MATHEMATICS AND EXPERIMENT

How to calculate areas without formulas?

Carène GUILLET[1], Marie-Line MOUREAU[2], Isabelle VOILLEQUIN[3]
IREM des pays de Loire, Faculté des sciences et des techniques, 2 rue de la Houssinière, Nantes, France
[1]carene.guillet@gmail.com, [2]marie-line.moureau@ac-nantes.fr, [3]isabelle.voillequin@hotmail.fr

ABSTRACT

In this paper, we are presenting a work carried out by a research group on the history of mathematics for secondary school teaching in France. The goal of this work is to introduce the notion of area from a historical perspective, using various media as ancient texts, software and concrete objects to manipulate. We show how this original approach increases the student's motivation and understanding. To do this, we present the historical texts worked on in our group and then explain how we have integrated them into our teaching at different levels and as transitions between the levels. In particular, we present some "integration machines", whose manipulation aroused our audience surprise, whether students' as well as teachers'.

Introduction

In our IREM[1] research group on history of mathematics, we have been working on the issue of experiment in mathematics since 2014. After studying the use of the balance in mathematics (Equipe GRHEM, 2018), we started to work on the notion of areas throughout history[2].

That's how we came up with the idea of introducing the notion of area and the associated calculations, based on historical methods. This approach has a cultural purpose since it helps to familiarize our students with elements of the history of mathematics. In addition, it seems useful to us for a better understanding of the students for three main reasons:

- They often calculate areas with formulas they have learnt without having understood their meaning, as a kind of application of a "magic formula". The historical method allows them to obtain results with the sole force of their reasoning and makes more concrete formulas learned by heart in the past.
- The curriculum in France does not make any link between the different approaches to the concept of areas: 11 to 15 year old students[3] learn it in the context of geometry, high school[4] students use integrations in the context of calculus. The history of mathematics makes it possible to connect those different points of view as it provides a real transition in the curriculum between successive levels.
- Initiating students to ancient ideas and techniques can improve their motivation in mathematics. For example, the use of machines like planimeters and integrators stimulates their curiosity and makes them connect the current methods to the oldest.

[1] Since the late 1860s, the *Instituts de recherche sur l'enseignement des mathématiques* (Research institutes on mathematics education) bring together primary, secondary and higher education teachers to conduct research on mathematics education and thus participate in teacher training. The authors of this article are high school teachers.

[2] The link between these two subjects was the study of Archimedes' text about his "mechanical method", which will be discussed later.

[3] Secondary school is called "collège" in France.

[4] High school is called "lycée" in France.

In this paper, we present a possible progression from the level “Seconde” (first grade in high school, with students aged from 15 to 16 years) to the level “Terminale” (last grade in high school, with students between 17 and 18 years old). In French curriculum, rectilinear areas are studied in primary school and in “collège”. The areas then only reappear in “Terminale” with the integral calculation. In between, there is a gap in the school programs that we are trying to fill.

This progression is based on well-known historical supports, extending over three distinct periods:

- In Antiquity: *The Elements* of Euclid, *Measurement of a circle* and the “mechanical method” of Archimedes.
- In the 17th century: *Geometria Indivisibilus* by Cavalieri and the *Treatise on the Indivisible* by Roberval.
- In the 19th century: integration machines such as planimeters and integrators.

Our work takes place in several stages: first of all, reading and analysis of texts in a research group, then reflection on how to apply it in class, followed by experimentation with our students, and finally feedback to fellow teachers via presentations in workshops and training courses.

The pedagogical work we present to you followed this process and was presented at a workshop at ESU-8 in Oslo.

1 Greek quadratures in “Seconde”

1.1 Areas in Euclid’s way

1.1.1 Text presentation

In Euclid’s *Elements*, a plane figure is a shape and a magnitude (its area). Contrary to what seems familiar to us today, however, the areas were not evaluated by numerical units of measurement. To measure areas consists in equaling them geometrically by comparison or by addition for example. Indeed, common notion 4 states that “things which coincide with one another are equal to one another” and common notion 2 indicates: “if equals be added to equals, the wholes are equal” (Heath, 1956, p. 224). To square a figure is to construct (with ruler and compasses) the side of a square of the same area.

In his *Elements*, Euclid states theorems that consist in comparing areas of rectilinear figures:

“Book I – Proposition 35 – Parallelograms which are on the same base and in the same parallels are equal to one another. (Heath, 1956, p. 326) […]

Euclid also gives construction methods for moving from one geometric figure to another geometric figure.

Proposition 44 – To a given straight line to apply, in a given rectilinear angle, a parallelogram equal to a given triangle. (Heath, 1956, p. 341) […]

Proposition 45 – To construct, in a given rectilinear angle, a parallelogram equal to a given rectilinear figure.” (Heath 1956, p. 345) (fig. 1.1)

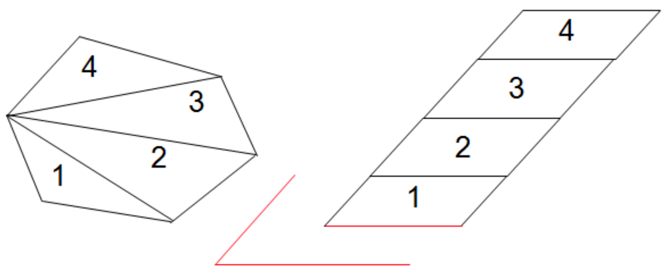


Figure 1.1: Euclid-Book I-Proposition 45[5]

Quadratures are therefore fundamental in this geometry since they make it possible to reduce any figure to a square, which can then be easily compared to other squares. The last proposal in Book II in fact gives the quadrature of any rectilinear figure:

"Book II – Proposition 14 – To construct a square equal to a given rectilinear figure." (Heath, 1956, p. 409)

To prove this last proposition, Euclid begins to construct a rectangle equal to the rectilinear figure, and then squares this rectangle.

In that way, one can square any rectilinear figure by decomposing it into triangles, each triangle being equal to a parallelogram, itself equal to a rectangle, finally equal to a square.

