USING HISTORY TO TEACH COMPLEX NUMBERS

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ABSTRACT

As usual we used textbook approach to introduce the imaginary unit *i*, whose square equals to -1. When class ended, a student asked if negative numbers could have logarithms. Does log(-1) exist? His question reminded us to reconsider how to make more sense of the imaginary unit *i*. There must be some more natural and intuitive alternative to define it. The historical development of mathematical concepts and the way that they develop in an individual mind are observed to be very much alike (Jankvist 2009). With an exploration of a history-based classroom practices and students' performances in non-routine problems solving, our research aims to examine when and how, and in what context to introduce the imaginary unit and what geometric intuition can serve to enhance students' understanding.

The case study took place with a class of 11th grades students (16-17 year old) in our high school. The theoretical framework used to study the teachers' classroom practices is the double approach (Vandebrouck 2012). We articulate both the analysis of students' and teachers' activities in order to identify, understand and interpret the link between the teaching of complex numbers using original sources and the corresponding students' activities. In the highlight of theoretical framework, we attempt to answer the following two research questions:

1. Can students' own solutions to a cubic equation together with Cardan's attempt, Leibniz's doubt and Bombelli's discovery pave the way for the introduction of the imaginary unit *i*?

2. Can geometrizing the imaginary unit by Wessell, Argand, and Gauss enable students to visualize the concept and acquire an intuitive understanding on dealing with non-routine, open-ended "real-life" challenges thereafter? The data that we collected consists of videos of classroom sessions and interviews with students.

Our research indicates that teachers themselves first need to be well equipped with the history of mathematics so as to better judge how students should acquire such knowledge. The concern about the logarithms of negative numbers raised by the student mentioned above can be encouraged to be an interesting after-class historical project. Meanwhile, we confirm that the utilization of original sources in mathematics classroom can activate students' engagement to make mathematical discoveries, when different strategies flash into their minds and a repertoire of diverse representations are compared and the best is then chosen. Also, we address some obstacles to overcome. In the presentation, a detailed teaching scenario with the historical package, teachers' implementations and students' activities as well as implications for teaching and research will be presented and discussed.

Questions discussed in our class included the following:

Question 1: Find the solution to the equation $x^3 = 15x + 4$.

Question 2: Calculate the above cubic equation with Cardano's formula; share what you discover.

Question 3: Leibniz considered the following situation: Let *x* and *y* be positive values, and $x^2 + y^2 = 4$ as well as $xy = 5\sqrt{5}$. Can you find the value of $x + y$ and *x*, *y* respectively? Explain why.

Question 4: With Bombelli's equation $\sqrt[3]{a+b\sqrt{-1}} + \sqrt[3]{a-b\sqrt{-1}} = c + d\sqrt{-1} + c - d\sqrt{-1} = 2c$, can you shed new light into $x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$ (this is what we got when we applied Cardano's formula to solve the cubic)?

Question 5: Comparing with the number -1, what conclusion can you draw about the algebraic and geometric representation of the imaginary unit *i*?

Question 6: Share your ideas about Wessel's work with your classmates and infer the geometric representation for $\sqrt{-1}$ (Ever since Wessel, then, multiplying two directed line segments together has meant the two-step operation of multiplying the two lengths, with length always taken to be a positive value, and adding the two direction angles. These two operations determine the length and direction angle of the product, and it is this definition of a product that gives us the explanation for what $\sqrt{-1}$ means geometrically).

Question 7: A treasure hunt with imaginary numbers (quoted from one two three… infinity by George Gamow.)

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TSCHIRNHAUS' TRANSFORMATION MATHEMATICAL PROOF, HISTORY AND CAS

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ABSTRACT

The paper addresses the potential of using history of mathematics in combination with ICT to illustrate the notion of proof and proving, including also the role of examples and counterexamples, to undergraduate students and relates such use to research findings regarding students' difficulties. The historical case used to illustrate this is Tschirnhaus' transformation from 1683, suggesting a presumed method for solving *n*-degree algebraic equations, and we draw on a small empirical example from a Roskilde University undergraduate mathematics student project report. Based on this example we further discuss the potential use of history in combination with CAS in a classroom setting drawing on the notions of justificational mediations, sociomathematical norms and scientific debates.

