HISTORY, EPISTEMOLOGY AND TEACHING MATHEMATICS

A challenging partnership?

Caterina VICENTINI[1,2,8,9,13,14] **, Nathalie CHEVALARIAS**[3,4,10,15]**,**
Kathleen M. CLARK[5,11,]**, Michel ROELENS**[6,7,12,16]

[1]Mathematics and Physics Teacher at Liceo Scientifico "Michelangelo Buonarroti", Monfalcone (GO), Italy
[2]Mathematics Teacher at Herschel Grammar School, Slough, London SL1 3BW, UK,
[3]Mathematics Teacher at Lycée Pilote Innovant International, Jaunay-Marigny,
[4]member of IREM Poitiers, France
[5]Associate Professor, Florida State University, Tallahassee, Florida, USA
[6]Lecturer of Geometry and Mathematics Didactics at University Colleges Leuven-Limburg, Faculty of Teacher Education, Diepenbeek, Belgium,
[7]Mathematics Teacher at Maria-Boodschaplyceum, Brussels, Belgium

[8]caterina.vicentini@goiss.it, [9]cvi@herschel.slough.sch.uk,
[10]nathaliechevalarias.irem@gmail.com, [11]kclark@fsu.edu, [12]michel.roelens@ucll.be

ABSTRACT

In this paper we approach an analysis of the partnership between history, epistemology and teaching in 'real maths classes'. We will do so by starting our consideration as much as possible under the 'real maths' teachers' perspective. The initial focus is on comparing and contrasting the situations in different countries. In the follow-up, however, we investigate a bit further, entering deeper into specific situations and suggesting good practices. To conclude, more theoretical features will be briefly discussed, as well.

1 Introduction by the panel coordinator

A typical danger in mathematics education research, and in my opinion the main reason why it has less effectiveness than it could have, is the general lack of open discussions taking into account teachers' feelings as well as "orthodox statistics" or academic literature on the subject. This field could benefit from a more humanistic and realistic setting if teachers with 30 or 40 years of classroom experience were respected when expressing their conceptions as much as the academic researchers usually are. I was fortunate enough to be able to learn this way of working by Nicolas Rouche at the Université Catholique de Louvain in Louvain-la-Neuve (BE) in the late 1980s. For this reason, I would like to dedicate this work to him. Moreover, I feel particularly thankful to Évelyne Barbin and all the French ESUs' participants for having let their inter-IREM conferences become European Universities in which the original spirit of Montpellier

[13] Panel coordinator.
[14] Also, Member of the British Society for the History of Mathematics; Professor on contract of History and Epistemology in Mathematics Education, Università degli studi di Udine, Italy; and Professor on contract of Teaching Maths in CLIL, UFR-FVG Trieste, Università "Ca' Foscari" di Venezia, Italy.
[15] Also, co-responsible of CII Epistemology and History of Maths, France.
[16] Also, Editor of *Uitwiskeling*, journal for mathematics teachers, Belgium.

1993 is still alive 25 years later.

I am particularly glad to also express my gratitude to the organizers who invited me. I am essentially a "general maths teacher" and therefore I could have been considered not entitled to coordinate a Plenary Panel dealing with general questions as the presence and the integration of History and Epistemology of Mathematics in the curricula. Furthermore, I have had the opportunity to invite at the table with me two other "general maths teachers": Michel and Nathalie. The presence of Kathleen has contributed to balance the point of view that we share.

The main aim of this panel has been to give a voice to the teachers, trying to discover in which measure they are keen on letting history and epistemology of mathematics enter their classrooms, whether they do it, and why. Then we concentrated on comparing and contrasting the situations in different countries in order to set the right directions for our research.

We moved from the analysis of a questionnaire we implemented in our particular countries and regions. We originally took into account the Flanders in Belgium and the Netherlands, France, Italy, and Florida and Massachusetts in the USA. Ultimately, we also had representation from China, as well. I would like to thank very much indeed Yue Zengcheng, PhD, from East China Normal University in Shanghai, who wrote me kindly proposing himself as a collaborator.

Eventually, following to the suggestion given by Costas Tzanakis, we decided not to discuss in detail the results of the questionnaires in these proceedings. We would rather prefer to emphasize the possible reasons which led to such results, and discuss some steps towards 'solutions' considering the different teaching and teachers' training situations, as well as the very recent changes in the programs or guidelines of some of the nations involved.

To conclude, I would like to express my gratitude to my dear friends Daniele Gouthier and Colette De Coster. Daniele is a very eclectic freelance mathematician, who teaches at the Master of Science in Communication at the Scuola Superiore di Studi Avanzati (SISSA) in Trieste. He helped me both with the questionnaires and with the review of this paper. Colette works as professor at Université de Valencienne. She recently came to Trieste as visiting professor at the local University. In such occasion I met her and while having a pleasant meal we informally discussed about the training courses for French young teachers held by her university.

2 The starting point of the panel: the questionnaire

As has already been stated, this panel report does not focus on the work of researchers in history of mathematics or in educational science. The main aim is to instead give a voice to the teachers, focussing on those who work with adolescent pupils. Therefore, we will discuss about the difficulties that teachers meet, the possible reasons of such difficulties and consequently propose some realistic steps which could help them to introduce history and epistemology in their lessons.

Our panel's starting point was a questionnaire we adapted for the regions involved. The questionnaire can be found in the conference site, where it was published to allow ESU8's prospective participants to express their opinion as well (https://esu8.edc.uoc.gr/esu-8-main-themes-plenary-sessions/).