1.1.2 Pedagogical application

Without presenting the text to the students, we proposed them exercises in which this method could be applied. They had to determine areas without calculating, but rather by counting tiles, making decompositions or puzzles (see exercises 1 and 2 in the appendix).

In our workshop, we presented these problems to the participants, and we asked them which level of high school students they could give it to. First of all, they were surprised that we proposed this kind of exercises to high school students, because these notions are taught in "collège" and even in elementary school. In our classroom experiments, solving those problems wasn't so easy for our fifteen years old pupils (see works 1 and 2 in the appendix).

One of us also gave exercise 1 to her students in "prépa ECT"[6]. That class includes students from technological education[7], aged from 18 to 20 years. While they were in high school, they didn't study a lot of mathematics (about three hours per week) and so it wasn't their main subject at school, and nor their favorite one! The teacher was surprised at the reaction of students to this exercise. First, they counted each tile of the rectangle, and after counting, realized it was only the multiplication of the length by the width, and remembering the formula, lastly said: "That's why!"[8]

Finally, even an exercise that may look simple is not necessarily so for students, when it uses reasoning and unusual figure manipulations instead of the implementation of formulas. Coming back to the source of the calculation of areas by a method related to history has allowed students to rediscover the formulas learned long ago by understanding

[5] This figure has been realized with a geometry software as well as figures # 2, 4, 5, 6, 7 and 8.
[6] In this two years course, students prepare entrance contests to major business and management schools.
[7] STMG series in France "Science et technologie du management et de la gestion" (Financial management science and technology).
[8] In "ECT" class, this work started the "integration" chapter. The other experiments were conducted in "Seconde" in half classes, over time devoted to research, and at any time of the year, since these concepts do not appear in the official program of this level.

them. This leaves us thinking about the all-powerful place of calculations in the programs, especially when they are meaningless for students.

1.2 Squaring parabola with Archimedes' method

1.2.1 Texts presentation

After having squared all rectilinear figures by Euclid's method, one may wonder how the Ancients were able to square curvilinear figures.

In his treatise *Measurement of a circle* (Heath, 1897), Archimedes solves the quadrature of the circle in that way:

"Proposition 1 – The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle" (Heath, 1897, p. 91) (fig. 1.2).

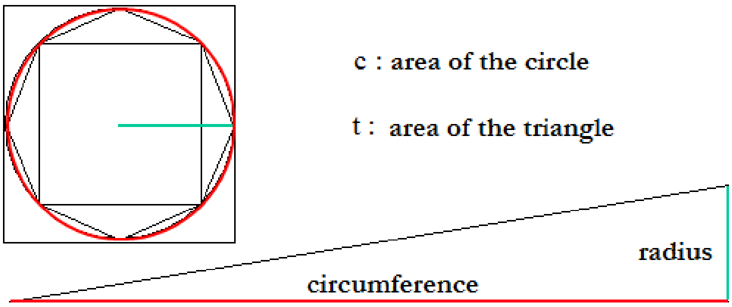


Figure 1.2: Squaring the Circle

In this text, Archimedes proposes a proof of the theorem based on a double *reductio ad absurdum*, which is a classical method in ancient Greek texts. Thus, the method of finding the result is not indicated.

In the 1630s, mathematicians made many reproaches to the Ancients for having hidden their method to find their results (Barbin, 1987, pp. 125-159). They tried to find a new "method of invention", which could solve any quadrature by giving the way they obtained them, and also for new curves, like the cycloid, defined by a kinematic process[9].

In this period, they didn't know that Archimedes gave one in a letter that had been lost for centuries, and discovered in 1906. The original Archimedes' text was copied around the 10th century on a parchment, which had been reused in the 12th century. The hidden text of the palimpsest had been entirely revealed by technological methods in the 2000s (Noel, 2008).

In this text, Archimedes describes his "mechanical method" and applies it among others to the quadrature of parabola (Heath, 1912).

"Proposition 1 – Let *ABC* be a segment of a parabola bounded by the straight line *AC* and the parabola *ABC*, and let *D* be the middle point of *AC*. Draw the straight line *DBE* parallel to the axis of the parabola and join *AB*, *BC*.

Then shall the segment *ABC* be $\frac{4}{3}$ of the triangle *ABC*." (Heath 1912, pp. 15-16) (fig. 1.3)

[9] See "Indivisibles Method" in section 2.

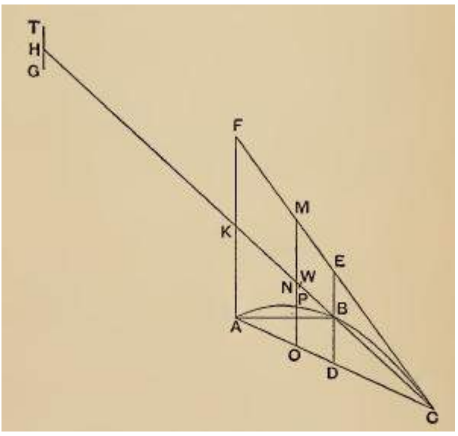


Figure 1.3: Archimedes' Mechanical Method (Heath 1912, p. 16)

He makes a both mechanical and mathematical proof, using the straight line *CK* as a lever around which he balanced the areas to be compared.

1.2.2 Pedagogical application

For teaching, the issues are the same: we can prove the result to our students in rigorous ways, but they won't understand the result if we don't give them a way to catch the "invention method" with which we found it. In this sense, working on Archimedes' mechanical method could be very instructive for our students.

We did not work on Archimedes' text with our students, because it seemed too difficult to address, especially because of the underlying properties of conics, which would require prior work to be introduced. On the other hand, we proposed an open problem to our students in "première" (aged from 16 to 17 years) aimed at conjecturing the area under a portion of a parabola.[10]

You can see the exercise and their works in the appendix (works 6 and 7). That area (below the parabola) represents a third of the area of the rectangle. Indeed, the area above the parabola covers four third of the triangle *ABC* (according to Archimedes' result), i.e. two third of the rectangle *ABCD*. This reasoning could already be the subject of an exercise in itself. Some students found the good ratio and others weren't too far from the result. Their methods were varied and imaginative.

