THE BICYLINDER OR BIRDCAGE OR MÓUHÉFĀNG GÀI

Combining a cultural approach with many other goals of mathematics education

Michel ROELENS
University Colleges Leuven-Limburg, Faculty of Teacher Education
& Maria-Boodschaplyceum Brussels, Belgium
michel.roelens@ucll.be

ABSTRACT

We want to introduce our students into some mathematical ‘culture’. This noble goal, however, has many competitors in the form of other important goals of mathematics education, and time is finite. We also want to (and have to) teach calculus methods and train the students’ algebraic skills. We want to show them that different methods can solve the same problem. We want to show the utility of mathematics in architecture and technology. We want to stimulate the students’ spatial insight and their ability to make sketches of spatial situations. We want to show them the relationships between different parts of mathematics. We would like to make them critical of manipulation by the media or the internet.
The bicylinder is an object that allows combining all these goals. We will discuss how the bicylinder can play an interesting role in mathematics courses for students between the ages of 15 and 18, in order to combine various goals and to make the mathematics courses more cultural and versatile without spending much extra time.

1 Describing the bicylinder

1.1 Orthogonal projections and ‘edges’

Consider two solid circular cylinders of equal radii, the axes of which intersect perpendicularly. While figure 1.1 shows their union, we are interested in their*intersection*.

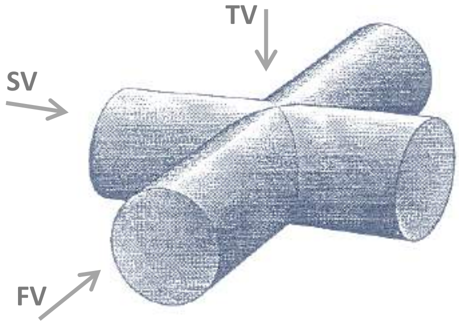


Figure 1.1: The two cylinders (Mathenjeans, 2006)

We call that intersection a bicylinder, but it has been given many names throughout history: 牟合方蓋 [Móuhéfānggài, double box-lid] (ZǔChōngzhī, 5th century),Steinmetz’ solid (Charles Proteus Steinmetz, 19th and 20th century), birdcage (Stannard, 1979), equidomoid (Ferréol, 2013)…

Showing the students only the union of the cylinders (figure 1.1), we ask them to imagine whatthe bicylinder looks like and to draw its perpendicular projections: a front view (FV), top view (TV) and a side view from the left (SV) (figure 1.2). If this causes them some difficulties, they can be comforted by this quote: "It takes an unusual gift of imagination to visualize this shape clearly" (Strogatz, 2010).

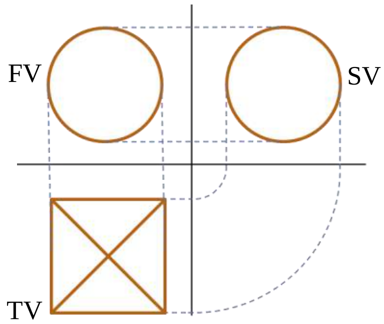


Figure 1.2: Orthogonal projections of the bicylinder

We also ask them the shape of the (curved) ‘edges’. It is a bit unusual to talk about edges when the figure is not a polyhedron, hence the quotation marks. One could say that the bicylinder has two ‘vertices’(at the top and at the bottom), four curved ‘faces’ and four curved ‘edges’ connecting the two ‘vertices’. Some students want to determine the shape of the ‘edges’ analytically. Taking the axes of the cylinders as x- and y-axes and calling the radius r, they determine the ‘edges’ with a system of equations. They solve by replacing the second equation by the difference of both and then factorizing.

$$\begin{cases} x^2+z^2=r^2 \\ y^2+z^2=r^2 \end{cases} \Leftrightarrow \begin{cases} x^2+z^2=r^2 \\ x^2-y^2=0 \end{cases} \Leftrightarrow \begin{cases} x^2+z^2=r^2 \\ (x-y)(x+y)=0 \end{cases} \Leftrightarrow \begin{cases} x^2+z^2=r^2 \\ x-y=0 \lor x+y=0 \end{cases}$$

The ‘edges’ are created by intersecting one of the cylinders with the two ‘vertical’ diagonal planes $x-y=0$ and $x+y=0$. Instead of analytically, this could just as well be discovered on the basis of symmetry. So the four ‘edges’ are halvesof two ellipses. The small axis of these ellipses is the height $2r$ of the bicylinder; the large axis is $2\sqrt{2}\,r$ (as you can deduce from the top view in figure 1.2).