The questions collected information about the teachers' background in the history of

mathematics, their epistemological awareness, the introduction of history of maths in their class practice, how and why they do it or why they do not.

Kathleen, with the assistance of two colleagues, was able to distribute the questionnaire to two different teacher populations professional networks. In total, 149 mathematics teachers, predominantly from Florida and Massachusetts, completed the questionnaire.

Nathalie analysed the situation in France. She first proposed the survey to all the teachers of Poitou-Charentes, the region where she teaches. She received approximately 230 responses. To get more feedback, she asked for help from colleagues in other regions. In the end, the questionnaire was given to teachers both in Paris and in Franche-Comté. In those two regions, she only managed to contact in-service training teachers; ultimately, the results of the survey were similar and she received 530 responses altogether.

As far as the Michel's work is concerned, he took into account the Flanders (the Flemish part of Belgium) and the Netherlands (with the collaboration of Jeanine Daems and Desiree van den Bogaart). In Flanders, the questionnaire was distributed on the website of *Uitwiskeling* (a journal for mathematics teachers that he has edited since 1984). In the Netherlands, the questionnaire was posted on the Facebook group of Dutch mathematics teachers. In total, 127 teachers responded.

Caterina submitted the questionnaire to 1500 teachers belonging to the national association "Mathesis", randomly spread around Italy with the help of Professor Salvatore Damantino, chair of the section of Udine. The response rate was very poor: only 3%. She underlines that it is very difficult to talk about "the Italian situation". Italy is a quite recent country that was dominated by cultures really distant from one another. It is neither as small as the Belgian Flanders or the Netherlands nor it has the shape of a hexagon like France. Even though it is one state, federal issues have been influencing the political choices and the personal feelings since its unification in 1861 and undoubtedly in a stronger way than before during the last 25 years. These facts are strongly related to quite a large number of different situations moving north, south and east or west within Italy. A sort of "average" is not really meaningful. However, even though Italy is neither as "systematic" as France, nor as clearly "split in parts" like Belgium or the USA, there are some features that are quite uniformly widespread and are worth being discussed.

In China, the questionnaire was implemented by Zengdeng through the WeChat app, a very popular social network in China (more than one billion people regularly use it). Furthermore, he also contacted the teachers he is linked to. Altogether he received 64 responses.

3 Evidence of the inquiry

We start our reflection from the general results from the questionnaire responses. However, since the sample of teachers interviewed was certainly not well representative of the entire population, and therefore our survey was certainly biased, we do not present the inquiry evidence as a statistic. We prefer to enlighten some interesting features that are nonetheless worth being extrapolated, because quite a strong core of similar answers around the occidental parts of the world involved. In China, due to a quite different cultural background, some differences arose.

In short, the common facts and feelings with very few exceptions are:

- Few teachers have followed a course in history of mathematics, either in their graduate program or in their initial training courses or master's; China was an

exception, where a course in history of mathematics is usually present and specially oriented to the introduction of the history in classrooms;
- Very few teachers are able to convey an epistemological position and teachers generally do not know what epistemology is;
- The majority is keen on making use of history of mathematics in their daily teaching;
- There are general opinions that history will:
    - render mathematics more human and interesting,
    - underline that mathematics are an historically contextualized cultural product,
    - allow a deeper acquisition of the conveyed notions,
    - make both the learning and the teaching more fun;
- When teachers integrate history in their teaching it is most often in an anecdotal way (to present a new topic, or a mathematician), more rarely solving old problems and almost never working on original sources;
- The rather superficial presence of history is allocated to a lack of knowledge, of training, of resources or of time.

4 General observations

Even though we have examined different cultural contexts, there are some overall teaching features that seem meaningful to us in order to understand the homogeneous situation which generally emerged.

First of all, mathematics is not always taught by mathematicians. The reasons can be different according to the geographical region involved, however this is a general fact that has essentially to do with the worldwide shortage of mathematics teachers.

On the second hand, the presence of history of mathematics in textbooks is usually characterized by a sort of juxtaposition. History is not integrated and does rarely permeate the way of presenting mathematical evidences. It is often used to share anecdotes or to introduce a mathematician, and whenever the presence is more substantial, it is either exiled in single problems or stored in very small "boxes".

Sources directly coming from research are usually perceived as too difficult to be adapted to "real classes" by "real teachers". This is also likely due to the very trendy standardization to which teaching has increasingly undergone. As a consequence, a dangerous need is perceived: the necessity of what during the panel Michel defined with a Flemish expression "gesneden brood" that can be translated as "bread already cut in slices".

However, in China the link between research on history of mathematics in an educational perspective and teaching of mathematics appears to be quite stronger.

In summary, the importance of history of mathematics was not strongly emphasized, nor it was included in the national final exam of high schools in the countries in which a final exam exists; even though China seems to be an exception again. However, it must be said that especially in France, but also in a weaker way in Italy, the situation very recently has changed. New ministerial indications have been added, as we will see in the next section.

In China, although they do not emphasize explicitly the importance of history of mathematics, they strongly enhance mathematical culture, and history of mathematics is

generally recognized as a rather important part of this.

5 Context-specific remarks

5.1 Specific features of French context

As far as the initial training in concerned, it has to be said that teachers in "college" (pupils aged 11 to 15 years) and "lycée" (pupils aged 15 to 18 years) are normally recruited through a competitive examination. In the last 10 years, in addition to the competition, teachers are required to have master's degree, as well. Very often, but not always, they enrol in a master's degree related to teaching in their subject and they are allowed to participate to the competition during the first year of their master's studies. The competitive examination is national and even if each university organizes the master's program autonomously, there exists official national guidelines to define the training.

