmic equations and to justify the symmetry of the inverse functions' graphs in terms of the roles of variables. The questions aimed at the construction of a transparent mathematical meaning by the students concerning the concept of the logarithmic function and its relationship with the reverse exponential one.

2.2 Questionnaires

Two questionnaires were distributed to the students without advance warning, one week after the end of the intervention, that means about one and a half month since the beginning of the course. The aim was to examine at what extend students could use their knowledge, without preparation and repetition, to solve logarithmic equations and inequalities and to gain an understanding on the logarithmic function with proper application of logarithmic rules. At each step justification was required.

The first questionnaire was designed to enable students to express their personal views about: 1) the use of history in the didactic intervention, 2) the initial necessity leaded to the invention of logarithms as well as their use in modern reality and 3) the dynamically evolving character of mathematical concepts. The second questionnaire focused on finding out current students' knowledge of the logarithm and logarithmic function as well as its applicability to resolve related activities as already been mentioned.

3 Analysis of results

3.1 First and second worksheets - logarithms

Students took advantage of the texts' dynamics, through the manipulations they used. More specifically, the symbols were decoded and the relative information was constructed, the exponential process was used in experimentation and researching frameworks to resolve the activities, and at the same time, based on the contents of the texts, formal and non-formal arguments were developed during the justification of the manipulation of the symbols and of the procedures proposed by the students in the groups. Characteristically, concerning the question about the definition of logarithm as a functional expression, groups used initially the exact words of the text to answer the question and indeed the sentence in which the word logarithm had been written in capital letters, without mentioning the exponential relation that links the variables, which means that at this initial stage they were unable to clarify the definition. As it may be seen from the discussions in the groups, students probably carried away by a norm that 'requires' the formulation of a short and formal definition. As students continued working on the remainder activities the inadequacy of the initial, automated approach to the definition was highlighted. Groups were forced to revert to the definition of the text and to examine relations of variables and constants as well as the range of acceptable values that variables could get. They used specific numerical values and processed algebraic notation using generalized variables, even by changing labels of the variables of the text, thus the semiotic dimension of the work was developed. Groups produced a verbal expression of the definition:

- If the number rises to an exponent, then, the exponent is the logarithm of the outcome.

The typical exponential procedure and its rules, well-known to the students, were used as artifacts for constructing the definition in the developing of the instrumental dimension of work. A representative dialogue in a group concerning the phrase of the text *"Just as, given a number α, for any value of z, we can find the value of y [=* a^z*], so, in turn, given a positive value for y, we would like to give a value for z, such that* $\alpha^z = y$*"* follows:

- If I know z and also the α, then I can find y. If I know y then how can I find z? What does it mean?
- Then say that z is the logarithm of y. Is it a result of estimation or a correspondence?
- The logarithm counts the ratios. It is a number. Is it both?

Although the last question was not answered by the members of the group at this stage, the previous dialogue demonstrates a creative progress of the work towards both the view of logarithm as an object and the dual nature of the function as a process and result. The etymology of the word 'logarithm', which was a key element in the presentation of the logarithm historical evolution, seems to have contributed positively to the attempt of constructing, according to Tall & Vinner (1981), the appropriate 'concept image'. In parallel, the correspondence of arithmetic and geometric progressions was used by the groups to justify and verify the solution of logarithmic equations such as $\log_2 4 = x$, which initially were solved by using Euler's definition.

However, although the definition had already been clarified and was applied successfully, students were struggling to accept $\log_5 112$ as a number that was a solution to a given equation ($5^x = 112$). Similarly, the question of "what is equal to $\alpha^{\log_a u}$?" proved to be of high difficulty for all groups. This is a fact which demonstrates the kind of obstacles students need to overcome when dealing with the logarithmic concept. The groups were experimented by applying specific 'convenient' numerical values to the variables and by using the Euler's definition both as a tool and as a component of the reference framework for validating the results. In this way the discursive dimension of the work was developed. It is noteworthy however, the formation of reflective abstraction in students' learning as it characteristically came from a student's point of view about the outcome of $\alpha^{\log_a u}$:

- The outcome is u because the logarithm is an exponent ... That is, here is the power of base α which is raised to the exponent that equals u ... so the outcome is u.

The evolving development of the work from the first to the second worksheet indicated a dynamic course of internalization of the definition and integration of the exponential process into the object - logarithm. In the second worksheet students analyzed the equalities of the text and led to the construction of proof of the logarithmic rules by flexible manipulations of the definition:

- Since $\log y = z$, α is raised to z, so $\alpha^z = y$ ... $y^2 = (\alpha^z)^2$, $y^2 = a^{2z}$ ... $2z$ is the exponent to the base α and thus the log of y^2. That is, $\log y^2$ equals to $2\log y$.