Figure 1.3 shows a wooden model of the bicylinder. We deliberately don’t give it to the students from the beginning; we want to appeal to their imagination.

Figure 1.3: Bicylinder (Modellsammlung)

1.2 Making the bicylinder

If students want to make a paper model of the bicylinder, they have to know the shape of its ‘faces’ when developed in the plane. We will prove here that the plane development of a cylinder cut by an oblique plane is bound by one period of a sine graph. It can be shown with a paint roll (figure 1.4). It is alsoused by knitters: the pattern of a sleeve, which is more or less a cylinder segment, is made using (approximately) a sinusoid (figure 1.5).

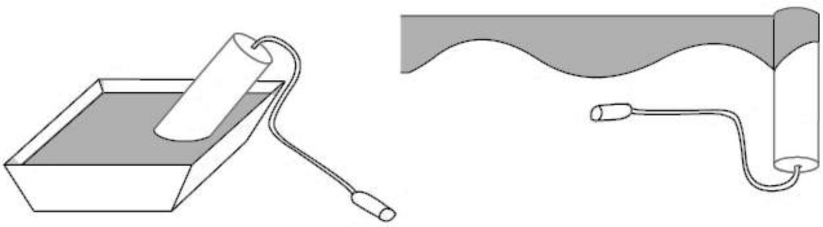
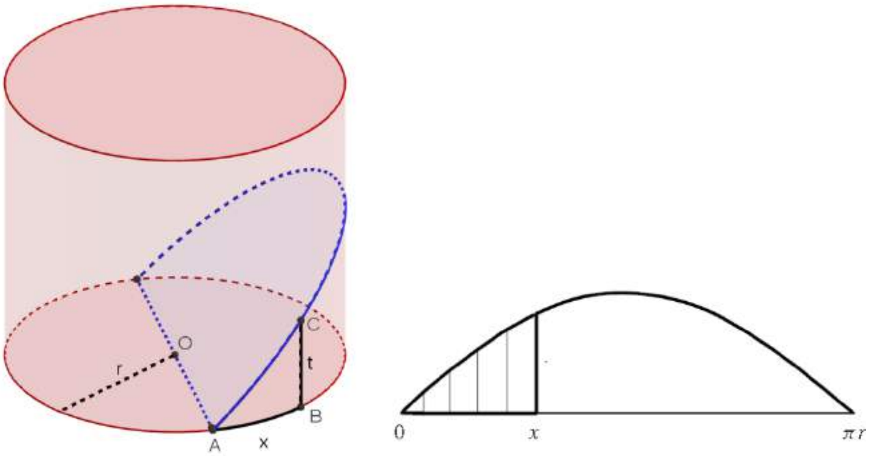


Figure 1.4: Paint roll experiment

Figure 1.5: Sleeve pattern

In order to prove that the plane development is delimited by a sinusoid, we use figure 6. By the symmetry it suffices to prove that the half ellipse (left part of figure 1.6) obtained by intersecting the volume with a plane, gives rise to half a period of a sine graph in the plane development. The solid consisting of the cylinder ‘under’ the half ellipse in figure 1.6a is called a *cylinder hoof*.



Figures 1.6a and 1.6b: Proof that we get a sinusoid, first part

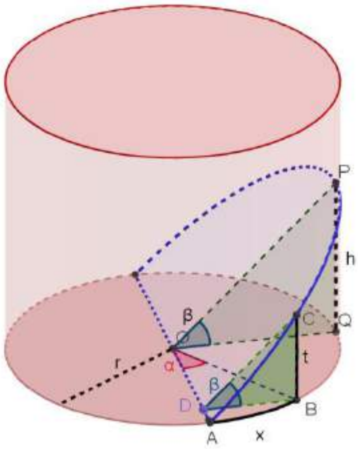


Figure 1.7: Proof that we get a sinusoid, second part

We prove that t is indeed a sine function of x (figure 1.7).

$$\begin{aligned} t &= |DB| \tan\beta && (\Delta\, DBC) \\ &= r \sin\alpha \tan\beta && (\Delta\, ODB) \\ &= r \sin\frac{x}{r} \tan\beta \\ &= h \sin\frac{x}{r} && (\Delta\, OPQ) \end{aligned}$$

Knowing this, the students can make a bicylinder from its plane development. In the case of the bicylinder, the height h of the cylinder hoof equals the radius r of the cylinders, so the development of two of the 'faces' is delimited by one period of the graphs$y = \pm r \sin\frac{x}{r}$ (figure 1.8), and in the same way the development of the other two 'faces' by $x = \pm r \sin\frac{y}{r}$ (not shown in figure 1.8).