The last reform of university in France introduced on average more history of mathematics in the training of the future teachers, but there can be many differences from one university to another. On the contrary, teachers recruited more than 10 years ago had had in general very little of history of mathematics in their courses, and some of them had no history at all.

Additionally, in order to face the shortage of mathematics teachers, numerous staff members are recruited without competitive examination to teach as temporary instructors just for 3-4 months, one year, or even more (depending on their status and the needs of the corresponding institution). In this case, teachers very rarely have had training in history of mathematics, and, if fortunate to do so, they have the possibility to undergo a training specially conceived for them; History of Mathematics is certainly not a priority.

What is more, there are also people who have a very different cultural background and "convert" to becoming mathematics teachers by taking from time to time some exams specially arranged for them. In this case, they may be interested in history because it allows them to understand a content they cannot cope with well enough, otherwise they are looking for didactical prompts ("bread already cut in slices").

Nowadays, history of mathematics is present either in the first or the second year of the master's degree. When introduced in the first year it brings to prospective teachers a minimum of culture in this field: names of mathematicians, their work, the fields of mathematics, the specificities and contributions of each period, but it is not very strongly linked with the programs and what can be taught to the pupils. However, if the training in the history of mathematics takes place in the second year of the master's degree, this is when students are interning with classes, and it usually starts from the examination of documents (e.g., primary sources) in order to construct teaching material to be tested in classrooms.

While the initial training in history of mathematics seems to have increased in recent years, in-service training has certainly decreased. In France it is quite easy to find online resources or books, especially edited by the *Instituts de Recherche sur L'Enseignement des Mathématiques* (IREMs), though not exclusively. However, it is not always simple to find the time to consult them and even more to understand them sufficiently while working independently. It is also not easy to "sort out" reliable from unreliable sources if someone does not have a minimal basic knowledge.

In textbooks, it is common for very little history of mathematics to be found and such

content is often limited to an image of a mathematician, his birth and death date, a small remark on his work, usually written in few lines and collocated in the page margin. Very rarely there are class activities or exercises entirely built on a historical situation.[17]

Therefore, teachers need to decide to undergo some voluntary training if they want to use original sources and they have to do additional work in case they want to become able to create activities based on history of mathematics.

The French curriculum does not require to teach precise aspects of history of mathematics but it does encourage teachers to use history:

> Elements of epistemology and history of mathematics naturally fit into the implementation of the curriculum. The knowledge of the name of some famous mathematicians, the time when they used to live and their contribution is part of the cultural background of any student with scientific training. The presentation of historical texts helps to understand the genesis and the development of some concepts. (Ministère de l'Education, 2010)

Recently the Villani and Torossian report (2018) reaffirmed the importance of the history and epistemology of mathematics:

> First, epistemology and history of the construction of mathematical notions, which bring a real didactic richness, are little taught in initial training. […] By taking advantage of history of mathematics, teachers place their teaching in the evolution thought. In addition, students are often sensitive to the "mathematics legend". Narrative can play a motivating role here. On the other hand, the epistemological lessons that emerge from history as the role of problems, the entanglement of concepts and techniques, the need of abstraction, etc., are obviously likely to contribute to training, in particular by overcoming short-sighted utilitarianism.

Apparently, this report has influenced some of the ministry decisions. In January 2019, new curricula were published and history of mathematics indeed takes a real important new place. The introduction indicates:

> It may be useful to illuminate the course with historical or epistemological contextual elements. History can also be seen as a fertile source of problems that clarify the meaning of certain concepts. The "History of Mathematics" items identify some possibilities in this direction. To support them, the teacher can rely on the study of historical documents. (Ministère de l'Education, 2019)

For each part of the curricula, some examples and indications are given. For example, in the part "Numbers and calculus" of the curriculum it is explained:

> The seemingly familiar notion of number is not self-evident. Two examples: the crisis caused by the discovery of irrationals by Greek mathematicians, the difference between "real numbers" and "calculator numbers". It is also important to highlight the gain in efficiency and generality brought by literal calculus, by explaining that a large part of mathematics could only develop once this formalism had stabilized over the centuries. It is possible to study ancient texts by authors such as Diophantus, Euclid, Al-Khwarizmi, Fibonacci, Viète, Fermat, Descartes and highlight their algorithmic aspects. (ibid.)

We still do not know what will be introduced in the textbooks, but hopefully teachers will be given more resources and activities to be encouraged to introduce history of

[17] Although a separate subsection for the United States context is not presented in Section 5, this description is true of textbooks in the US, as well. See Smestad, Jankvist, and Clark (2014) for additional details.

mathematics in their classes.

5.2 Specific features of Flemish context

The Belgian Flanders have a strong tradition in mathematics education, and still produce good results in comparative studies as PISA. However, they are facing problems that could influence future issues. The main critical points are the lack of a master's degree in mathematics and future teachers who have to learn everything.

In Belgium, to teach to pupils aged 15 to 18 years, a university master's degree is required. In the last decades, Belgium has been facing a shortage of mathematics students, accompanied by a shortage of mathematics teachers. Therefore, people having a master's in other subjects such as economics, biology, engineering, etc. often teach mathematics. Usually they do not feel a deep interest in mathematics and its history, and often they are less creative in doing mathematics and teaching it than mathematicians. They are more comfortable with ready-to-use materials and, as a consequence, publishers have been starting to provide these prompts. These kind of resources include exercises and tests with all the solutions, pre-prepared slides for projection on smartboards and other similar means. As a consequence, their creativity is not enhanced and tends to be more and more reduced.