Keywords: History of mathematics, original sources, CAS, mathematical proof and proving, counterexamples, "the mathematical underworld", justificational mediations.

1 Introduction

It has been pointed out previously that HPM research often do not make use of general mathematics education research frameworks (Jankvist et al., 2015). This is a pity for at least two reasons. Firstly, mathematics education research frameworks have a variety of lenses to offer HPM. Secondly, it may make it easier to communicate HPM research to the rest of the mathematics education community. In this paper, we shall address the topic of mathematical proof, examples and counterexamples under the assumption that HPM has much to offer in this regard – and that digital tools may also have a significant role to play.

Before we begin, a warrant should be provided to the reader, which is that this is a theoretical paper. Taking our departure in what the literature states regarding students' difficulties regarding mathematical proof and counterexamples, we outline a selection of theoretical constructs to potentially be used in relation to using elements of the history of mathematics in this endeavour. We also address elements of potentially using digital technologies in this respect, this in particular in relation to an illustrative and empirical example which serves as an invitation to think further. The example draws on the history of the so-called "Tschirnhaus' transformation" from 1683.

2 Students' difficulties with mathematical proof

As phrased by Duval (2007, p. 137), "Proof constitutes a crucial threshold in the learning of mathematics. Why do so many students not succeed in truly crossing it?" One reason is given by the Education Committee of the EMS (2011, p. 51) in their series of "Solid Findings" articles: "Mathematical thought concerning proof is different from thought in all other domains of knowledge, including the sciences as well as everyday experience; the concept of formal proof is completely outside mainstream thinking." Dreyfus (1999, p. 94) has observed that “[i]ndeed, research results on students’ conceptions of proof are amazingly uniform; they show that most high school and college students don’t know what a proof is nor what it is supposed to achieve. Even by the time they graduate from high school, most students have not been enculturated into the practice of proving or even justifying the mathematical processes they use”. Besides the differences to everyday reasoning, students’ difficulties with the notion of proof also stem from that they may have never learned what counts as a mathematical argument (Dreyfus, 1999). This is backed by Duval (2007, p. 159), who distinguishes two kinds of failures on students’ behalf: (1) “Dysfunctions in valid reasoning, such as status confusion, non-distinction between a statement and its converse, etc.” (2) “Gaps of deficiencies in the progress of a proof”. (For Duval, “status confusion” not only refers to different forms of reasoning in mathematics, but also to the different status of statements within a proof, e.g. hypothesis, property, and conclusion).

One type of status confusion is that of proof vs. example. Harel and Sowder (2007) have coined the notion of *empirical proof schemes*. The empirical proof schemes come into play when using examples to justify the truth of general (universal) statements, which is what Bell (1976) refers to as empirical justifications. Balacheff (1987) distinguishes between two subcategories: the naïve empiricism, which consists in the checking of assumed special cases, e.g. the first two or three instances; and the crucial experiment, which consists in checking for assumed general cases, e.g. if it is true for the numbers 1, 17, and a billion, it is true for all natural numbers.

Counterexample is yet another difficulty for students related to mathematical proof and reasoning, and one which is also subject to status confusion. Iannone and Nardi (2005) distinguish three, at times conflicting, roles that counterexamples play in learning and doing mathematics: (1) In the affective role the “counterexample has to be emotionally convincing for the students (strengthen their certainty). If it is based on what the student perceives as some minor technicality [...] the counterexample may not convince the student that the proposition is false.” (Nardi, 2008, p. 89) (2) The cognitive role “consists in conveying to the students that all counterexamples are the same, as far as mathematical logic is concerned; that a single counterexample can refute a proposition; that a proposition does not need to be always false in order to be false; and, that one occasion of falsity suffices.” (p. 89) (3) The epistemological-cum-pedagogical role “has to do with what can be learned from a good counterexample – for example in mathematics we use counterexamples to identify which elements of a false statement would need to be amended in order to transform this statement into a theorem.” (p. 89) (see also Peled & Zaslavsky, 1997).