In our workshop, we presented these exercises to participants, and asked them to think about a way to study Archimedes' "mechanical" proof with students. Indeed, we are still looking for some relevant and efficient schoolwork to propose on this subject in our class. We created a GeoGebra animation which details all the steps of the proof. That could be a good tool to build an educational sequence[11].

This pedagogical sequence is still a "work in progress" to be achieved but we are sure it might be rewarding for pupils to study that method. That exercise refers to the classical theme of the Archimedean lever in a particular way, and in a mathematical proof. It also appeals to the intuitive notion of balance while requiring strong geometric knowledge. It could show to students that mixing mathematics and mechanics is original and can be effective.

[10] The experiment was carried out in half classes, over time devoted to research, and at any time of the year, since these concepts do not appear in the official program of this level.
[11] The GeoGebra figure is available in following the link: https://www.geogebra.org/classic/chqspxrg.

2 In “Première”[12]: Cavalieri’s Indivisibles Method

2.1 Text presentation

In 17th century, the Italian mathematician Bonaventura Cavalieri publishes *Geometria indivisibilibus continuorum nova quadam ratione promota* (1635) (Cavalieri, 1653). In this treatise, he presents his new method called “Geometry of indivisibles” in these terms:

“If, between the same parallels, any two figures are constructed, if inside them, any straights lines are drawn equidistant from the parallels, and if the portions included in any one of these lines are equal, then the plane figures are also equals to each other” (fig. 2.1)

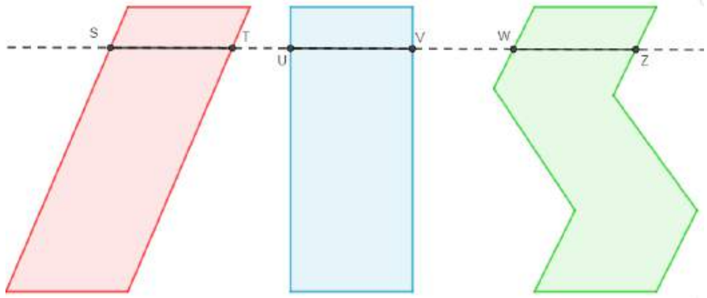


Figure 2.1: Cavalieri’s Indivisibles

This new process is quite simple to understand and is easily applicable to students.

2.2 Pedagogical application

We proposed an exercise, based on that principle, to our classes in level “première” (students aged from 16 to 17 years). Its statement is contained in the appendix as “Exercise 3”, as well as one student’s work (work 3). As you can see in question 1, this young student didn’t notice that the three areas were the same. All the other students made the same mistake. In question 2, a few pupils realized that the figures had the same area.

Students made also a construction on GeoGebra software to visualize the proof of the circle area with indivisibles method, proposed by Cavalieri (fig. 2.2):

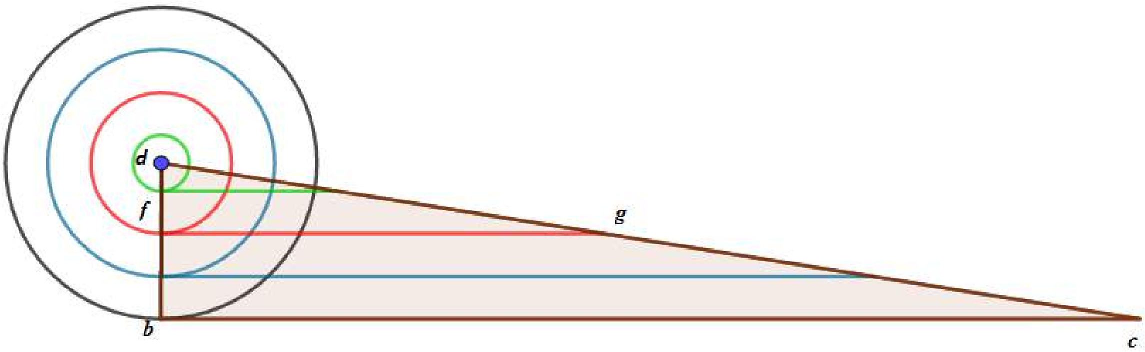


Figure 2.2: Squaring a Circle by Cavalieri’s Method

Our students easily built it on GeoGebra and were able to experience the indivisible thanks to the “trace” function of the software. They were thus able to understand visually the formula of the area of a disc.

We showed exercises and student works to the participants of the workshop. They could also watch and manipulate the GeoGebra animation about the circle area. We discussed about the relevance of that kind of exercise and the contribution of such a

[12] Second grade in high school, with students aged from 16 to 17 years.

method to the training of pupils.

Compared to the proof of the same theorem by Archimedes, this way of proving ~~is~~ could be more intuitive and understandable for the students. A double proof by the absurd does probably not have a meaning for them, contrary to this very visual evidence, which allows them to understand the profound reason for this result. We plan to build a new learning sequence in this sense, with a parallel study of Archimedes' and Cavalieri's proofs, in order to verify this theory.

2.3 To go further

To be complete on that subject, and go further with certain students, it is also possible to talk to them about the criticisms encountered by indivisibles method in the 17th century. Indeed, Cavalieri's contemporaries like Guldin[13] criticized his method because it generated paradoxes. The two most famous ones are the followings (fig. 2.3, 2.4 and 2.5):

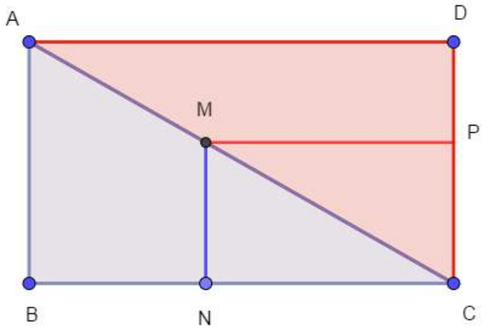


Figure 2.3: Paradox 1

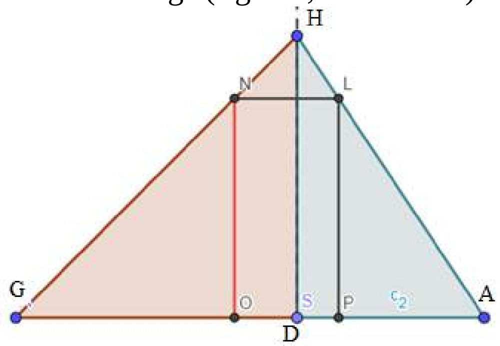


Figure 2.4: Paradox 2

"Do blue and red triangles have the same area?"