Alternatively, in order to teach to pupils aged 12 to 15 years, a professional bachelor's degree is needed. Many teachers start their bachelor courses with a poor background in mathematics. Their view of the subject is almost always limited to techniques and algorithms. Teacher trainers are consequently called upon to do a big work. They must: teach problem solving, spatial insight, reasoning, proving, speaking and writing about mathematics, recognizing mathematics in different contexts and something about the historical and cultural aspects of the mathematics they are supposed to use while teaching.

5.3 Specific features of Italian context

As already mentioned, in Italy mathematics is also not always taught by mathematicians. Teachers working in "scuola media inferiore", where pupils are aged between 11 and 14 years, can have a master's in biology, natural sciences, geology, chemistry, physics, and many other subjects since they are supposed to teach general sciences too. Professors working in "scuola media superiore", where students are aged between 14 and 19 years, on the contrary, need to have a master's in Mathematics, Physics, Engineering or Computer Sciences (and in rare cases, Economics).

What is more, as far as initial teacher training is concerned, the actual in-service teacher population is composed by essentially three different categories of people:

(i) Those who have been employed via a "concorso ordinario";

(ii) Those who have been employed via a "concorso riservato"; and

(iii) Those who have been employed after having had a university training followed by a thesis.

The teachers employed via a "concorso ordinario" only had to study their subjects and the laws that concern the school work; those who took part in a "concorso riservato", generally speaking, also had to attend some didactical courses; the youngest ones who had university training were supposed to study pedagogy, didactics and quite rarely, as already seen, history and epistemology courses, as well.

Another important remark to understand the Italian situation is related to the number of

hours dedicated to mathematics instruction in schools. In the "Liceo Scientifico opzione Scienze Applicate", the most scientific Italian school, students have only five hours of mathematics per week in the first year and only four hours in the following four years. To understand the historical reasons of this inadequate instructional time, the paper "Personal and Social Conscious and Unconscious Backgrounds in Mathematics Education (Vicentini, 1994) could be a useful reading.

Furthermore, in Italy the curricula are defined on a nationwide basis. However, since some years ago, the ministerial document has been named "indicazioni ministeriali", which means ministry's indications. Therefore, the ministry underlines the importance of certain contents, but does not oblige the teacher to deal with all the suggested topics. What is more, the "*independence*" and the "*full freedom*" of teachers' choices are underlined. To sum up, these indications are rather general. (see MIUR, 2010) Therefore, as a matter of fact, in Italy a teacher legally has a large autonomy in choosing what to teach and how to teach it. Practically speaking however, not many teachers use this legal freedom and there are many reasons for such a choice. To mention just one, in the final State exam the mathematics exam paper is the same for all the different types of "liceo scientifico" and it is issued from Rome for the entire country. Moreover, teachers do not feel they are really entitled to choose what to teach; therefore, they tend to adapt the didactical tradition. It must also be said that it is compulsory to "adopt" a textbook and there are only two syllabi that reach all together the majority of the students of Italian high schools (students aged between 14 and 19 years): one edited by Zanichelli in Bologna, whose authors are Massimo Bergamini, Graziella Barozzi and Anna Trifone; and the other edited by Petrini-De Agostini in Novara, whose author is Leonardo Sasso (recently together with Claudio Zanone).

As a result, what really happens in Italian classes is a melting pot of didactical tradition, of what is present in the two textbooks quoted above, what is written in the guidelines together with a few ideas coming from research in mathematics education (that can be in several cases very good, but do not generally thrill the majority of the teachers).

Concerning the presence and integration of the history of mathematics in the guidelines, the ministry refers to history, but mainly in the general part of the document in which all the subjects are mentioned. It is written: "According to the Lisbon's indicators the various subjects should be studied in a systematic, historical and critic way by reading, analysing and translating literary, philosophical, scientific texts as well as essays" (MIUR, 2010), but very few mathematics teachers usually read the general part. Therefore, the integration of the history of mathematics in the development of the regular curriculum is "de facto" not enhanced since the majority of teachers usually avoid a careful study of the entire document; instead, they concentrate on the section concerning their subject.

On the other hand, we must observe that very recently the ministry, which is about to change the structure of the State exam, has released the so called "quadri di riferimento" (landmark frameworks). In these frameworks it is clearly stated that "the problems may have an abstract or practical character and also contain references to classical texts or significant historical moments related to Maths" (MIUR, 2018); therefore, perhaps the importance granted to history of mathematics will increase in the near future (Rogers et al., 2015).

A final observation relates to the teaching materials other than textbooks. In Italy we do not generally have institutions like IREMs in France. We have the NRD instead, i.e. the

Nuclei di Ricerca Didattica, allocated in mathematics departments within some universities, in which "general maths teachers" together with mathematicians and researchers in mathematics education cooperate to enhance didactics of mathematics and to produce didactical materials they then disseminate. However, these institutions are not specifically devoted to history or epistemology of mathematics; therefore, good material is not easily available. This is especially true of learning how to use original sources, which is quite difficult. (Though, there exists the book by Dematté and Furinghetti (2004), whose title (in English) is *Doing Mathematics with Historical Documents*, and an important site edited by Giorgio Bagni, which was last updated in 2009, the year of Giorgio's death (http://www.syllogismos.it/).)

6 Suggestions in view of a more promising future

In our opinion, a course about history of mathematics is not enough and perhaps also not the most important despite being strongly recommended.

What really enables teachers to let history permeate their teaching is having a quite extensive experience of dealing with historical original sources in their pre-service and in-service training. Using them while working in groups on the design of workshops or for lessons in which historical aspects of mathematics are re-discovered; to solve together historical problems; or translating, interpreting and contextualizing original sources into a portion of the curriculum. Moreover, it is fundamental for the teacher to give such types of lessons firstly to their peers in order to test them and afterwards to pupils, with the guidance of an expert supervisor.