If, for example, the textbooks do a poor job in enabling students to distinguish between different forms of reasoning – and their status – in mathematics, the job is left entirely to the teachers. Often, they might do this by asking students to “explain” and “justify” their reasoning. But this is a task which is related also to the students’ perception (or beliefs) of mathematics and what it means to do mathematics. Dreyfus (1999, p. 106) says: “...the requirement to explain and justify their reasoning requires students to make the difficult transition from a computational view of mathematics to a view that conceives of mathematics as a field of intricately related structures. This implies acquiring new attitudes and conceiving of new tasks: The central question changes from ‘What is the result?’ to ‘Is it true that...?’”.

3 Question and hypothesis

Indeed, it is a “solid finding” that empirical proof schemes are widespread among students. Still, this also builds on a rather rough picture of what mathematics is – e.g. abstract vs. concrete – and often underplays the role of examples. Teaching students a sharp distinction between examples (that serve the purpose of generating ideas for propositions) and proofs (that serve the purpose of validating these propositions) might lead to a somewhat sterile image of what mathematics is – e.g. a collection of proved theorems as well as experience in finding theorems by investigating examples and proving them deductively. The history of mathematics has something to offer here: Firstly, in terms of original sources on methods and examples. Secondly, in terms of providing students with an idea about how mathematics has evolved over time (Jankvist, 2015).

Hence, the aim of this paper is to *explore the relationships between proof, examples and counterexamples through a use of history and historical sources in the classroom*. The hypothesis we build upon is that we firmly believe that digital technology may have a role to play in such an exploration (and we shall return to this later). We shall provide one example of an original source, or excerpt of one such, that may to some degree illustrate this. The source is that of Tschirnhaus' transformation from 1683, where he proposed a method which he apparently believed could be generalized to solve algebraic *n*th degree equations.

4 Tschirnhaus' transformation

Ehrenfried Walther von Tschirnhaus (1651-1708) was a German philosopher and mathematician, who among other things in the summer of 1675 established a very close relationship with Leibniz (1646-1716) while in Paris. Tschirnhaus is acknowledged for four major contributions:

- On catacaustics (1682)
- On quadrature or integration (1683)
- On the Tschirnhaus transformation (1683)
- His (philosophical) logic, *Medicina mentis* (1687)

We are concerned with the third, which is entitled *Methodus auferendi omnes terminos intermedios ex data æqvatione*, which in English translates to *Method for eliminating all intermediate terms in a given equation*.

With reference to Descartes' *Geométrie*, Tschirnhaus builds on the fact that it is always possible to eliminate the second degree term of any third degree equation. He considers a third degree equation in *y*:

$$y^3 - qy - r = 0 \text{ and } z = y^2 - by - a,$$

for appropriate a and b. By eliminating y between these two, Tschirnhaus finds “the resulting equation” which is what is displayed in Figure 4.1.

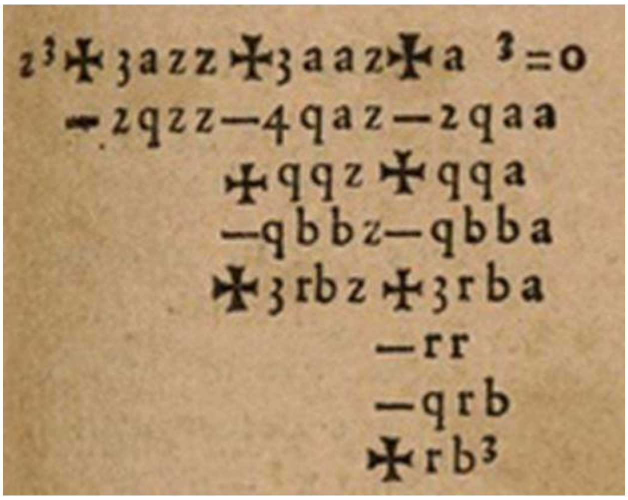

Figure 4.1: Excerpt from *Methodusauferendi terminus intermedios ex data aequatione*

The columns in Figure 4.1, arranged according to the degree of the *z*-term, are the key to Tschirnhaus' observations. Thez^2 terms in column 2 are eliminated if

$$3a - 2q = 0.$$

The *z*-terms in column 3 are eliminated, if

$$3a^2 - 4qa + q^2 - qb^2 + 3rb = 0,$$

a second degree equation in *b* with $a = 2q/3$. Tschirnhaus thus arrives at an equation of the form

$$z^3 - t = 0,$$

which has roots$\sqrt[3]{t}$, $\omega\sqrt[3]{t}$, and $\omega^2\sqrt[3]{t}$.