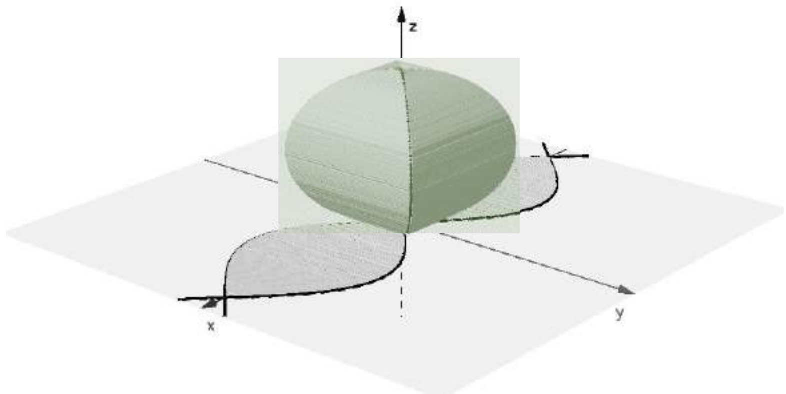


Figure 1.8: The bicylinder and the development of two of its 'faces'

2 Calculating the volume of the bicylinder

2.1 As did Archimedes?

Archimedes announces in his preface to *The Method* that the volume of a bicylinder is 2/3 of the volume of the circumscribed cube. His proof is lost but we have his proof of the volume of a cylinder hoof, which is in a non-lost part of *The Method*. With eight cylinder hoofs, we can construct a bicylinder (figure 2.2), so it is likely that Archimedes used this idea for determining the volume of the bicylinder.

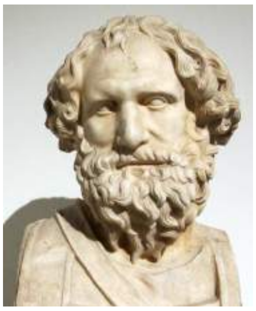


Figure 2.1: Archimedes of Syracuse (3rd century BC)

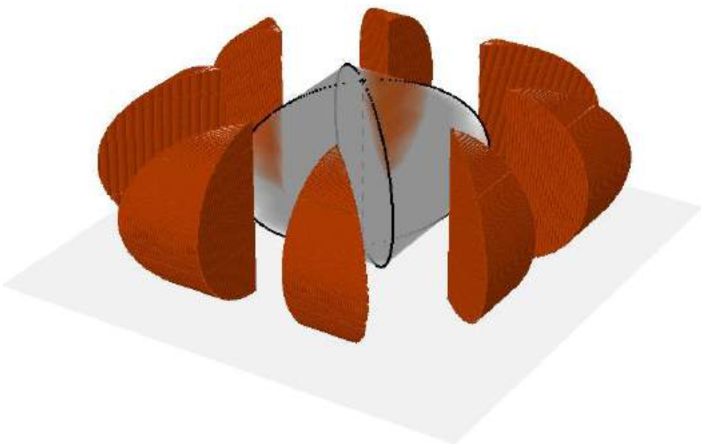


Figure 2.2: Eight cylinder hoofs form a bicylinder

Before going into the details of his proof for the volume of a cylinder hoof, let's say a few words about *The Method*.

Archimedes sent his papyrus roll with *The Method* to Eratosthenes at the famous library of Alexandria (in present-day Egypt). Later, the work was copied on parchment sheets for better preservation. Together with some other works of Archimedes, the sheets were knit together into a booklet, a *codex*. This codex disappeared until it was found in 1906 in a monastery in Jerusalem. In the 13th century, the monks had scrapped Archimedes'original texts and drawings to replace them by prayers. Archimedes' codex had become a *palimpsest*, a recycled piece of parchment. The codex was stolen in the course of the twentieth century and reappeared in 1998 in a sales hall, where it was sold by auction. A mysterious Mr. B bought it for $2,200,000. Nobody knows who Mr. B is, although there are some speculations. Fortunately, he allows scientists to study the codexusing UV- and X-rays. For more details about the story of this palimpsest, see Netz and Noel (2009).