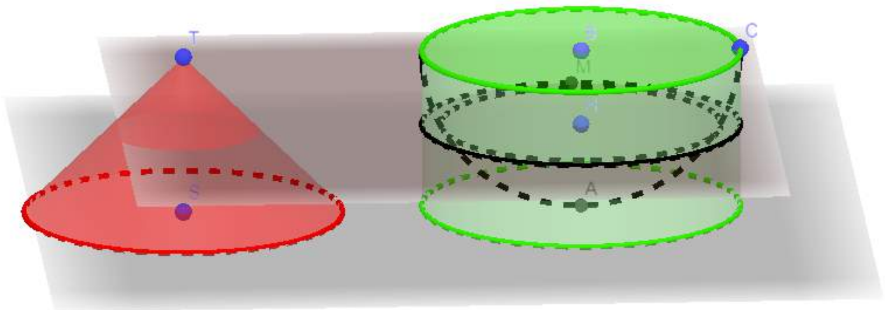


Figure 2.5: The Bowl's Paradox: "Is a point equal to a circle?"[14]

On these few examples, we can see how Indivisibles' method led mathematicians to ask questions about infinity and the infinitely small. That kind of queries can arouse students' curiosity, make them think about the way science works and progresses, the barriers people have had to overcome to build mathematics as we know them today. A good source to understand how Cavalieri improved his method to solve this kind of paradoxes is his letter to Torricelli of April 5th 1693 (Roberval, 1693, pp. 283-302).

[13] Paul Guldin (1577 – 1643) was a Swiss Jesuit, mathematician and astronomer.

[14] Paradox treated by Galileo (Galileo 1914). He proved that the cone and the bowl have the same volume, which gave, passing to the limit, the equality between the disc and the point.

3 Transition from “Première” to “Terminale”: Roberval’s indivisibles method

3.1 Text presentation

The last step in our progression in high school curriculum is the presentation of Roberval’s proof for squaring the parabola.

Still in the 17th century, Gilles Personne de Roberval[15] developed his own indivisibles method, in response to a problem put by Marin Mersenne[16]. Indeed, in the 1630s, Mersenne asked Roberval to determine the area of a portion of a cycloid. To answer him, Roberval devised a method of invention of the tangents and a method of quadratures with the help of the indivisibles. He applied this method to “all the curves” known at the time, solved the problem of the cycloid posed by Mersenne, and then obtained the quadrature of the parabola. He showed that the area of the portion of the parabola is equal to two-thirds of the area of the rectangle. (Walker, 1932, pp. 181-182) (fig. 3.1)

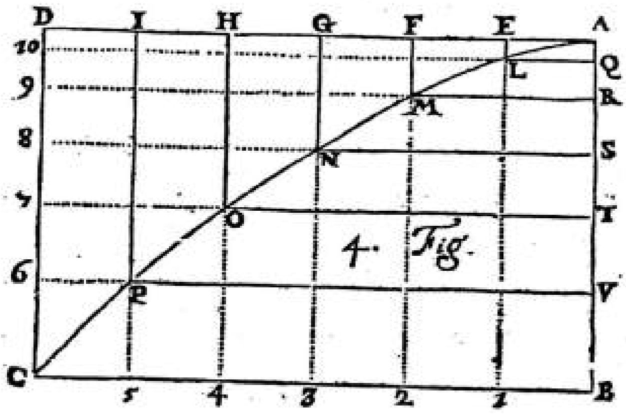


Figure 3.1: Roberval’s Quadrature of Parabola (Walker, 1932, p. 181)

His proof was based on the Indivisibles principle and a formula of sum, which he established a little before in his treatise.

3.2 Pedagogical application

We introduced this to our students in three steps:

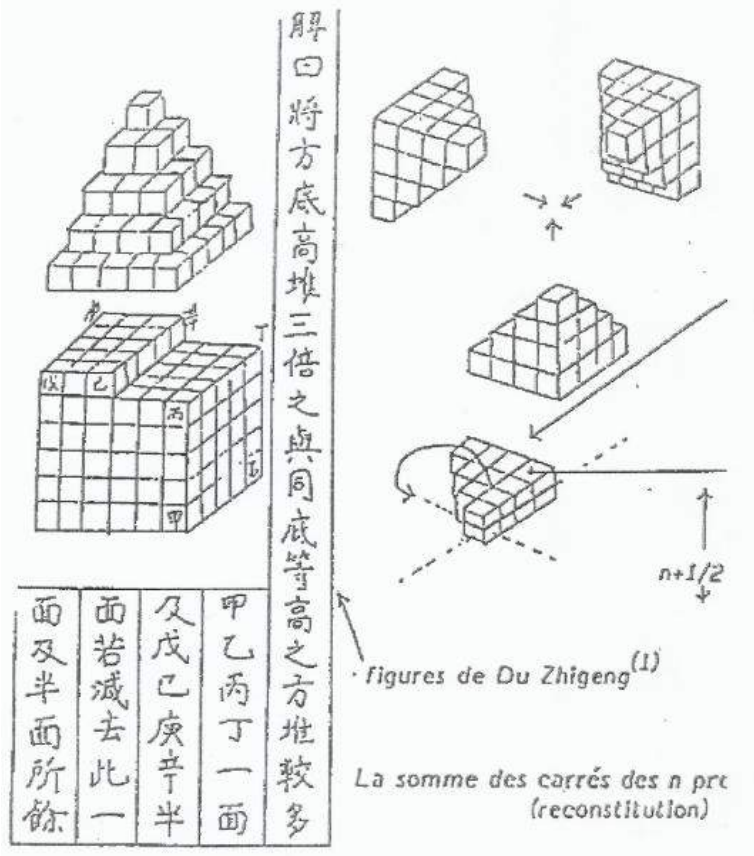
- Firstly: they had to work on the sum of squares formula. They had at their disposal wood pyramids to assemble in order to form a rectangle parallelepiped. From this 3D puzzle, they had to guess the formula of a sum of squares (pict.3.1 and 3.2).

