EPISTEMOLOGICAL BELIEFS ABOUT MATHEMATICS

Challenges and chances for mathematical learning: Back to the future

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ABSTRACT

The transition from school to university is connected to a variety of some problems for many students. This can be attributed to different beliefs about mathematics in school and university. While mathematics teaching at school allows knowledge to be developed on the basis of real objects and empirical working methods, mathematics at universities is characterized by a rigorous axiomatic structure. The successive detachment of the connection to real objects has also occurred in the history of mathematics. From this situation, conclusions can be derived for the teaching of mathematics at school and university. The transition from school to university seems to be facilitated by the use of digital media in processes of concept development and the systematic thematisation of different beliefs about mathematics.

1 A challenge in mathematics education

When mathematics teachers as students move from school to university and then again when moving from university training back to school to teach mathematics they are often confronted with various problems. Felix Klein describes this situation as "double discontinuity":

> *"The young university student found himself, at the outset, confronted with problems, which did not suggest, in any particular, the things with which he had been concerned at school. Naturally he forgot these things quickly and thoroughly. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and, since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honoured way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching."* (Klein, 1908)

Witzke, Struve, Clark & Stoffels (2016) describe a seminar at university level that focuses on the first discontinuity, the transition from school to university. In an empirical study, they put the following question to the participants: *What is the biggest difference or similarity between school and university mathematics?*

One male participant answered: "The fundamental difference develops as mathematics in school is taught empirical-perceptual (ger.: anschaulich), whereas at university there is a rigid modern-axiomatic structure characterizing mathematics. In general, there are more differences than similarities, caused by differing aims".

Many similar statements from other participants could be found. Thus, the problem of the transition from retrospective student viewpoint is closely connected with the "differentness" of mathematics. These differences concern the aspects clearness, level of abstraction, evidence, formal rigor and axiomatic structure. The result is a clear distinction between school and higher education mathematics regarding its character.

2 Beliefs about mathematics – A theoretical framework

Looking at the challenges presented in the previous section, one question is particularly obvious:

How do we develop mathematical knowledge (further)?

The answer to this question is crucially related to our conceptions of mathematics. The notion about the beliefs of mathematics provides a good basis for this description.
According to Schoenfeld, the way someone works on a mathematical problem depends strongly on his beliefs about mathematics:

> *"One's beliefs about mathematics [...] determine how one chooses to approach a problem, which techniques will be used or avoided, how long and how hard one will work on it, and so on. The belief system establishes the context within which we operate […]"* (Schoenfeld, 1985, 2011)

Steiner emphasizes the influence of the conception of mathematics on concepts for teaching and learning:

> *"Concepts for learning and teaching of mathematics [...] often implicitly are based on certain aspects of a philosophy of mathematics"* (Steiner, 1987)

Green describes teaching as the modification of the belief system of learners:

> "*The activity of teaching, at least in the sense of instructing, might therefore be defined as the effort to reconstitute the structure of our belief systems so that the number of core beliefs and belief clusters are minimized, the number of evidential beliefs are maximized, and the quasi-logical order of our beliefs is made to correspond as closely as possible to their objective logical order*."(Green, 1971)

How to build up mathematical knowledge, how to handle it and whether one is successful seems to depend essentially on the individual conceptions of mathematics (mathematical world view, attitudes, beliefs). The term beliefs of mathematics is frequently used in literature:

> "*Psychologically held understandings, premises, or propositions about the world that are thought to be true.*" (Philipp, 2007)

> "*Belief System: One's 'mathematical world view', the set of (not necessarily conscious) determinants of an individual's behavior about self, about the environment, about the topic, about mathematics.*" (Schoenfeld, 1985)

> "*individual's beliefs [...] as subjective, experienced-based often implicit knowledge and emotions on some matter or state of the art*" (Pehkonen & Pietilä, 2003)

The different explanations show the diversity of the term belief (cf. Rezat, 2009). Mathematics education is characterized by various beliefs about mathematics (cf. Grigutsch, Raatz & Törner, 1998, Schoenfeld, 2011, Witzke & Spies, 2016):

- Scheme-Aspect: Mathematics is a system consisting of rules, formulas and algorithms.
- Formalism-Aspect: Mathematics is characterized by logic, formal rigidity and precise technical terminology. It is the formal-abstract science.
- Process-Aspect: Mathematics is seen as a creative and constructive process.