In *The Method*, Archimedes determines areas and volumes in a revolutionary way for Greek mathematics: he determines an area by considering an infinity of line segments and a volume by considering an infinity of flat slices. Much later, this idea will become Cavalieri's principle (Bonaventura Cavalieri, 17^{th} century) and integral calculus. Sometimes he also uses the physical idea of a balance with which he 'weights' the slices, but this is not the case in the proof about the cylinder hoof.

How did Archimedes determine the volume of a cylinder hoof (i.e. of one eighth of a bicylinder)? We follow the ideas of his proof, but in a very anachronistic way, using today's algebraic notations.

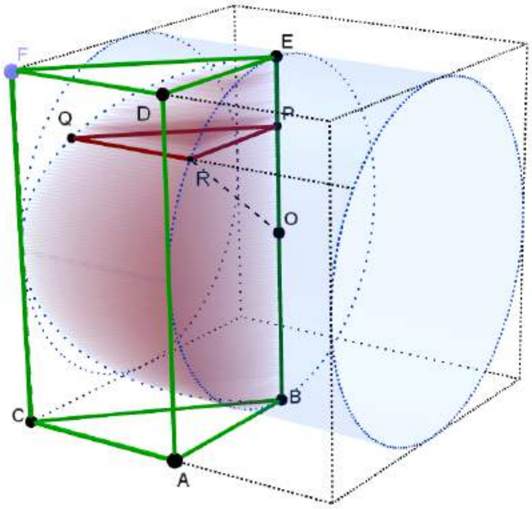


Figure 2.3: The cylinder hoof as locus of triangular slices

In a cube of edge 2, a cylinder of radius 1 and height 2 is inscribed (figure 2.3).The prism $ABC.DEF$ is one eighth of the cube. The cylinder hoof obtained as the intersection of this prism with the cylinder is one eighth of the bicylinder.Archimedes considers this hoof as the 'locus' of the variable horizontal triangle PQR as P varies on the segment $[BE]$ (see figure 2.3). The students calculate the area of the triangle ABC, the variable area of the triangle PQR (as a function of $x = |OP|$) and the proportion of both areas.

$$\left.\begin{array}{r} \text{area}(ABC) = \dfrac{1}{2} \\ |QR| = |PR| = \sqrt{1-x^2} \Rightarrow \text{area}(PQR) = \dfrac{1}{2}(1-x^2) \end{array}\right\} \Rightarrow \frac{\text{area}(PQR)}{\text{area}(ABC)} = 1 - x^2$$

The proportion of these areas is a quadratic function of x. The graph of a quadratic function is a parabola. This is not the way the Greeks of the time of Archimedes considered a parabola, but in a different, more geometric way Archimedes came to the same idea. Then he constructs a point S on the segment $[PR]$ such that

$$\frac{|PS|}{|BA|} = \frac{\text{area}(PQR)}{\text{area }(ABC)} = 1 - x^2.$$

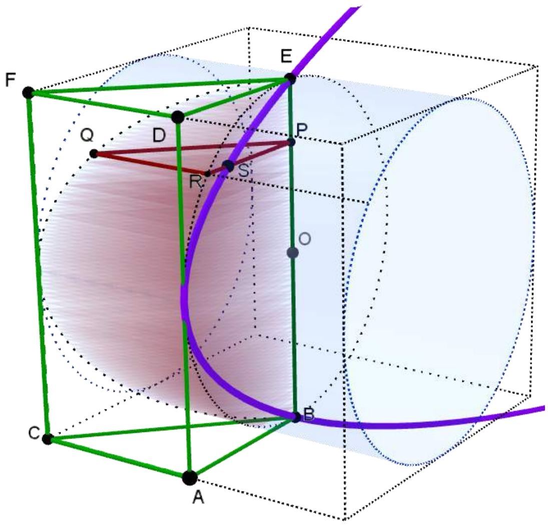


Figure 2.4: The parabola added in figure 2.3

If x varies, the point S moves on a parabola and the variable segment $[PS]$describes a parabola segment inscribed in the rectangle $ABED$ (figure 2.4).