What is more, the work based on original sources is surely very interesting and rich, but it can be too challenging for prospective teachers or pupils who have to learn the mathematical content in parallel with navigating the original source material. Having to simultaneously combine various goals, we rarely have the time to focus on the analysis of the old texts. Therefore, it can be a good idea to do it from time to time as an optional work.

Alternatively, we might consider ourselves allowed to treat historical contents in an anachronistic way, e.g. explaining a historical method in modern language, using equations written in our modern way, visualizing with GeoGebra even though the software did not exist, etc. Of course, we have to warn students that it is anachronistic and emphasize the moments in which the anachronism is particularly strong, clarifying in which period the tools we are making use of are born and by which civilization they were introduced. In China, this way of working is already a reality: the purpose of integrating history of mathematics into teaching is to achieve certain teaching goals and a rather consistent relationship between history and teaching has already been built (Wang, Qi, & Wang, 2017).

7 Some "virtuous" examples?

7.1 How Nathalie usually integrates History of Mathematics in her courses

Nathalie considers herself a "normal" teacher; she feels she has no lessons to give to colleagues on an "ideal" integration of the history of mathematics. She is used to think about how to make use of history through discussions with the members of the IREM's

commission on epistemology and history of mathematics, who mainly are researchers.

However, in her classes, she does as much she can, with the time and the constraints she has. Sometimes she takes the time to read a historical text with the pupils or to construct short exercises from a historical situation, but often she only presents the context and the historical interest of a notion or problem (Chevalarias, 2016; Chevalarias & Minet, 2012). On certain occasions, she only gives a few elements from history, but she believes that knowing a bit of history of mathematics allows her to approach the content she teaches in a different way and stimulates her to be more attentive to the difficulties students encounter in approaching notions that we know have taken centuries to be built up and still are not trivial.

7.2 What Michel and his colleagues try to do while teaching at the professional bachelor and what he does in high school in Brussels

As far as the professional bachelor is concerned, since students generally enter the training with a poor vision of what mathematics is, thinking of mathematics as a magic box to solve exercises, he and his colleagues decide to teach alternative methods: sometimes theoretical, often through problem-solving, visualizing, etc. Here and there they mention the origins and stress the differences between old and new concepts. As an example, they underline the distance between Greek notion of number and today's concept of number; or they show the Arabic origin of trigonometry and the role of astronomy in its invention… but in the end the context and the historical side notes are easily forgotten.

In the third and last year of the teacher training, they let students develop teaching material by themselves. They are meant to design a workshop lasting one hour about a historical mathematical topic, in which they have to zoom out (sketch the cultural context) and zoom in (let the participants work on a proof or calculation or a figure…).

Concerning the teaching in high school instead, an observation, which undoubtedly does not apply only to Belgium, is the very diverse, multicultural student population. Taking into account the also very diverse cultural background of the branches of mathematics taught, it seems very important to let the pupils know which part of mathematics comes from ancient Greece, or from the Arabic/Islamic tradition from the Middle Ages, and what we have inherited from other centuries and cultures, also in view of motivating students with roots in Islamic countries to work hard and consider the possibility of going ahead studying mathematics and science at university.

7.3 How Caterina usually treat history of Maths in her lessons

While teaching in high school Caterina acts in essentially in two parallel ways. On one side, she lets history permeate her courses, on the other side, she makes use of original sources. By "general" integration of history she means that the historical perspective allows her decide the order of topics, the grade of emphasis which has to be given to a notion or a problem, which type of language which is worthwhile to use in situations, and the degree of "hybridization" among different branches of mathematics which is functional to a deeper understanding of a concept (e.g. Euclidean geometry, Cartesian geometry, linear algebra).

The use of sources is done either during the normal timetable or during special extracurricular and volunteer seminars on history and epistemology of mathematics open to all the students of the school, according to the situation. Very often she also encourages

students to produce something ‘concrete’: a mathematical social play to be presented in a festival, an exposition of sources in a gallery, a theatrical show to be acted in some theatre, etc. (Vicentini, 2001; 2004; 2007)

While teaching to future teachers at university, first Caterina introduces a few general elements of epistemology and history, presents some points of view on didactics of mathematics, and then she imitates what Nicolas Rouche did in his course of “Méthodologie de l’enseignement” in Louvain-la-Neuve (BE), in letting future teachers work in groups in order to examine an original source and build a series of lessons making use of this source (cf. Rouche et al., 2006; 2008).

Eventually, in order to pass the exam, prospective teachers are supposed to present and discuss a complete unit based on historical material. The main phases of this unit must include an introduction to the topic, the historical presentation of the source, the guided reading of the source, some exercises and problems built from the original text, some possible hermeneutics hints for the students, some guided social activities on the source, some individual activities on the text, a summary of the important notions acquired, and the contextualization of the notions acquired in the school program (Bagni & Vicentini, 2008).

8 Closing remarks by the panel coordinator

I think some observations of a different kind and relevance are worth making before closing this quite long (and I hope never boring) paper. Let’s sketch them in a very synthetic way.