- Application-Aspect: Mathematics is a tool for applications in the natural sciences and everyday life.
- Empirism-aspect: Mathematics describes a universe of discourse in reality. It is a natural science.

A formal-abstract view on mathematics is beside others represented at the university level. According to the frequently used textbook for calculus courses at university Heuser (2009), the central properties of mathematics are the brightness and sharpness of the concept formation, the pedantic care in dealing with definitions, the rigor of proofs and the abstract nature of mathematical objects that you cannot see, hear, taste or feel.

At least since Hilbert it is possible to see mathematics as an archetype of formal science with an axiomatic structure that is detached from reality. He developed mathematics as a science of uninterpreted abstract systems (focus on structures) with an absolute notion of certainty (internal consitency) (e.g. Hilbert, 1899). Thus, “the umbilical cord between reality and geometry has been cut” (Freudenthal, 1961). Geometry has become pure mathematics and the question of whether and how it can be applied to reality is answered just as in any other branch of mathematics.

The axioms are no longer self-evident truths; in fact, it does not even make sense to ask for their truth. This does not mean that there are no real applications or interpretations of the theories.

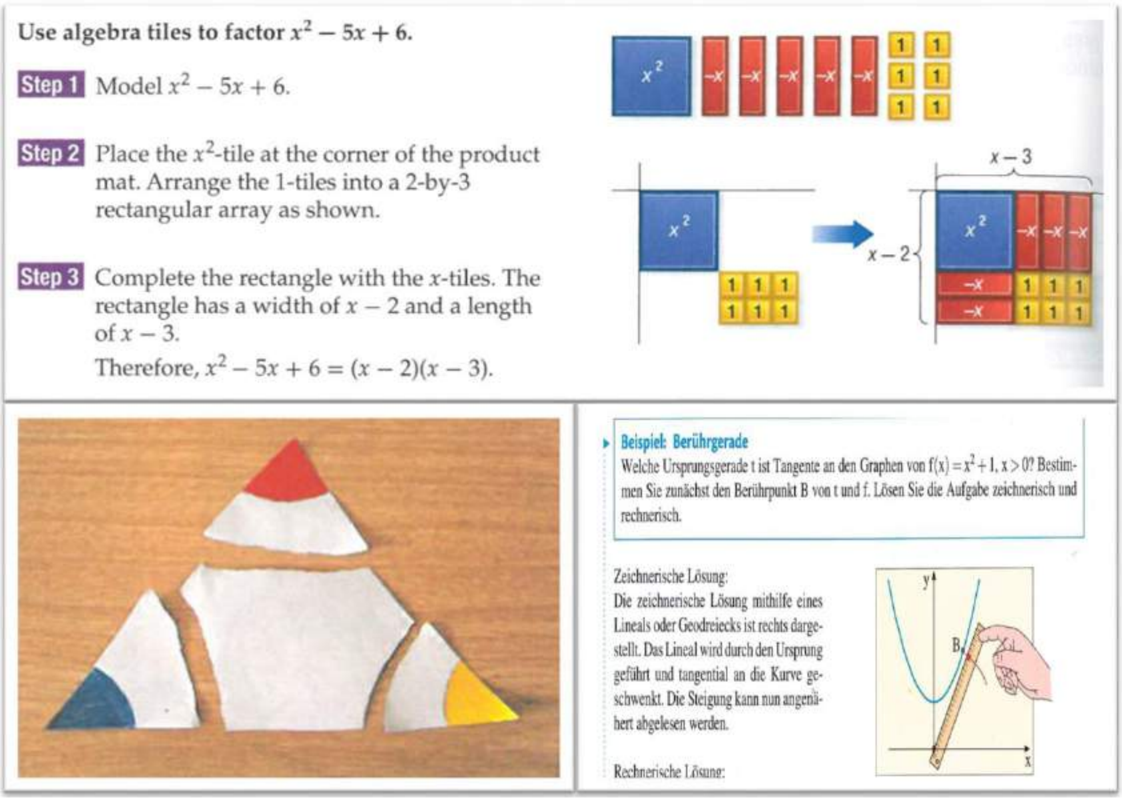
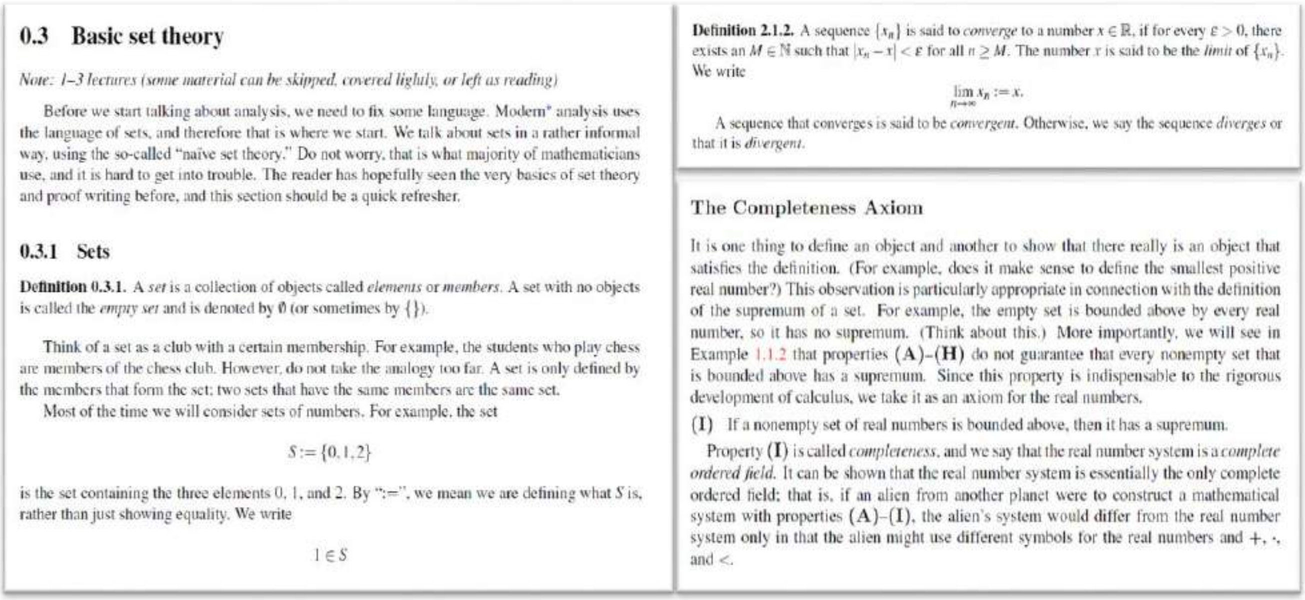


Figure 2.1: Empirical approaches to mathematical concepts and theorems in schoolbooks



0.3 Basic set theory