Archimedes knows from one of his other works (*The Quadrature of the Parabola*, theorem 14) that the area of a parabola segment inscribed in a rectangle equals 2/3 of the area of the rectangle. Since each slice (line segment) of the parabola segment represents a slice (triangle) of the cylinder hoof and each slice (line segment) of the rectangle $ABED$represents a slice (triangle) of the prism$ABC.DEF$, he deduces, 'integrating' all these slices, that the volume of the cylinder hoof equals 2/3 of the volume of the prism.

So the volume of the whole bicylinder is

$$8 \cdot \left(\frac{2}{3} \text{ volume(prism)}\right) = \frac{2}{3} \cdot \left(8 \cdot \text{volume(prism)}\right) = \frac{2}{3}\text{volume (cube)} = \frac{16r^3}{3}$$

Surprisingly, the formula for calculating the volume of bicylinder does not contain a factor π. Archimedes: "Unlike spheres, cones and cylinders, this object is equal [in volume] to asolid figure bound by plane figures." (Introduction to *The Method*, cited in Hogendijk, 2002).

2.2 As did (not) Liú Huī

Figure 2.5: Liú Huī (3rd century)

Liú Huī wrote his famous Commentary on the *Jiǔzhāng Suànshù* [Nine Chapters on the Mathematics Art] (2nd century BC), wherein he added explanations and proofs to the Nine Chapters. He considers a bicylinder and its inscribed sphere (figure 2.6).

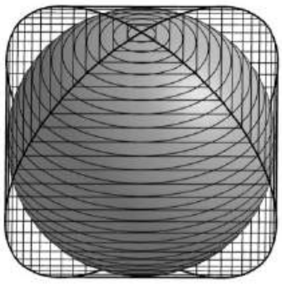


Figure 2.6: A bicylinder and its inscribed sphere (Cadav92, 2014)

He uses 'horizontal' slices. Each slice of the bicylinder is a square. The slice of the sphere at the same height is a circle inscribed in that square. So the proportion of the slices at a same height is always $\frac{4}{\pi}$. He deduces that the volume of the bicylinder is $\frac{4}{\pi}$ times the volume of the sphere. This is an early use of what we now call Cavalieri's principle (Bonaventura Cavalieri, 17thcentury). This principle says: *if at any height the 'horizontal' cross-sections of two solidsare in a fixed proportion, then the volumes of these solidsare in the same proportion*.

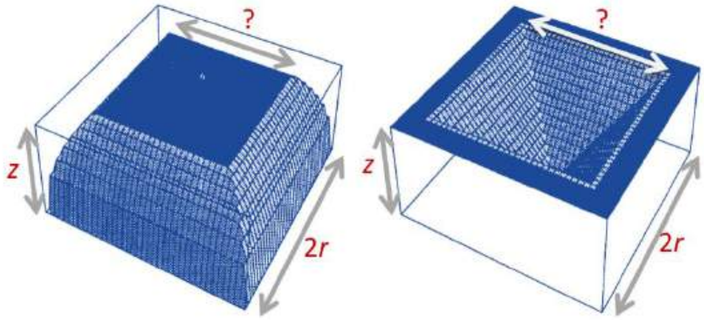
Our students can use this proportion to calculate the volume of the bicylinder from the volume of the sphere, getting the same result as in 2.1 (i.e. 2/3 of the volume of the circumscribed cube). But, unlike our students, LiúHuī did not dispose of the volume of a sphere. On the contrary, he considered this discovery of the proportion $\frac{4}{\pi}$as a step towards finding the volume of the sphere, if he would be able to find the volume of the bicylinder first.

2.3 As did ZǔChōngzhī

Figure 2.7: Zǔ Chōngzhī (5th century)

Zǔ Chōngzhī succeeded in finding the volume of the bicylinder, using a cube, a pyramid and the later principle of Cavalieri, already used by Liú Huī.

In figure 2.8a, you see the upper half of a bicylinder cut by a horizontal plane. The part above this plane has been removed. In figure 2.8b, you see a half cube hollowed by an inscribed pyramid (top down). This is also cut by a horizontal plane. The part above this plane has also been removed.



Figures 2.8a and 2.8b: Proof of the volume of a bicylinder by Zǔ Chōngzhī (De Temple, 1994)

Using a front view of figure 2.8a, the students can show that the side of the square slice at height z equals $2\sqrt{r^2 - z^2}$. So, the area of this square is $4(r^2 - z^2)$. On the other hand, the side of the section of the pyramid at height z (figure 2.8b) equals $2z$. Therefore, the area of the square 'ring', the slice at height z in figure 2.8b, equals $4r^2 - 4z^2$. Because the areas at height z are equal for each value of z, the students conclude by the principle of ZǔChōngzhī and Cavalieri that the volumes in figure 2.8a and 2.8b are equal. So, the volume of the half bicylinder is equal to 2/3 of the volume of the half cube. By symmetry, the volume of the whole bicylinder is also equal to 2/3 of the volume of the whole cube, as in 2.1.