As remarked by Kracht and Kreyszig (1990, pp. 17-18), in "the remaining portion of his paper, Tschirnhaus discussed expressions *a* and *b* suitable for eliminating the third term in an equation of degree 4, 5, or 6 from which, just as in the cubical case, the second term is already absent" and that the "paper was supposed to explain a general method, and one wonders to what extent the author himself believed his claim: '*Et sic idem processus observaturadtres, qvatuor, qvinqve & terminus auferendos*'. In the continuous correspondences between Tschirnhaus and his old friend, Liebniz, the latter certainly expressed his doubts in relation to the generality of Tschirnhaus' method. Already as early as in 1678 or 1679, Leibniz challenged Tschirnhaus' idea:

> Concerning your ... method for finding the roots of an equation, which for solving
>
> $$x^5 + px^4 + qx^3 + rx^2 + sx + t = 0$$
>
> consists in assuming
>
> $$x^4 + bx^3 + cx^2 + dx + e = q \qquad \text{[}q\text{ should read }y\text{]}$$
>
> and then eliminating *x* by *y* and ... eliminating the middle terms in the resulting equation. ... I do not believe that it will be successful for equations of higher degree, except in special cases. I believe that I have a proof for this. (Leibniz, 1678-79, quoted from Kracht & Kreyszig, 1990, p. 27)

As noted by Kracht and Kreyszig, Leibniz' last sentence of course refers to the transformation under consideration, not to a forerunner of Abel's famous proof for the insolvability of the quintic by radicals.

5 An empirical case from a mathematics undergraduate program

In a Roskilde University mathematics bachelor's thesis on algebraic equation solving from Cardan (1501-1576) to Cauchy (1789-1857), three students describe and discuss Tschirnhaus' transformation in the light of other and related events in the history of algebraic equation solving (Backchi, Jankvist & Sağlanmak, 2002). Besides the original source from 1683, they relied on the research by Kracht and Kreyszig (1990) as well as Tignol's (2001) book about Galois theory. The students work through both the historical presentation, i.e. that of Tschirnhaus, of the transformation, and the modern by Tignol, which relies on calculating determinants and using matrices.

As pointed out by the students, the reason that Tschirnhaus' method works for n=3 is that "the system of the n-1=3-1=2 equations, in this case only leads to a second degree equation in b, since (n-1)!=(3-1)!=2" (Backchi et al., 2002, p. 43). For n=4 and n=5, the intermediate equation "will in worst case become an equation of degree (4-1)!=3!=6 or (5-1)!=4!=24" (p. 43).

From our perspective the interesting aspect of this thesis is the way that the students "test" Tschirnhaus' transformation. The students use CAS (*Maple*) to empirically "test" Tschirnhaus' transformation for n=4 and n=5 in order to further their understanding of what "goes wrong". For n=4 they obtain a third of page long expression for the intermediate equation, visually illustrating the inefficiency of the method (pp. 152-153). For n=5, *Maple* crashes before completing the calculations (p. 165).

Surely, the students "test" can also be done in CAS without the modern day notation and use of matrices introduced by Tignol (2001), although it undoubtedly will be a more cumbersome task. Our purpose in this particular paper, however, is not to discuss and compare modern day mathematical notation to that of previous times. Rather we seek, as previously mentioned, to illustrate that digital technologies can be used to empirically "test" mathematical conjectures, as a kind of "*technological un-likelihood test*". This is to say, the technology does not provide a traditional counterexample in the usual sense, but it can play the role of illustrating to the students why it is unlikely that a given mathematical conjecture holds - in the example above, because the intermediate equations simply increase so much in complexity.And unlike modern day mathematics, the history of mathematics is full of conjectures which has already been proven wrong, and which for that reason can act as illustrative examples in the teaching and learning of mathematics.