1. Unusually, in a references list, one can find more items than those strictly quoted. This is due to the will of giving to those potential readers who are teachers the possibility to find ready-to-be-used classroom sources possibly in their mother tongue. What is more, some additional, theoretically important references are present to consent an enrichment of the teachers’ personal theoretical framework.
2. The situation in China appears to be somewhat different to the occidental one. Maybe this fact allows an explanation for why “more than one in four students in Beijing-Shanghai-Jiangsu-Guangdong (China), Hong Kong (China), Singapore and Chinese Taipei are top-performing students in mathematics, meaning that they can handle tasks that require the ability to formulate complex situations mathematically, using symbolic representations” (PISA 2015-OECD).
3. The introduction of history in classrooms can help teachers to ‘row against’ what Freudenthal called the widespread “anti-didactical inversions” present in the curricula all over the world (Barbin, 2015; Freudenthal, 1999).
4. We focused more on history than on epistemology since according to the inquiry results, historical issues are more explicitly felt by teachers compared with epistemological ones (D’Amore, 2004).
5. We neither addressed hermeneutical problems that naturally arise while using history in general and original sources in particular (Bagni, 2006; 2009; Rorty, 1979), nor provided a reflection under the semiotics’ point of view (Peirce, 1989). Nevertheless, I am strongly convinced that a thorough analysis should deal with this perspective.
6. We only lightly touched upon cultural issues (D’Amore, Radford, & Bagni, 2006) and we did not approach interdisciplinarity (Battisttutti & Vicentini, 1994).
7. A more massive political and institutional presence of teachers and researchers seems

suitable in view of amending the curricula.

To conclude, I kindly thank Alain Bernard from France for his participation in the discussion on this topic with an interesting remark. Quoting the paper, "No, I Don't Use History of Mathematics in my Class. Why?" by Man Keung Siu, Alain criticized the term "use" in our questionnaire, proposing to substitute it with the verbs "integrate" or "permeate". Obviously, under the point of view of the HPM research, I agree with him. Nonetheless, given the results we acquired, I wonder if the majority of teachers interviewed would have noticed the difference and answered in a different way.

While underlining that to improve the situation in real classes we have no need to be too demanding; meaning (among other things) that the entire HPM research community should not bother too much about a sort of 'contamination' of their results. The distance between the 'ideal' and the 'real' class being quite large, the first step has to be a sort of 'breaking the ice', and I would almost say 'no matter how'. Let's take as an example the widespread use of the English language: the more it is spoken, the less it is orthodox BBC English. This seems to be an unavoidable consequence.

Often in conferences, while talking about the effectiveness of research, a tacit assumption is silently implied, i.e. that teachers have to do something more to align with researchers' work. In my opinion the converse is also important: researchers should worry a bit more about the large integration of their research, not being too afraid of 'contaminations' going in the sense of 'adaptability' to real situations (Artigue, 2014).

Man Keung Siu himself, in the introduction to the same paper quoted by Alain Bernard, suggests: "instead of harbouring a preconceived view one should join the company of school teachers and listen with an open mind to what they have to tell about their classroom experience*"* (2006, p. 268)

I am glad to thank all the audience members during the panel for their open, rich and very kind participation. Now I feel comfortable in loudly confessing that I was really worried by the embarrassing silence that could have occurred due to my decision to manage the situation much more as a reflection's starting point than as a presentation of ready-made results. In my opinion a panel should always be only loosely structured, otherwise instead of being one of the rare brainstorming official occasions in a conference, it runs the risk of becoming something like a sequence of multiple talks, losing a bit its appeal. In Italy we call panels 'round tables' and in a round table there are no privileged seats.

Finally, I offer all participants of ESU with the wish that instructional practice and HPM research will be able to move one towards the other in the fastest and most harmonious way possible. The meeting of these two essential aspects will hopefully be able to strengthen each other, in the same way as the experience of teaching and learning of mathematics.

REFERENCES

Artigue, M. (2014). *The challenging relationship between fundamental research and actions in mathematics education*. In G. Kaiser (Ed.) *Proceedings of the 13th International Congress on Mathematical Education*, ICME 13 Monographs (pp. 145-163). Cham: Springer.

Bagni, G. T. (n.d.). *History and Hermeneutics for Mathematics Education*. http://www.syllogismos.it.

Bagni, G. T. (2006). *Linguaggio, Storia e Didattica della Matematica*. Bologna: Pitagora.

Bagni, G. T., & Vicentini. C. (2008). History and Epistemology of Calculus and Algebra, Celebrating Leonhard Euler's Tercentenary. Cooperative Learning and Effectiveness of Perspective Teacher Training. In É. Barbin, N. Stehlíková, & C. Tzanakis (Eds.). *History and Epistemology in Mathematics Education: Proceedings of the 5th ESU* (pp. 439-440). Prague: Vydavatelský servis.

Bagni, G. T. (2009). *Interpretazione e didattica della matematica. Una prospettiva ermeneutica.* Bologna: Pitagora.

Barbin, É. (1994). Het belang van geschiedenis voor de wiskundige vorming. *Uitwiskeling 10*(3), 1-7.

Barbin, É. (2015). Philosophies and theories behind history and education: thirty years after Hans Freudenthal. In É. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU* (pp. 21-39). Copenhagen: Danish School of Education, Aarhus University.

Battistutti, M., & Vicentini, C. (1995). Achille, la tartaruga e l'interdisciplinarità, *Quaderno 21 del NRD dell'Università di Trieste,* Trieste.

Bergamini, M., Trifone, A., & Barozzi, G. (from 1999 to 2019). Various Maths Textbooks. Bologna: Zanichelli.

Berlinghoff, W. P., & Gouvêa F., Q. (2016). *Wortels van de Wiskunde. Een historisch overzicht voor leraren en anderen*. Amsterdam: Epsilon Uitgaven.

Bkouche, R., Charlot, B., & Rouche, N. (1991). *Faire des mathématiques: le plaisir du sens*. Paris: Armand Colin.