Here we worked in half a cube; the original proof of ZǔChōngzhī divides the cube in eight parts, but this does not change much. (See Papillon, 2012, Lam, 1985 or Antony, s.d.).

2.4 As we do

The ‘normal way’ for our students to calculate the volume of a solid, is using an integral.Unlike the majority of the textbook exercises on volumes with integrals, the bicylinder is not a solid of revolution. The horizontal slices are squares, so the volume is the integral of the area of a square slice as a function of the height z (figure 2.9).

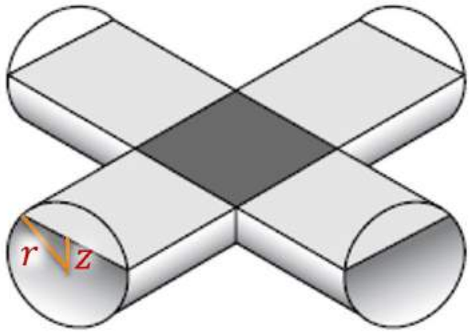


Figure 2.9: Square slice at height z

Again, the side of the square slice at height z is $2\sqrt{r^2 - z^2}$, so the volume is

$$\begin{aligned} \text{volume (bicylinder)} &= \int_{-r}^{r} \left(2\sqrt{r^2 - z^2}\right)^2 dz \\ &= 4\int_{-r}^{r} (r^2 - z^2)dz \\ &= 4\left[r^2 z - \frac{z^3}{3}\right]_{-r}^{r} \\ &= 4\left(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}\right) \\ &= \frac{16r^3}{3} \end{aligned}$$

3 Calculating the surface area of the bicylinder

As we saw in paragraph 1.2, the flat development of the bicylinder of radius $r$is delimited by the sine graphs $y = \pm r \sin\frac{x}{r}andx = \pm\, r \sin\frac{y}{r}$ (figure 1.8 in 1.2). The surface area of the bicylinder is the area of its development:

$$\begin{aligned} \text{surface area(bicylinder)} &= 8r\int_{0}^{\pi r} \sin\frac{x}{r}\,\mathrm{d}x \\ &= 8r\left[-r\cos\frac{x}{r}\right]_{0}^{\pi r} \\ &= 8r(-(-r) + r) \\ &= 16\, r^2 \end{aligned}$$

Again it is striking that there is no π in the formula! The surface area is simply the area of a square with side $4r$. The fact that the area of the bicylinder is equal to the area of a square of side$4r$ is culturally interesting. It means that the quadrature is possible here, the constructionof a square with the same area as the bicylinder by means of ruler and compass, starting from the given radius r. The quadrature of the circle is one of the famous construction problems of Greek antiquity, which in the 19th century has been shown to be impossible to solve with ruler and compass.

The area of the bicylinder is $2/3$ of the surface area of the circumscribed cube $(6 \cdot (2r)^2 = 24r^2)$. For the bicylinder and its circumscribed cube the proportion of the volumes equals the proportion of the surface areas!

Hogendijk (2002) explains how the surface area can be derived from the volume, without integrals.

4 Applications of the bicylinder

Cross vaults and the joining of cylindrical pipes are an obvious application of meeting cylinders, although it is more the union than the intersection of the cylinders (figures 4.1 and 4.2).

Figure 4.1: A cross-vault (Glaeser, 2007)

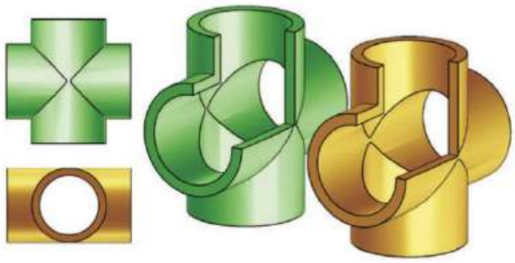


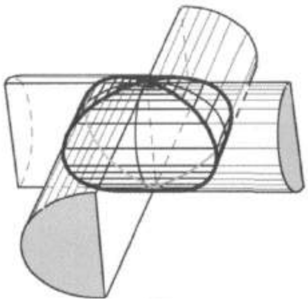
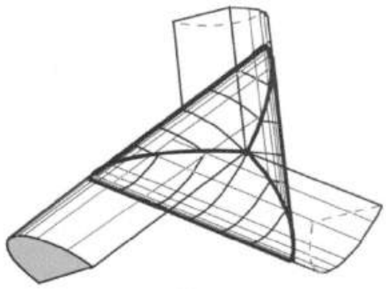
Figure 4.2: Pipes meeting at a right angle (Glaeser, 2007)