6 The "mathematical underworld"

With outset in this example, it makes sense to look at the distinction between deductive and inductive reasoning in mathematics. The classical picture that we as educators try to convey is that of mathematics as a pure deductive discipline, where inductive or experience-based reasoning is "wrong", "problematic" or "superfluous". However, the example of Tschirnhaus' transformation – as it is described above – can help us unpack this notion of mathematics as "pure", or phrased in another way; introducing Tschirnhaus' transformation and using CAS to explore his conjectures allows us to open up another layer of mathematical practice in relation to education. The students in the example above do not follow the classical image of mathematics. Rather they explore through computer work the likelihood of Tschirnhaus' hypothesis in an informal way.

If we distinguish examples from counterexamples, and again from proofs and conjectures, the mainstream story of mathematics is that examples allow for inductive reasoning, but only in a heuristic manner. Nothing is proved by example! Hence, examples are thought of as ways to generate ideas for further exploration and formulation as theorems and proofs (Johansen & Sørensen, 2014, p. 140). On the other hand, counterexamples are part of deductive reasoning as ways of rejecting propositions or proving converse statements (by contradiction or counterexamples). This means that the only "allowed" inductive reasoning in a mainstream view of mathematics is in relation to heuristic treatments of examples. But this mainstream story leaved out two forms of reasoning that are relevant, and that we believe it is worthwhile focusing on; namely inductive reasoning in relation to proofs and in relation to counter-examples. Our interest here is howfailed attempts to proof, or difficulties with conducting calculations, build mathematical intuition, both within the individual mathematician and mathematics student and in the mathematical society at large.We know from the literature on mathematical practice that inductive reasoning in mathematics is not limited to the heuristic work with generating conjecture through examples. Several autobiographical accounts (e.g. Thurston, 1994) as well as work in the philosophy of mathematical practice (Misfeldt & Johansen, 2015) point to the importance of development of strategic ideas and intuitions about routes to be pursued in mathematical research as well as dried out areas and problems in which one can be stuck for so long that it challenges one's career.

This phenomenon, i.e. that mathematicians and the mathematical society are accumulating inductive knowledge about proofs and counterexamples, is on the one hand a critically important part of mathematical practice, and on the other hand not a part of the official mainstream story about mathematics. We shall refer to such use of inductive reasoning and knowledge generation in relation to mathematics in ways that are not solely related to heuristic treatment of examples as "*the mathematical underworld*". This term is close to what Reuben Hersh (building on Goffman 1978) describe as the backside of mathematics (Hersh, 1997). The backside of mathematics is defined in relation to the front side, and where the front side is described as formal and precise ordered and abstract, the backside is fragmentary, informal, intuitive and tentative (p. 36). Hersh describes how the philosophy of mathematics tends to not see the backside of mathematics (of course new approaches in the philosophy of mathematical practice tries to address this) and we can add to this that mainstream teaching of mathematics, has the same problem. How do we address the backside of mathematics in teaching? The informal and intuitive nature makes the backside hard to point to in a precise manner. We can define the "mathematical underworld" as the object of teaching when trying to address the backside of mathematics in teaching. The underworld is where the backside lives and can be studied. In that sense, the mathematical underworld represents knowledge generation in the mathematical field that transcends the clear distinction between modes of discovery and of justifications.

This leads us to the question of how to understand and teach the mathematical reasoning process in a way that does not neglect this mathematical underworld. In order to do that we base ourselves in a model developed with an outset in Lakatos and Polya and published in the proceedings of the previous ESU (Misfeldt, Danielsen & Sørensen, 2015). This model describes an ideal reasoning process, focusing on the relation between inductive reasoning (from example to conjecture) and deductive elements such as counterexamples and refinements of proofs.