The semiotic dimension of the work developed without any problems in the manipulation of the symbols and the decoding of the relations of the text. The use of the definition activated the instrumental dimension of the work and the validation of the construction of proof contributed to the development of the discursive dimension. Characteristically, a student stated:

- If I know the logarithm of two numbers then I know the logarithm of their quotient ... because if I know that the logarithm of y is equal to z then α^z is y and in the same way α^x

is v. If I divide them, the result is α^{z-x} ... that is, z-x is the exponent of α, hence the logarithm of the quotient.

It should be noted that groups distinguished the simplification of estimations in Euler's rules due to the reduction of multiplication to addition and division to subtraction, which was the basic reason leaded to the invention of logarithms and a significant element in the presentation of the historical evolution of logarithmic concepts. Hence, the correspondence of arithmetic and geometric progressions was brought to the forefront of the groups' work one more time. This finding was used by students for justifying the usefulness of logarithmic rules.

3.2 Third worksheet – logarithmic function

In the third worksheet, except of the Euler's text with the definition of function as analytical expression and a reference to the concept of the inverse function, a function of exponential formula ($I = I_0 \cdot 10^R$, $I_0 > 0$) involving the earthquake magnitude (R) and intensity (I), in the context of a real problem in the field of seismology was also included. Groups were asked to decide if this expression was a function according to Euler's definition and to define the dependent and independent variables. They created functional expressions that were either the result of a specific application of their knowledge in context of a discipline like physics, e.g. the displacement in a uniform linear motion, or were given in the typical form of a functional symbolism, e.g. $y = z + 1$. They also focused on the uniqueness of the y value (value of function) obtained for each z value. Groups used elements of the history of logarithms, trying to relate the meaning of function with the Euler's and Napier's logarithmic approximations. They found out that there is a co-variation of variables in both approaches. Groups analyzed the functional symbolism of the text and used it in their experimental efforts. The discussion focused on the role of independent and dependent variables and constants in the expressions that students presented. They had no problem to recognize the relationship between magnitude and intensity of earthquakes as a functional expression and to identify dependent and independent variables. They stated:

- Changing values of magnitude results in change of corresponding values of intensity ... the intensity is depended on the magnitude that changes. The intensity is y and the magnitude is z.

They compared and contrasted the Euler's definition with the one which they had already been taught regarding function as correspondence of values. Specifically, they said:

- Each numerical value of x is assigned to only one numerical value of y. In the text x is z. Saying that for any value of z is defined a single value for the function, it means that every x is assigned to a single y that is the value of the function.

Afterwards the preceding process, the members of the groups concluded that the given relation of earthquake magnitude and intensity was consistent with Euler's definition of function:

- It is an expression formed by the variable R and constants ... is the way in which the intensity changes when the magnitude changes.

The given relation was also examined as co-variation of variables, in which R is arithmetic and I geometric progression. It was found that the arithmetic change in values of magnitude results in a geometric change of intensity. The result was the expected appearance of logarithms while shifting variables as it was requested in the worksheet:

- *R* is arithmetically altered and *I* geometrically … *R* is the logarithm of intensity.

Groups agreed that shifting the variables leads to a new expression that retains the properties of function definition mentioned in the text:

- The switch of the variables leads to the inverse function.

At the same time they criticized the phrase of the text: *"if y is any function of z, then z will be a function of y"*. They argued that this may not be the case and justified their claim using contradictory examples like $y = z^2$.

Subsequently, the work was transferred to the digital environment of the Geogebra mathematical software. Students constructed graphical representations by matching pairs of independent and dependent variables to points using the Cartesian axes. Conversely, they managed to match the graph appeared on the screen as an illustration of exponential function points, where the coordinates were reversed, to the appropriate logarithmic formula despite the initial difficulties faced at this point. Groups had to return to the text and the definition:

- The expression solved with respect to the dependent variable is the formula of function ... now the dependent is magnitude, *R*.

At each step of the work students combined the elements of the historical text concerning the concept of function and the reverse function, with the supervision of multiple representations of function (formula, graph, value table) in the digital environment of Geogebra in order to accomplish graphical and algebraic resolution of the worksheet's activities:

- In the spreadsheet (value table) there is just a single value of the dependent variable for each value of the independent one, either when the dependent is *I* or when is *R*. The same is apparent in graphs too, as each number on the horizontal axis (horizontal coordinate) corresponds to a single point of the graph.