Picture 3.1: Students at work

Picture 3.2: Students have found

[15] Gilles Personne de Roberval (1602 – 1675) was a French mathematician.
[16] Marin Mersenne (1588 – 1648) was a French cleric, scholar, mathematician and philosopher.

They could also watch a video about this “Chinese puzzle”[17]. Furthermore, they had an iconographic document to refer to (doc. 3.1).



Document 3.1: Sum of Square

To further study this formula, they had an exercise to do on a spreadsheet[18]. The point was to see that a sum of squares is not so far from the third of the cube of the highest number of the sum, and that the difference is all the smaller as the number of terms is big.

- Secondly: they had to read and try to understand Roberval’s text (Walker, 1932; Roberval, 1693) and to illustrate it by completing a GeoGebra animation[19]. The idea was to build and see the indivisibles with the help of dynamic geometry, from the figure of the text.
- Thirdly: they had to read and try to catch the idea of the end of Roberval’s text, in which he extended his result to other power functions. Exercises were thus proposed to train students to calculate areas below curves in different configurations.

This sequence is the opportunity to make a transition between the “Première” curriculum (students aged from 16 and 17 years), and the “Terminale” one (a year later)[20]. Indeed, in “Terminale” class, students work a lot on integral calculation. These activities were intended to prepare students to develop reasoning on integrals, without resorting to calculate primitives.

[17] [Math Help]. (2015, Jan 14). *Sum of squares* (video file). Retrieved from: https://www.youtube.com/watch?v=kZTFrv3vRgg.

[18] You can see the spreadsheet following the link: https://docs.google.com/spreadsheets/d/1-dNkYiK_16IXT1Aa7yLitB9R_ZxxMlAAzq4gxmA90Z0/edit?usp=sharing.

[19] You can see the GeoGebra file following the link: https://www.geogebra.org/classic/a5436kdu.

[20] That’s why this sequence was experimented at the end of “première” and ended in some classes at the beginning of “Terminale” (the third step).

Participants of the workshop had access to all the pedagogical material: pyramids, videos, documents, GeoGebra and spreadsheets animations. For example, they could experience the difficulty to make the puzzle of the Chinese pyramids (pic. 3.3).

Picture 3.3: Making the puzzle at the workshop

They also could notice that, working on these exercises, some students had found by themselves the primitive formula of power functions. Some works are shown it in the appendix (works 4 and 5).

This part of our work is at the margins of our theme "how to calculate areas without formulas", because Roberval's proof requires to use the sum of squares formula, and is definitely more computational than Cavalieri's or Archimedes' approaches. However, we wanted to introduce it because it is a very rich teaching sequence for our students. It mixes the reading and understanding of historical texts with activities on spreadsheets and GeoGebra software as well as manipulation of concrete material. We had obtained results beyond our expectations, from the point of view of acquisition as well as the one of students' motivation and interest.

4 In all classes: squaring with instruments and machines

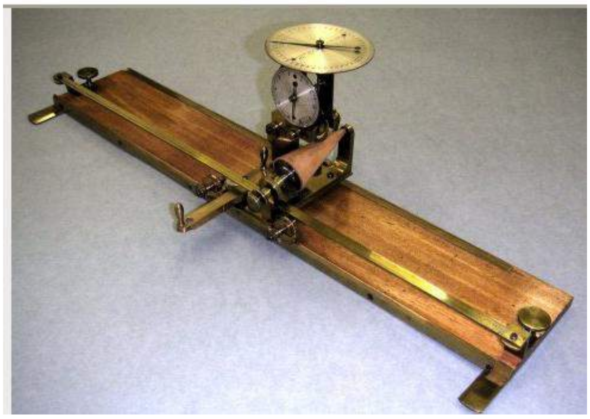
4.1 Machines presentation

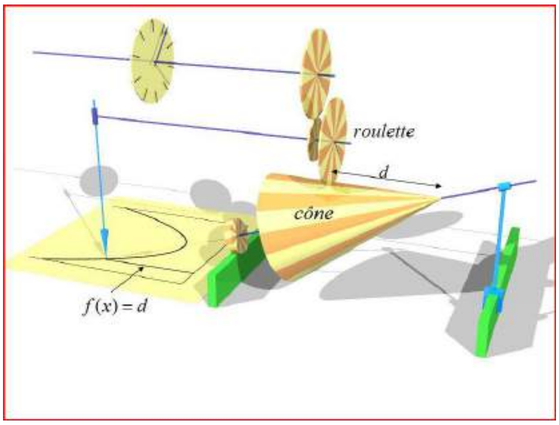
Squaring with machines is a method almost forgotten nowadays. The use of computers has made it obsolete. But it was a fairly common way in the 19th century. The need was great in many fields like: computing stress in civil engineering, evaluation of the average power of a machine, calculation of the property tax, measure of the extent of a forest from a cadastral map. All those operations require the determination of the area of a surface.

Economic, technical and industrial issues were therefore important.

As early as 1814 engineers invented machines like planimeters to measure the area of a surface by going along its contour and like integrators to draw an integral curve. These machines were used until the 1970s (Tournès, 2003; Gatterdam 1981).