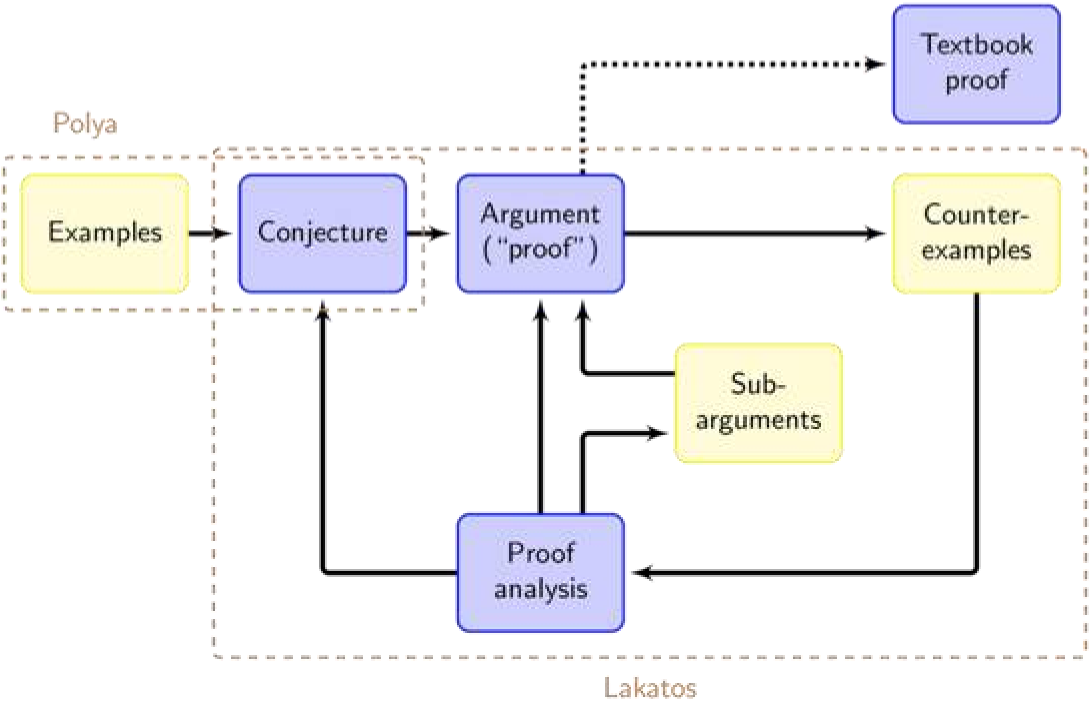

Figure 6.1: This figure taken from (Misfeldt et al., 2015, p. 437), describes the reasoning process in mathematics from ideation to proof. The model shows the interplay between proofs and counterexamples, based in Lakatos, but the model does not capture inductive reasoning that is not involved in working with examples. The yellow labels designate inductive reasoning, and are intimately connected to examples.

The model in Figure 6.1 formalizes the mainstream image of the role of examples in mathematical proof work (Misfeldt et al., 2015). But the role of inductive and instrumented reasoning in the mathematical underworld is not clearly described here. If we apply the model to the way the students use CAS to work with Tschirnhaus' transformation, we see the phenomenon these students experience is not completely captured by the model, and in particular by the wording applied in the model. The attempt to create examples arguing for the potential conjecture fails and this sparks interest in understanding the limitations of the methods rather than proving the general claim.

We now take a look at some of the circumstances related to how the considerations discussed above may find their application in a classroom setting. We do so by outlining a selection of theoretical constructs that we deem potentially relevant for such an endeavour. These are Misfeldt's and Jankvist's (2018) notion of justificational mediations, Yackel's and Cobb's (1996) notion of sociomathematical norms, and finally Alibert's(1988) notion of scientific debate.

7 Justificational mediations

The use of CAS for learning mathematics is often understood with the instrumental approach to mathematics learning (Trouche, 2005), which focuses on how students transform CAS tools to personal instruments. One critical concept in the instrumental approach is that of mediation that designates the way in which tools support goal directed activities and hence mediate between a student and his/her goal. The literature highlights a critical distinction between epistemic and pragmatic mediations referring to whether the students aim at understanding certain phenomena or at solving specific tasks (Artigue, 2002).

When working with the use of CAS in the context of proofs and proving activities, we have described four core questions/aspects about how CAS mediate proving (Misfeldt & Jankvist, 2018). Aligning with the instrumental approach, this can be conceptualized as using CAS for justificational mediations (Misfeldt & Jankvist 2018). These four aspects of justificational mediations are:

1. Does the CAS use establish truth? This is the core function of a justificational mediation. To what extent does the CAS output act as warrant in an argument?
2. Does the CAS use allow interaction and experimentation? This highlights the degree to which students can change parameters, explore phenomena etc., and therefore to what extent the students still have agency when working with CAS in relation to proofs.
3. Is the argumentation inductive, deductive or authoritarian? What type of proof scheme is in play and what type of warrant do CAS provide.
4. Does the argument highlight important aspect of the proof or the mathematical relationships?