Note: 1–3 lectures (some material can be skipped, covered lighly, or left as reading)

Before we start talking about analysis, we need to fix some language. Modern* analysis uses the language of sets, and therefore that is where we start. We talk about sets in a rather informal way, using the so-called "naive set theory." Do not worry, that is what majority of mathematicians use, and it is hard to get into trouble. The reader has hopefully seen the very basics of set theory and proof writing before, and this section should be a quick refresher.

0.3.1 Sets

Definition 0.3.1. A *set* is a collection of objects called *elements* or *members*. A set with no objects is called the *empty set* and is denoted by $\emptyset$ (or sometimes by $\{\}$).

Think of a set as a club with a certain membership. For example, the students who play chess are members of the chess club. However, do not take the analogy too far. A set is only defined by the members that form the set; two sets that have the same members are the same set.

Most of the time we will consider sets of numbers. For example, the set

$$S := \{0, 1, 2\}$$

is the set containing the three elements 0, 1, and 2. By ":=", we mean we are defining what S is, rather than just showing equality. We write

$$1 \in S$$

Definition 2.1.2. A sequence $\{x_n\}$ is said to *converge* to a number $x \in \mathbb{R}$, if for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq M$. The number x is said to be the *limit* of $\{x_n\}$. We write

$$\lim_{n\to\infty} x_n := x.$$

A sequence that converges is said to be *convergent*. Otherwise, we say the sequence *diverges* or that it is *divergent*.