One machine has been invented in 1814, by Johann Hermann in Germany. It had been named a "Cone planimeter" (pic. 4.1 and 4.2):





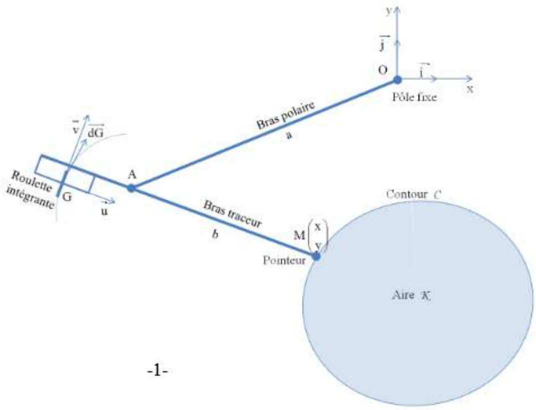
Picture 4.1: Cone Planimeter Picture 4.2: Cone Planimeter's Scheme

A stylus follows the curve, whose equation is $y = f(x)$, that delimits the surface, whose area one wants to measure. When the stylus holder moves with respect to the carriage in the direction (Ox), it causes an identical displacement of the carriage, thanks to a small disk. When the stylus holder moves with respect to the carriage in the (Oy) direction, it drives a disk, secured to a small toothed wheel, which rolls without slipping on the cone. The set of two toothed wheels is a speed reducer and drives the dial hand. For an elemental shift along the x-axis, the wheel rotates (with a coefficient close depending on the dimensions of the device) at an angle $f(x)dx$ and, for a movement along a segment [x_0, x], it rotates from a total angle: $\int_{x_0}^{x} f(t)dt$.

You can see how it works on the video linked here :

http://images.math.cnrs.fr/Un-planimetre-a-cone.html (Ghys & Leys 2009)

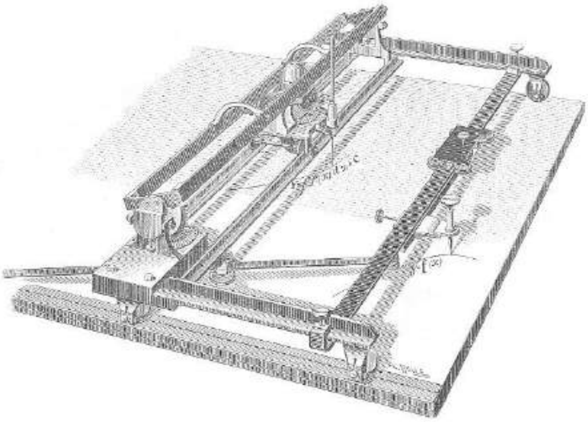
Another one was called "polar planimeter", invented by Johan Amsler in Swiss in 1854 (pic. 4.3 and 4.4):



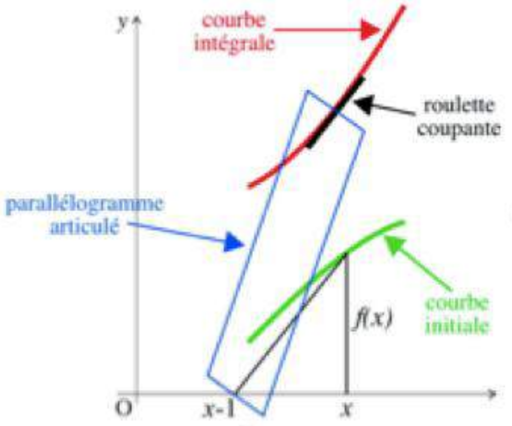
Picture 4.3: Polar Planimeter Picture 4.4: Polar Planimeter's Scheme

When the pointer moves, the wheel rotates. The number of turns is proportional to the component perpendicular to AM of the pointer movement. When the curve is traveled, the number of turns of the wheel is proportional to the length of the curve. Finally, by a calculation in which we use Green's theorem which makes it possible to transform a curvilinear integral into a double integral, we show that the number of turns of the wheel is proportional to the area of the domain. (Gatterdam, 1981).

There is also another one, made by Abdank-Abakanowicz in Poland in 1876, baptized "integrator" (pic. 4.5 and 4.6) (Abdank-Abakanowicz, 1886):



Picture 4.5: Integrator



Picture 4.6: Integrator's Scheme

This instrument draws an "integral" curve of the curve whose area it delimits. The tangent of the angle alpha is the ordinate $f(x)$. The other side of the triangle is orthogonal to the side of the articulated parallelogram. Parallelism "preserves" orthogonality, the inclination of the tangent to the integral curve is equal to the ordinate $f(x)$.

You can see one copy there made in Lego technics :

http://www.nico71.fr/integraph-graphic-planimeter/ (Nico71'Creations 2017)

4.2 Pedagogical application

In our classrooms, we intended to educate our students to the difficulty of determining non-geometric form areas. So we proposed them this exercise (doc. 4.1)[21].

The method of the grid works, but is not very practical to apply, and gives approximative results, hence the need for a more efficient technique that are the machines.

We have bought a polar planimeter to use with our students in our classes. It has been tested to determine the area of a square, and then of a circle. As these areas can be obtained by calculation, it was possible to check the result after manipulation. It was reliable to the nearest ten-thousandth.

Participants of the workshop in Oslo really enjoyed manipulating this material, and were also surprised at the accuracy of the result (pic. 4.7).

The use of a simple machine to get the result of an area precisely is surprising for the students. It is an original and practical method.

Thereafter, it is possible to deepen this exercise with the best students by proving the mathematical principles underlying these machines.

[21] This sequence was experimented in "ECT" class.

IV. How to calculate area of a non-rectilinear figure

1) by a finer and finer grid?

In Figure 1, area unit is a tile. Give a frame of area S delimited by the curve.