Both the second and the fourth aspects are to some extent related to epistemic mediations. Still, we suggest that these four aspects of justificational mediations more or less capture the important aspects of using CAS tools in proving activities.

Looking at the case of the students' work with the Tschirnhaus transformation, we can make the following observations related to the four questions/aspects:

1. The CAS use does not establish truth in the usual sense – what we do learn from the use of CAS, however, is that the computations of the Tschirnhaus transformation in the case of fourth and fifth degree polynomials become very lengthy and complicated and do not seem to work in the sense of providing a mathematical result. This is not the same as establishing truth in a classical sense.
2. The use of CAS is a case of experimentation, in the sense that the reported activity, more or less is an experiment where the students could – and did – play with attempts to calculate the Tschirnhaus transformation for specific polynomials of higher degree than 3. In the case, there is a direct continuity between calculating the general case (as attempted in the student project) and experimenting with specific examples.
3. The argumentation here is both deductive and inductive. In a sense the attempt is to calculate the general Tschirnhaus transformation, for polynomials of degree 2, 3, 4, …. In cases where this strategy works, the argument is straightforward deductive and fulfils all mathematical standards in the classical sense. Yet, when this plan breaks down (from polynomials of degree 4 and onwards) the argument is different. The CAS breakdown is not by any means a valid mathematical argument

for the impossibility of Tschirnhaus' hypothesis. Still, it does support the idea that this hypothesis might not hold water, in an inductive fashion. This type of non-valid/non-kosher mathematical knowledge is what we have designated "the mathematical underworld".

4. The relation to the main ideas in the proof of the Tschirnhaus transformation is slightly awkward here; since we are looking at a negative result (the general Tschirnhaus transformation cannot be calculated). However, CAS provide a good idea about the reason why the result is unobtainable.

8 Sociomathematical norms

Sociomathematical norms were observed and named by Yackel and Cobb (1996), who in a teaching situation noticed that aspects which could neither be described as purely mathematical norms nor purely as classroom social norms were in play. Yackel and Cobb (1996, p. 461) define sociomathematical norms as "normative aspects of mathematics discussions specific to students' mathematical activity" and describe the difference to social norms as "The understanding that students are expected to explain their solutions and their ways of thinking is a social norm, whereas the understanding of what counts as an acceptable mathematical explanation is a sociomathematical norm. Likewise, the understanding that when discussing a problem students should offer solutions different from those already contributed is a social norm, whereas the understanding of what constitutes mathematical difference is a sociomathematical norm".

Sociomathematical norms are negotiated between the students and the teacher, and may thus vary from classroom to classroom. This negotiation builds on already "taken-as-shared" perceptions within the classroom, and as such they are "… intrinsic aspects of the classroom's mathematical microculture. Nevertheless, although they are specific to mathematics, they cut across areas of mathematical content by dealing with mathematical qualities of solutions, such as their similarities and differences, sophistication, and efficiency. Additionally, they encompass ways of judging what counts as an acceptable mathematical explanation." (Yackel & Cobb, 1996, p. 474). Hence, explanations and justifications are themselves made the objects of reflection.

The case highlights a type of inference in mathematics, different from the established and accepted reasoning. This "mathematical underworld" poses an educational problem, since exploring the Tschirnhaus transformation and realizing that there seems to be a breakdown when attempting to calculate the transformation for the general polynomial of degree 4 and 5, is not a proof or even an established mathematical result. However, it does constitute a type of mathematical knowledge that should be object for teaching, namely establishing sociomathematical norms that both allow for discussing and working with the type of phenomena described as the "mathematical underworld", while avoiding the type of student mistakes and misconceptions that easily result from working with inductive reasoning and non-formal mathematical work. This dilemma calls for specific teaching strategies allowing students and teachers to discuss the epistemological status of various pieces of knowledge and intuitions about mathematics. We suggest that the combination of historical sources and CAS is a good context for developing such discussion, and in the following section we will describe one pedagogical strategy that develops this discussion.