The Completeness Axiom

It is one thing to define an object and another to show that there really is an object that satisfies the definition. (For example, does it make sense to define the smallest positive real number?) This observation is particularly appropriate in connection with the definition of the supremum of a set. For example, the empty set is bounded above by every real number, so it has no supremum. (Think about this.) More importantly, we will see in Example 1.1.2 that properties **(A)**–**(H)** do not guarantee that every nonempty set that is bounded above has a supremum. Since this property is indispensable to the rigorous development of calculus, we take it as an axiom for the real numbers.

(I) If a nonempty set of real numbers is bounded above, then it has a supremum.

Property **(I)** is called *completeness*, and we say that the real number system is a *complete ordered field*. It can be shown that the real number system is essentially the only complete ordered field; that is, if an alien from another planet were to construct a mathematical system with properties **(A)**–**(I)**, the alien's system would differ from the real number system only in that the alien might use different symbols for the real numbers and $+$, $\cdot$, and $<$.

Figure 2.2: Formal-abstract representations in lecture notes for analysis at the university level

In contrast, this clear distinction between reality and mathematics does not take place for school mathematics. Hefendehl-Hebeker (2016) states in this context:

> "*The concepts and contents of school mathematics have their phenomenological sources predominantly in our surrounding reality. […] All in all the ontological bounding to reality is in place because of the educational and psychological purposes and aims of school. School mathematics barely surpasses the conceptual niveau and state of knowledge of the 19th century […]. Mathematics as a scientific discipline has today become a network of highly specialized abstract sub-areas.*"

This considerations lead to the following research thesis:

Research thesis I: *„Mathematical knowledge of pupils is generated in a constructive process - through interaction and the work with the offered learning material."* (cf. Bauersfeld, 1983)

At school, mathematics appears as an empirical science of concrete objects, it is not an abstract science of uninterpreted systems of terms as in modern mathematics. The empirical character of school mathematics (argumentation, models, experiments, term, etc.) is on epistemic grounds comparable to the character of natural sciences. Argumentations are based on real objects. This results in the following thesis:

Research thesis II: *If mathematics is consequently taught with the support of visual representations and illustrative material, students acquire an empirical belief system about mathematics. It is a theory about these representations – a quasi - 'natural science'.*

This kind of mathematics describes a universe of discourse in physical reality. The notion of truth relies in empirical facts gained through observation and experiments. Nevertheless, empirical mathematics needs logical reasoning to avoid a pure empiricism and pure phenomenology. The empirical characteristic is a fundamental difference to the above described university mathematics. The question arises, if this ‘non-abstract’ point of view is a reasonable one for the developing of mathematical knowledge.

3 Epistemological beliefs about mathematics in the past

A first possible answer to the above mentioned question is provided by an insight into beliefs in the history of mathematics. Substantial pieces of historical mathematics can be reconstructed as empirical mathematics (e.g. Witzke, 2009) with the help of structuralism (cf. Balzer, Moulines & Sneed, 1987).