Groups concluded that logarithm is a monotonic increasing function when base α is greater than1 and a decreasing one when $\alpha < 1$ both in a graphical way, using the software, and in an algebraic way using the definition and suitable numerical values. They also studied the domain and the set of values of the logarithmic function as a reverse co-variation of the exponential and explained the symmetry of the graphs about the line $y = x$:

- The horizontal coordinates of the points on the logarithmic graph are the vertical ones of the points of exponential function. This is because reversion of functions means reversion of the variables.

The development of the discursive genesis of the work, as a result of attempting to produce adequate logical explanation, was either based to the exponential process and the operation of its rules

- If $c < k$ then $\log c < \log k$ because if $\log c = x$ and $\log k = y$, since $c < k$, it is $\alpha^x < \alpha^y$ so $x < y$,

or to direct references to logarithm as a number - exponent:

- If $b > c$ then $\log b > \log c$ because greater power means greater exponent when there is the same base higher than one.

Groups analyzed relationships and dependencies between elements of graphs, assigned the values of dependent and independent variables to point coordinates and used the elements of the enriched referential framework, i.e. the definition of the logarithm and the logarithmic function set of values, in developing lines of argumentation to support their conjectures. The use of the software promoted the development of the instrumental and discursive dimensions of work.

4 Conclusion

The knowledge of historical elements concerning the logarithmic concept prompted the use of the definition as a tool for the working on the activities as well as for the decoding of the symbolism, developing the instrumental and semiotic dimensions of the work and helping, in that way, the integration of the exponential process into the subject - logarithm. The correspondence of arithmetic and geometric progressions and the etymology of the word ‘logarithm’ were complementarily used along with the latest Euler’s definition when attempting to deal with the activities required in the worksheets. Although this process was not followed by all groups and in every phase of the work, the knowledge of the historical evolution of the concept, since the initial role of logarithm as a tool and the necessity leaded to its invention until the latest definition and its functional form, provided a more wholistic approach to mathematical concepts, which favored the development of a conceptual understanding. According to students' responses to the questionnaires, the familiarization with the historical evolution activated their interest and triggered their motives, as through the presentation of historical data, frequently asked questions by the students about the necessity of existence of the mathematical concepts and their usefulness in modern reality were answered.

The second pole of HM utilization in this didactic approach concerned the study of historical texts. It is true that students’ responses to the questionnaires have shown that they faced difficulties processing the texts. This is a fact that could be expected as, through the texts, they dealt with concepts at their source, studying in a different ‘language’, more authentic and less ‘refined’ than these they are used to in school textbooks. They developed a kind of dialectical relationship with texts. As it appeared in the transcripts, the ‘rhetorical’ formulation of mathematical relations in a significant part of the texts was a challenge for students, depriving the "facilitation" of an automatic and purely procedural perception of concepts. As shown in the course of the work during the intervention and in the transcripts, the collaborative teaching method functioned supportively to overcome the difficulties that arose. The cognitive abilities of the text were exploited by the groups, not only through study and analysis, but even through criticism of the provided information such as the prerequisites for the inverse function. Therefore, historical texts functioned as cognitive tools in the course of students' conceptual understanding of logarithm and logarithmic function as they were used as levers of work development along the axes of the semiotic, instrumental and discursive dimensions. In parallel with using historical texts, the digital software functioned in an auxiliary way for studying the logarithmic function and its properties, thus its contribution to this instructional approach was considered positive by the students. The transcribed conversations in the groups and the successful respond to the activities of the questionnaires have demonstrated a satisfactory degree of integration of the exponential process into the mathematical object - logarithm as well as the approximation of the logarithmic function as inverse co variation.

At the same time, students seemed to realize that mathematical discipline is an evolving product that is linked to the general social context in which it develops. Therefore, from a macroscopic point of view it can be said that HM contributed, in this case, in students' approaching to mathematics not as a mockery of definitions, theorems and exercises of unjustifiable existence, but as a human cultural product. However, it is natural that the use of HM specifically in a particular subject, in this case the logarithm, is not sufficient for an overall and conscious review of students' opinion about mathematics discipline. This was partly reflected in some of the students' responses who, on the one hand, agreed about the evolving nature of Mathematics but, on the other hand, they still doubted whether the didactic intervention influenced significantly their views. Despite the fact that this research is a case study which may have limitations in generalization, we could say that the utilization of HM facilitates the teaching of logarithms in secondary education and also provides an alternative way to approach mathematics from a different perspective.