The roofs of the Château de Cheverny, one of the Châteaux of the Loire valley, have the form of half bicylinders (figure 4.3).

Figure 4.3: Chateau de Cheverny (Papillon, 2012)

5 Generalizations of the bicylinder

A first generalization consists of inventing analogue solids with a different number of curved 'faces' than four. It is easy to imagine making one with 6 (or $8, \ldots, 2n$) 'faces'by intersecting 3 (or $4, \ldots, n$) cylinders, with equal radii and axes lying in one plane and intersecting in one point at equal angles of 60° (or$45°, \ldots, \frac{180°}{n}$). Is it also possible to obtain an odd number of 'faces'? The number of cylinders is half of the number of 'faces' and one cannot use half cylinders... Why not actually? In figure 5.1a the familiar bicylinder is made in a different way and this way can be generalised to an odd number of 'faces'.

Figures 5.1a and 5.1b: (Apostol & Mnatsakanian, 2004)

On the website Mathcurve (Ferréol, 2013) these solids are called polygonal *equidomoids*. The bicylinder is a quadrangular equidomoid. On that site it is claimed that the dome of the Cathedral of Florence is a (half) pentagonal equidomoid (that has been vertically stretched a little). In order to convince us of this, they place the following two figures next to each other (figure 5.2).

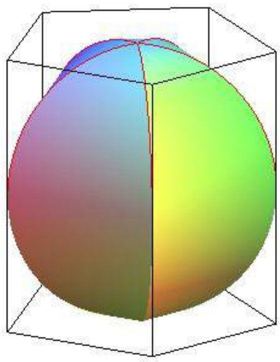



Figure 5.2: The dome in Florence and a pentagonal equidomoid (Ferréol, 2013)

However, you can clearly see in the picture that the dome has more than five 'faces' since four of them are visible. By entering the cathedral and looking upwards you find out that the dome is octagonal (figure 5.3).

Figure 5.3

Another generalisation is to take three cylinders with the same radius and with the axes that intersect in one point and are two to two perpendicular. The intersection of these cylinders is a*tricylinder*. This is a very interesting object, a curved rhombic dodecahedron, but we will not discuss it here (figure 5.4). Moore (1974) mentions applications in crystallography of the tricylinder and intersections of more than three cylinders, when due to increased temperature or decreased pressure a polyhedral crystal gets curved ‘faces’.

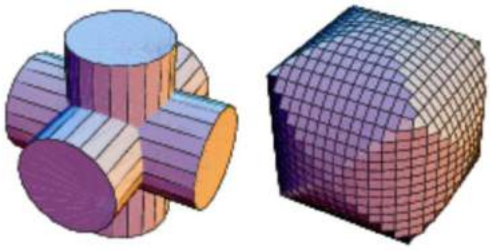


Figure 5.4: (Weisstein, 1999-2018)

6 In the classroom

In my classroom, I introduced the bicylinder as an exercise on the calculation of volume with an integral (2.4). In order to know what area they had to integrate, they had to imagine how the object is like (1.1). Only afterwards I confronted the students with other, historical, ways of finding the same result. In my esu8-workhop and in this article, I follow the chronologic order. You can find the worksheets of the workshop online: https://esu8.edc.uoc.gr/1112-2/.

What I like about the bicylinder, is that it includes a ‘normal’ textbook-like exercise (calculate its volume (2.4) or its surface area (3) with an integral), but that it goes further. Other methods than integral calculus are possible and have been discovered through history (2.1, 2.2, 2.3). The bicylinder is an challenge for the students’ spatial insight (1.1, 1.2) and it has applications in architecture and technology (4). It can even remind the students to be critical of manipulation by the media or internet (5). It is an object with a rich cultural history and it provides exemplary access to important mathematicians and important highlights in the history of mathematics.

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