Buchini, M., & Vicentini, C. (2017). Le equazioni di secondo grado con Lagrange, un insegnamento per competenze. *Periodico di matematiche,* vol. *9*, serie XIII anno CXXVII, 71-81.

Buchini, M., & Vicentini, C. (2018). Tweedegraadsvergelijkingen bij Lagrange, *Uitwiskeling 34*(2), 11-17.

Chevalarias, N. (2016). Comment calculer ses chances de gagner ? ... réponses historiques. In *Enseigner les mathématiques en seconde. Statistiques et probabilités : quatre parcours*. IREM de Poitiers, groupe lycée, 95-115.

Chevalarias, N., & Minet, N. (2012). Des séances "Maths-Histoire" en classe de seconde; *REPÈRES - IREM. 86*. http://www.univ-irem.fr/exemple/reperes/articles/86_article_579.pdf.

Chevallard, Y. (1985). *La transposition didactique: du savoir savant au savoir enseigné*. Grenoble: La pensée sauvage.

D'Amore, B. (2004). *Le basi filosofiche, epistemologiche e concettuali della didattica della matematica*. Bologna: Pitagora Edizioni.

D'Amore, B., Radford, L., & Bagni, G. T. (2006). Ostacoli epistemologici e prospettive socioculturali, *L'insegnamento della matematica e delle scienze integrate, 29B*(1), 11-40.

Demattè, A. (2015). History in the classroom: educational opportunities and open questions. In É. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU* (pp. 335-352). Copenhagen: Danish School of Education, Aarhus University.

Demattè, A. (2015). Yes, I do use history of Mathematics in my class. In É. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU* (pp. 511-525). Copenhagen: Danish School of Education, Aarhus University.

Demattè, A., & Furinghetti, F. (2006). *Fare matematica con i documenti storici.* Trento: Iprase.

ESU 8 website https://esu8.edc.uoc.gr/esu-8-main-themes-plenary-sessions/

Freudenthal, H. (1958). Sur la responsabilité humaine du mathématicien. *Mathematica & Pedagogia 15*, 38-40.

Freudenthal, H. (1999). *Didactical Phenomenology of Mathematical Structures.* New York: Kluwer Academic Publishers.

GEM (from 1993 to 1996). *De question en question 1,2,3*. Bruxelles: Didier-Hatier.

Gilbert, T., & Rouche, N. (2001). *La notion d'infini, L'infini mathématique entre mystère et raison*. Paris: Ellipses.

Gilbert, T., & Rouche, N. (2004). *L'infinito matematico fra mistero e ragione. Intuizioni, paradossi, rigore*. Bologna: Pitagora edizioni.

Hauchart, C., & Rouche, N. (1987). Apprivoiser l'infini, Louvain-la-Neuve: CIACO.

Jankvist, U. T. (2009). A categorisation of the 'whys' and 'hows' of using history in mathematics education. *Educational Studies in Mathematics 71*, 235-261.

Kaenders, R., & Weiss, Y. (2018). Algebra without Context is empty, Visualizations without Concepts are Blind, In K. M. Clark, T. H. Kjeldsen, S. Schorcht, & C. Tzanakis, (Eds.), *Mathematics, Education and History toward a Harmonious Partnership* (pp. 121-141). ICME 13 Monograph Series. Cham: Springer.

Lakoff, G., & Nunez, R.E., (2000), *Where Mathematics Comes From. How the Embodied Mind Brings Mathematics into Being*. New York: Basic Books (Perseus Books Group).

Lakoff, G., & Nunez, R. E. (2005), *Da dove viene la matematica. Come la mente embodied dà origine alla matematica.* Torino: Bollati Boringhieri.

Ministère de l'Éducation nationale, de l'Enseignement supérieur et de la Recherche. (2010). *Bulletin officiel spécial n° 9 du 30 septembre 2010*. http://cache.media.education.gouv.fr/file/special_9/21/1/mathsS_155211.pdf. Accessed: 22 November 2018.

Ministère de l'Éducation nationale, de l'Enseignement supérieur et de la Recherche. (2019). *Bulletin officiel spécial n°1 du 22 janvier 2019.* https://cache.media.education.gouv.fr/file/SP1-MEN-22-1-2019/95/7/spe631_annexe_1062957.pdf. Accessed: 10 March 2019.

MIUR (Ministero dell'Istruzione, dell'Università e della Ricerca). (2010). *Indicazioni Nazionali riguardanti gli obiettivi specifici di apprendimento per il liceo scientifico* http://www.indire.it/lucabas/lkmw_file/licei2010///indicazioni_nuovo_impaginato/_Liceo%20scientifico%20opzione%20Scienze%20Applicate.pdf

MIUR (Ministero dell'Istruzione, dell'Università e della Ricerca). (2018). *Quadro di riferimento per la redazione e lo svolgimento della seconda prova scritta dell'esame di Stato. Percorsi liceali: liceo scientifico opzioni scienze applicate* http://blog.deascuola.it/uploads/sites/5/2018/11/QDR-liceo-scientifico-LI02-22-11.pdf.

Peirce, C. S. (1989). *Semiotica*. Torino: Einaudi Paperbacks 115.

PISA 2015-OECD https://www.oecd.org/pisa/pisa-2015-results-in-focus.pdf

Roelens, M. (2008). The volume of a pyramid through the ages. To slice or not to slice, that's the question. In É. Barbin, N. Stehlíková, & C. Tzanakis (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 5th ESU*. Plzen: Vydavetelsky Servis.