9 Scientific debates

Under the heading of generating *scientific debate* Alibert (1988, p. 32) provides three steps for making this happen in the mathematics classroom:

1. The teacher initiates and organizes the production of scientific statements by the students. These are written on the blackboard without any immediate evaluation of their validity.
2. The statements are put to the students for consideration and discussion. They must come to decisions about their validity by taking a vote; each opinion must be supported in some way, by scientific argument, by proof, by refutation, by counterexample.
3. The statements that can be validated by a full demonstration become theorems; those found to be incorrect are preserved as "false-statements" associated with appropriate counter-examples.

In an empirical example of a classroom scientific debate described by Alibert (1988), students came up with counterexamples to a proposed statement, which then resulted in new hypotheses, empirical observations, derived conjectures, and eventually an argument which required a formal proof.

Although our small empirical example with the undergraduate student report is not one of classroom interactions, we do see a resemblance to the activities. Here, however, it was neither the students nor the teacher who as part of step 1 provided the scientific statements, but the history of mathematics! This approach could easily be transformed into a classroom activity, where the teacher posed a mathematical conjecture from the history of mathematics on the blackboard. In the case of Tschirnhaus' transformation, the voice of Leibniz could play the role in framing the scientific debate.

In our empirical example, CAS came to act as a justificational mediator in terms of judging the potential validity of Tschirnhaus' transformation. The same could certainly be the case in a classroom setting. Also, here CAS would have a significant role to play in order to guide the students' discussions in step 2. One sociomathematical norm to be established here would be that a "technological un-likelihood test" certainly is not the same as an actual counterexample. This is to say, step 2 of the scientific debate provides, as mentioned above, the opportunity to discuss the epistemological status of mathematical statements and arguments, i.e. the combination of both history and technology here comes to serve as a way of counteracting the "status confusion" as described by Duval.

Step 3 in Alibert's scientific debates would in our example correspond to the students' reasoning of the degree of the intermediate equations in Tschirnhaus' transformation. Depending on what historical conjecture, false or true, was posed by the teacher in step 1, step 3 will result in a theorem with a proof or an actual counterexample.

Such scientific debates may be used to establish "healthy" sociomathematical norms in a classroom by: developing students' deductive proof schemes; assisting in discarding their empirical proof schemes and/or external conviction proof schemes; illustrating the difference between proofs that prove and proofs that explain; illustrating the difference between different types of mathematical arguments (cf. Dreyfus) and the status confusion of which Duval talks.

10 Final remarks

The example described above as well as the way that it can be viewed through the lenses of three different mathematics education frameworks allows us to suggest that the relation between history of mathematics and the use of technology can be mutually fruitful.

As mentioned, the history of mathematics is rich on examples of conjectures that turned out not to be true. By having students look at and work with such examples they may come to grasp not only some of the differences in mathematical statements, but also the very need for formal proof to begin with. In that sense, the history of mathematics has a role to play in assisting students in overcoming the "status confusion" of which Duval's speaks. But apart from motivating the need for formal proving, historical examples can also open up new perspectives on what types of mathematical knowledge is relevant to address from an educational perspectives.

In this paper, we have brought light on the "mathematical underworld" of inductive reasoning in relation to hypothesis testing and theorem proving. Hereby we have come closer to understand types of reasoning and knowledge that have played (and still are playing) a role in developing mathematics, but are not considered as part of mainstream mathematics. We have been able to activate "computer evidence" to open up and discuss historical examples that would be too laborious to address using classical algebraic methods. We have labelled this computer evidence as justificational mediations, aligned with the instrumentational approach of mathematics education (Trouche, 2005). Furthermore, we have used the concepts of sociomathematical norms as well as the didactical idea of scientific debates to unfold the pedagogical challenges that the mathematical underworld poses to teaching. By developing sociomathematical norms that value working with examples and counterexamples, and in a complex way includes inductive reasoning, we argue that it is important to fully grasp the nature of mathematics as a discipline (e.g. Jankvist, 2015) and that historical sources in combination with digital tools are useful for that.

Yet, the problems of status confusion and development of empirical proof schemes persist, and hence we end the paper by suggesting deliberate teaching strategies based in scientific debates to support students' understanding of all the different natures of mathematical knowledge that has been and still are important for mathematical practice.

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