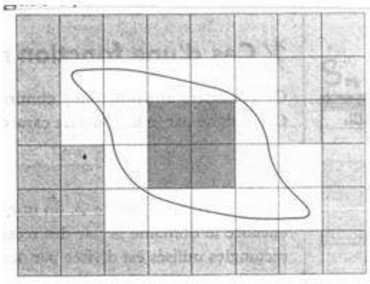
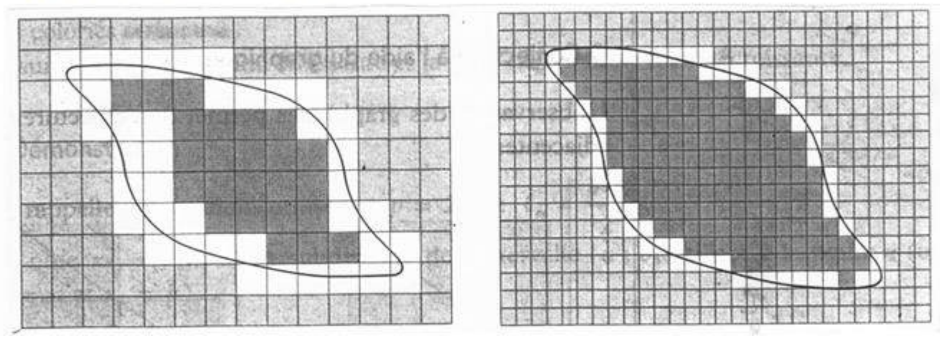


Figure 1

In Figures 2 and 3, we have squared the plan more and more finely:
In Figure 2, each square represents ¼ area unit, then 1/16 area unit for Figure 3.
Then give two more surroundings of S.



Document 4.1: Non Rectilinear Areas

Picture 4.7: Using Polar Planimeter at the workshop

5 Conclusion

Throughout the progression we have presented, the most important feature is the fact that the students were able to grasp what an area is, beyond the simple formulas applied hitherto without understanding them. Showing them historical methods, in their great diversity, is a way to achieve this. The originality and the beauty of the reasoning put into effect can more win their adhesion to mathematics in a more general way.

In addition, these activities can be very well integrated into French high school curricula: from elementary geometry taught in “Seconde”, to function and sums concepts seen in “¨Premiere”, until integrations learned in “Terminale”. They also create links between the “collège” and high school, as well as being transitions between the different levels of high school.

As they were often carried out in small groups, sometimes in the computer room, the sessions were well experienced by the students, they motivated and interested them. They appreciated the manipulation of objects that allowed them to anchor their knowledge on concrete experience, which is rather rare in the current teaching. The diversity of media (historical texts, objects, videos, software) gave them opportunities to express broader skills than in traditional mathematical activities. We were able to see the main benefits of these sequences: a more concrete meaning given to the areas, a better understanding of the formulas and calculations, and an easier access to the integral calculation.

For the coming periods, we plan to continue this approach on another subject, that of tangents. Here again, it is a theme that closely combines geometry and analysis, that poses real difficulties of understanding in high school and that would benefit from the contribution of history.

We hope that the participants of the workshop and our readers will find this project inspirational, and will have as much pleasure in transmitting these notions as we had.

REFERENCES

Abdank-Abakanowicz, B. (1886). *Les Intégraphes*, Paris: Gauthier-Villars.

Équipe GRHEM (2018). *Expériences et théorie : la balance et le poids suspendu d’Archimède à Camus*. In É. Barbin, D. Bénard, G. Moussard. *Les mathématiques et le réel. Expériences, instruments, investigations* (ch. 1, pp. 15-32). Rennes: Presses Universitaires de Rennes.

Barbin, É. (1987). Heuristique et démonstration en mathématiques: la méthode des indivisibles au XVIIe siècle. *Fragments d’histoire des mathématiques tome II* (pp.125–159). Paris: APMEP.

Cavalieri, B. (1653). *Geometria indivisibilibus continuorum nova quadam ratione promota.* Bologna: Ex-typo-graphia de Duciis.

Chevalier A. (1995). Introduction aux aires et volumes dans une perspective historique. In F. Lalande, F. Jaboeuf, & Y. Nouazé (Eds.), *Actes de la première Université d’Été Européenne sur l’Histoire et Épistémologie dans l’Éducation Mathématique*. (pp. 143-155). Montpellier: IREM de Montpellier, Université Montpellier II.

Galileo, G. (1914). *Dialogues concerning two new sciences*, trans. Crew, H., De Salvio, A. New York : Macmillan.

Gatterdam, R. W. (1981). The planimeter as an Exemple of Green’s Theorem. *The American Mathematical Monthly, 88*(9), 701-704.

Heath, T. L., (1912). *The method of Archimedes*. Cambridge: Cambridge University Press.

Heath, T. L. (1956). *The thirteen books of Euclid’s Elements* (p.326). New York: Dover Publications.

Heath, T. L., (1897). *The work of Archimedes*. New York: Cosimo classics.

IREM de Basse-Normandie Cercle d'Histoire des sciences (1999). *Aux origines du calcul infinitésimal*. Paris: Ellipses.

Mercier, J-P. (2012). Le volume de la pyramide chez Euclide, Liu Hui, Cavalieri et Legendre. In É. Barbin (Éd.), *Les mathématiques éclairées par l'histoire* (pp. 91-112). Paris: Vuibert.

Merker C. (2004). La méthode des indivisibles racontée lors d'un stage. *Repères-IREM*, N°54, pp. 57-76.

Noel, W., & Netz, R. (2008). *Le codex d'Archimède – Les secrets du manuscrit le plus célèbre de la science*. Paris: JC Lattès.

Noel, W., & Netz, R. (2007). *Archimedes Codex : How a Medieval Prayer Book is Revealing the True Genius of Antiquity's Greatest Scientist.* Boston: Da Capo Press.

Nouet, M. (1992). Histoire des mathématiques en classe de terminale, *Repères-IREM*, N°9, p.15-33.

Roberval, G. P., (1693). *Lettre à Torricelli*, in *Divers Ouvrages mathématiques et de physique par MMrs les savants de l'Académie Royale des Sciences*) (pp. 283–302). Paris: Imprimerie Royale.

Tournès, D. (2003). *Du compas aux intégraphes: les instruments du calcul graphique*. Paris: Repères IREM.

Walker, E. (1932). *Quadrature of the Parabola, A study of the Traité des Indivisibles of Gilles Personne de Roberval*. New York: Bureau of Publications, Teachers College, Columbia University.