REFERENCES

Artigue, M. (1992). Functions from an algebraic and graphic point of view: cognitive difficulties and teaching practices. *The concept of function: Aspects of epistemology and pedagogy*, *25*, 109–132.

Artigue, M. (1999). The teaching and learning of mathematics at the university level. *Notices of the AMS*, *46*(11), 1377–1385.

Barbin, É. (2015). Philosophies and theories behind history and education: thirty years after Hans Freudenthal. *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU*, p. 21–40, Barbin, É., Jankvist, U., & Kjeldsen, T. H. (Eds.), Copenhagen: Danish School of Education, Aarhus University.

Dawn Leigh Anderson, D., L., Berg, B., Ann Willis Sebrell, A., W., Smith, D., W., (2004). Exponentials and Logarithms, *Historical Modules Project***,** Mathematical Association of America.

Fauvel, J. (1991). Using history in mathematics education. *For the learning of mathematics*, *11*(2), 3–6.

Fried, M. N. (2008). History of Mathematics in Mathematics Education: a Saussurean Perspective. *The Mathematics Enthusiast*, *5*(2), 669–703.

Furinghetti, F. (1997). History of mathematics, mathematics education, school practice: Case studies in linking different domains. *For the learning of mathematics*, *17*(1), 55–61.

Gagatsis, A., Elia, I., & Mousoulides, N. (2006). Are registers of representations and problem solving processes on functions compartmentalized in students' thinking? *Revista Latinoamericana de Investigación en Matemática Educativa*, *9*(1), 197–224.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for research in Mathematics Education*, *25*(2), 116–140.

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *Journal of Mathematical Behavior*, *17*(1), 123–134.

Jankvist, U. T. (2009a). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational studies in Mathematics*, *71*(3), 235–261.

Jankvist, U. T. (2009b). History of modern applied mathematics in mathematics education. *For the Learning of Mathematics*, *29*(1), 8–13.

Kieran, C. (2007). Learning and Teaching Algebra at the Middle School through College Levels: Building Meaning for Symbols and their Manipulation. In F. K. Lester (Ed.). *Second Handbook of Research on Mathematics Teaching,* (pp. 707–762). Charlotte NC: Information Age Publishing Inc.

Lagrange, J. B., & Psycharis, G. (2014). Investigating the potential of computer environments for the teaching and learning of functions: A double analysis from two research traditions. *Technology, Knowledge and Learning*, *19*(3), 255–286.

Minh, T. K., & Lagrange, J. B. (2016). Connected functional working spaces: a framework for the teaching and learning of functions at upper secondary level. *ZDM*, *48*(6), 793–807.

Panagiotou, E. (2014). History of Logarithms and its Pedagogical Dimensions. *Epistimes Agogis*, 92–113, (in Greek).

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1–36.

Sierpinska, A. (1992). On understanding the notion of function. In E. Dubinsky & G. Harel (Eds.), *The Concept of Function: Elements of Pedagogy and Epistemology*, (pp. 23–58). Washington, DC: Notes and Reports Series of the Mathematical Association of America.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics teaching*, *77*(1), 20–26.

Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. *Educational Studies in Mathematics, 12*(2), 151–169.

Thomaidis, Y. (1986). Origin and Applications of Mathematics Teaching Theory - The Example of Logarithms. *Euclid C*, *13*, 1–30, (in Greek).

Tzanakis, C., & Thomaidis, Y. (2000). Integrating the close historial development of mathematics and physics in mathematics education: Some methodological and epistemological remarks. *For the Learning of Mathematics*, *20*(1), 44–55.

Tzanakis, C., & Thomaidis, Y. (2012). Classifying the arguments and methodological schemes for integrating history in mathematics education. In B. Sriraman (Ed.). *Crossroads in the history of mathematics and mathematics education*, (pp. 247–295). Charlotte NC: Information Age Publishing Inc.

Tzanakis, C., Arcavi, A., de Sa, C. C., Isoda, M., Lit, C. K., Niss, M., de Carvalho J. P., Rodriguez M., & Siu, M. K. (2002). Integrating history of mathematics in the classroom: an analytic survey. In J. Fauvel, & J. van Maanen (Eds.). *History in mathematics education: the ICMI study*, (pp. 201–240). Dordrecht: Springer.

Vagliardo, J. (2006). Substantive knowledge and mindful use of logarithms: A conceptual analysis for mathematics educators. *Focus on Learning Problems in Mathematics*, *28*(3), 90+.