Roelens, M. (2018). Archimedes en het volume van de bol. *Pythagoras 58*(2), 22-26.

Roelens, M., & de Graaf, W. (2015). Workshop on the Use and Mathematics of the Astrolabe. In É. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU.* Copenhagen: Danish School of Education, Aarhus University Denmark.

Roelens, M., & Van den Broeck, L. (2015). Veeltermvergelijkingen van vroeger tot nu. *Uitwiskeling 31*(4), 12-38

Roelens, M. (2018). Archimedes en het volume van de bol. *Pythagoras 58*(2), 22-26.

Rogers, L., Weiss-Pidstrygach, Y., Barnett, J. H.; Guillemette, D., & Metin, F. (2015). The question of evaluation and assessment of experiences with introducing history of mathematics in the classroom, Panel Discussion 2. In É. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU* (pp. 59-76). Copenhagen: Danish School of Education, Aarhus University.

Rorty, R., (1979*), The Philosophy and the Mirror of Nature*. Princeton: Princeton University Press.

Rouche, N., in collaboration with De Laet, L., Docq, C., Gantois, J. Y., Hauchart, C., Tancré, M., & Tossut, R. **(**2006). *Du quotidien aux mathématiques, Nombres, grandeurs, proportions*. Paris: Ellipses.

Rouche, N., in collaboration with Cuisinier, G., De Laet, L., Docq, C., Gantois, J. Y., Hauchart, C., Tancré, M., & Tossut, R. (2008). *Du quotidien aux mathématiques, Géométrie*. Paris: Ellipses.

Sasso, L. (in some cases together with Zanone, C.). (from 2006 to 2019). Various Maths textbooks, Novara: De Agostini.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coins. *Educational Studies in Mathematics 22*, 1-36.

Sfard, A. (2008), *Thinking as communicating: Human development, the growth of discourses, and mathematizing,* Cambridge: Cambridge University Press.

Sfard, A. (2009). *Psicologia del pensiero matematico, il ruolo della comunicazione nel processo cognitivo*, Trento: Erickson.

Siu, M. K. (2005). No, I don't use history of mathematics in my class. Why? In F. Furinghetti, S. Kaisjer, & C. Tzanakis (Eds.), *Proceedings of HPM2004 & ESU4* (pp. 268-277). Iraklion: University of Crete.

Smestad, B., Jankvist, U. T., & Clark, K. (2014). Teachers' mathematical knowledge for teaching in relation to the inclusion of history of mathematics in teaching. *Nordic Studies in Mathematics Education*, *19*(3-4), 169-183.

Tall, D. (2013). *How humans learn to think mathematically. Exploring the three worlds of mathematics.* Learning in Doing: Social, Cognitive and Computational Perspectives. Cambridge: Cambridge University Press.

Vicentini, C. (1994). Personal and social conscious and unconscious backgrounds in mathematics Education, theory and practice in mathematics education. In L. Bazzini (Ed.), *Proceedings of the Fifth International Conference on Systematic Cooperation Between Theory and Practice in Mathematics Education* (pp. 231-240). Pavia: ISDAF.

Vicentini, C. (1998). Percorso didattico alla riscoperta della trigonometria. In *Matematica é/e Cultura, la bellezza della teoria, il sapore delle applicazioni, Atti del Congresso nazionale Mathesis 1998* (pp. 301-310). Bologna: Pitagora editrice.

Vicentini, C. (2001). Si les mathématiques m'étaient contées... In P. Radelet-De Grave & R. Brichard (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 3rd ESU* (pp. 355-366). Leuven and Louvain-la-Neuve: Université catholique de Louvain.

Vicentini, C. (2003). Si può 'fare matematica' in prima media? *Progetto Alice 2003-III*, *4*(12), 597-615, Roma: Pagina.

Vicentini C. (2004). Once upon a time mathematics... (part 2). In F. Furinghetti, S. Kaisjer, & C. Tzanakis (Eds.), *Proceedings of HPM2004 & ESU4* (pp. 296 -305). Iraklion: University of Crete.

Vicentini, C. (2007). From π to aleph through the theatre: a way to avoid didactical frauds and pseudo-structured assurances, *La matematica e la sua didattica*, *21*(1), 111-115.

Vicentini, C. (2008). 'Echte wiskunde' in de eerste graad, *Uitwiskeling 34*(3), 8-18.

Vicentini, C. (2012). *Ocalogik*, https://www.premiocesarecancellieri.it/2012-lavori-premiati-concorso-cancellieri/

Vicentini, C. (2012). Si può migliorare il ragionamento giocando? In *Progetto Alice 13*(38), 337-351, Roma: Pagina.

Vicentini, C. (2014). Ocalogik: al spelend je redeneervermogen verbeteren, *Uitwiskeling 30*(3), 16-21.

Vicentini, C. (2015). Playing with Euler. In É. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *History and Epistemology in Mathematics Education: Proceedings of the 7th ESU* (pp. 373-387). Copenhagen: Danish School of Education, Aarhus University.

Vicentini, C (in progress). *Storia ed epistemologia della matematica, CLIL e il gioco della 'briscola*

pitagorica'.

Villani, C., & Torossian, C. (2018). *21 mesures pour l'enseignement des mathématiques.* http://cache.media.education.gouv.fr/file/Fevrier/19/0/Rapport_Villani_Torossian_21_mesures_pour_enseignement_des_mathematiques_896190.pdf. Accessed: 22 November 2018.

Wang, X., Qi, C., & Wang, K. (2017) A categorization model for educational values of history of mathematics: an empirical study. *Science & Education, 26*(7-9), 1029-1052.