WEBOGRAPHY

Nico71'Creations (2017, Feb 11). *Integraph: Graphic Planimeter.* http://www.nico71.fr/integraph-graphic-planimeter/

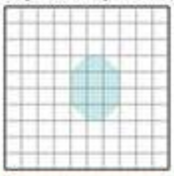
Ghys, E., & Leys, J. (2009, Nov 4). *Un planimètre à cône.* http://images.math.cnrs.fr/Un-planimetre-a-cone.html

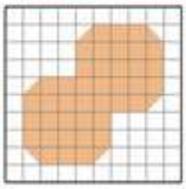
[Math Help]. (2015, Jan 14). *Sum of squares* (video file). Retrieved from: https://www.youtube.com/watch?v=kZTFrv3vRgg.

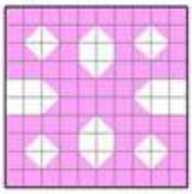
ACTIVITE 1 CALCULS D'AIRES SANS FORMULE
(EXERCISE 1) CALCULATIONS OF AREAS WITHOUT FORMULAS)

I. Calculez l'aire des zones colorées de chaque grille. Un bloc représente 1 unité carrée (u^2).
(Calculate area of colored areas of each grid. One block represents 1 square unit (u^2)).

1)

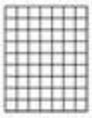

2)


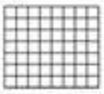
3)


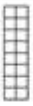
4)


II. Calcul de l'aire d'un rectangle (Calculating area of a rectangle)

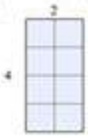
a) Calculez l'Aire. Chaque □ = 1 unité (u^2).
(Calculate area. Each □ = 1 unit (u^2))

1)


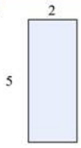
2)


3)


b) Calculez l'aire du rectangle en le décomposant par carreaux. Comme le premier exercice.
(Calculate area by breaking it down by tiles. Like the first exercise.)

1)


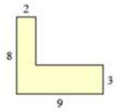
2)

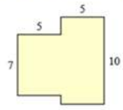

3)


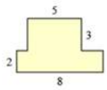
4)

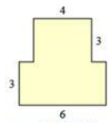

III. Calcul de l'aire d'une figure rectiligne
(Calculation of area of a rectilinear figure)

a) Calculez l'aire totale de chaque figure. Les mesures sont en millimètres (mm).
(Calculate total area of each figure. Measurements are in millimeters (mm))

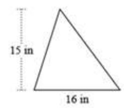
1)


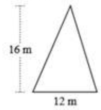
2)


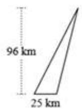
3)


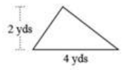
4)


b) Calculez l'aire de chaque triangle. Les triangles ne sont pas à l'échelle.
(Calculate area of each triangle. Triangles are not to scale.)

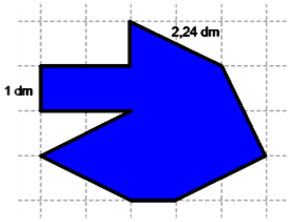
1)


2)


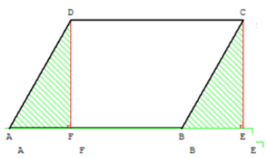
3)


4)


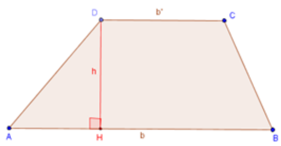
c) Calculez l'aire de cette figure en la décomposant
(Calculate area of this figure by breaking it down)



d) Calculez l'aire du parallélogramme ABCD
(Calculate area of parallelogram ABCD)



e) D'une manière analogue, calculez l'aire du trapèze ABCD
(In a similar way, calculate area of trapezoid ABCD)



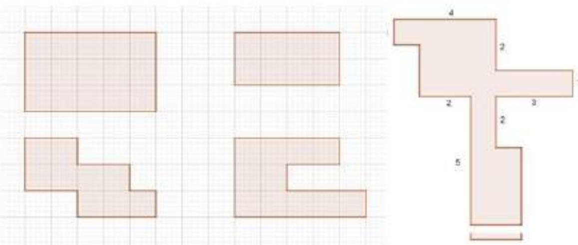
f) Aire sous une courbe rectiligne (Area under a straight curve) :

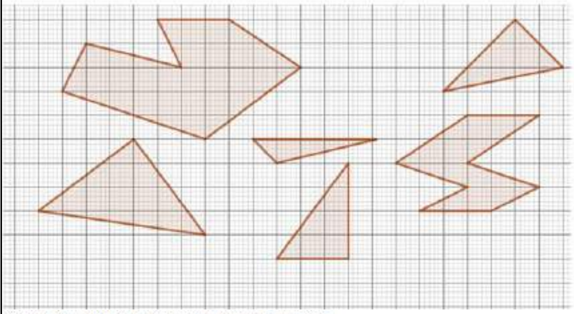
Calculez l'aire sous la courbe représentative de la fonction f définie par $f(x)=\begin{cases} x & si\ 0\le x\le 2\\ 2 & si\ 2\le x\le 5\\ \dfrac{x-1}{2} & si\ 5\le x\le 7\\ -x+10 & si\ 7\le x\le 10\end{cases}$

(Calculate area under representative curve of function f defined by $f(x)=\begin{cases} x & \text{if } 0\le x\le 2\\ 2 & \text{if } 2\le x\le 5\\ \dfrac{x-1}{2} & \text{if } 5\le x\le 7\\ -x+10 & \text{if } 7\le x\le 10\end{cases}$)

Activité 2 (Exercise 2): Aires rectilignes (Rectilinear areas)

Déterminer les aires de chacune de ces surfaces, en unité de quadrillage
(Determine areas of each of these surfaces in a grid unit.)

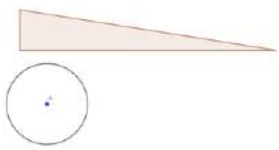




Exercise 3: To compare areas without computing

1. Sort these surfaces in ascending order of their areas

2. Of this triangle and disk what is the surface of bigger area?



Students' works :