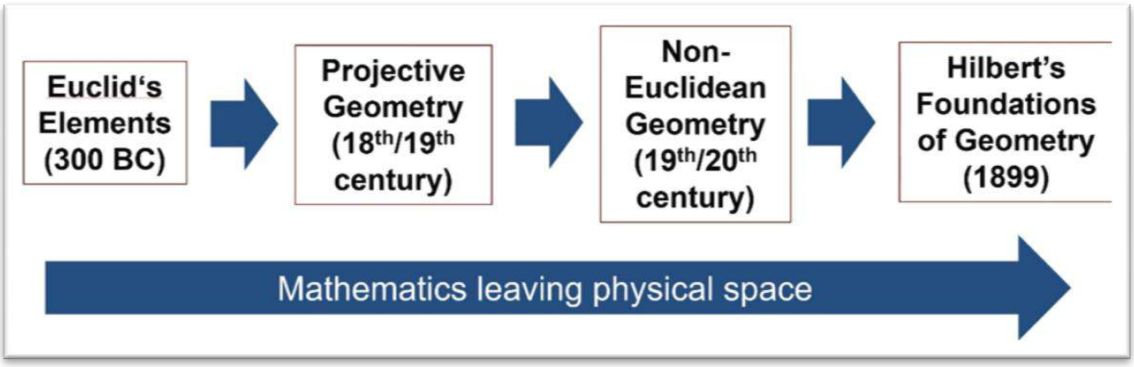
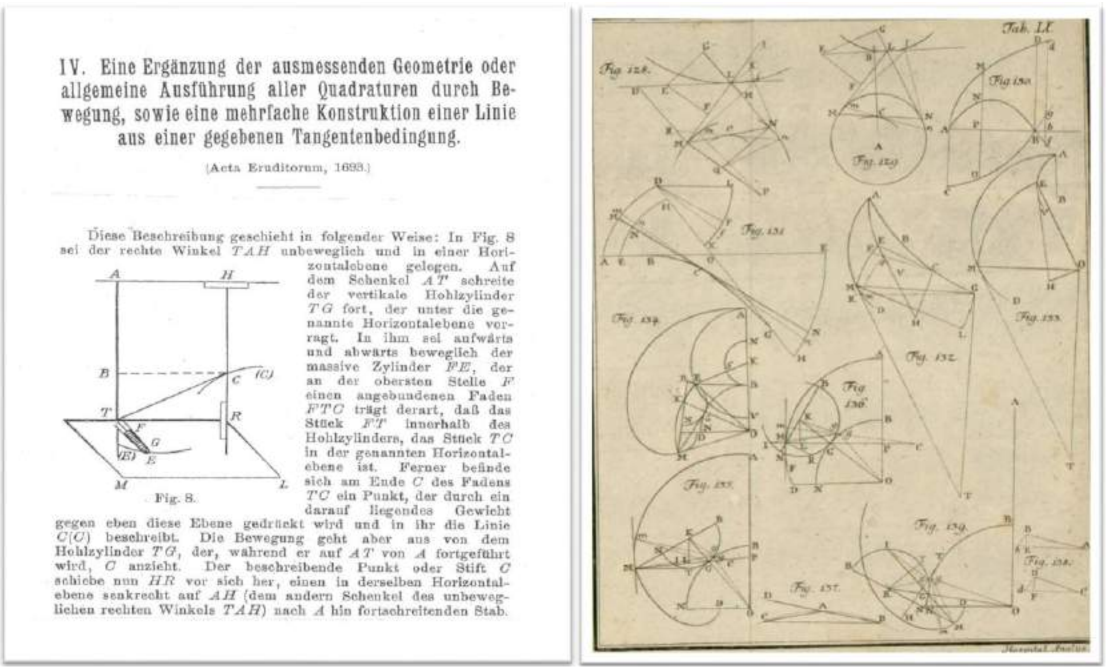


Figure 3.1: Development of geometry in the history of mathematics



IV. Eine Ergänzung der ausmessenden Geometrie oder allgemeine Ausführung aller Quadraturen durch Bewegung, sowie eine mehrfache Konstruktion einer Linie aus einer gegebenen Tangentenbedingung.

(Acta Eruditorum, 1693.)

Diese Beschreibung geschieht in folgender Weise: In Fig. 8 sei der rechte Winkel TAH unbeweglich und in einer Horizontalebene gelegen. Auf dem Schenkel AT schreite der vertikale Hohlzylinder TG fort, der unter die genannte Horizontalebene vorragt. In ihm sei aufwärts und abwärts beweglich der massive Zylinder FE, der an der obersten Stelle F einen angebundenen Faden FTC trägt derart, daß das Stück FT innerhalb des Hohlzylinders, das Stück TC in der genannten Horizontalebene ist. Ferner befinde sich am Ende C des Fadens TC ein Punkt, der durch ein darauf liegendes Gewicht gegen eben diese Ebene gedrückt wird und in ihr die Linie $C(C)$ beschreibt. Die Bewegung geht aber aus von dem Hohlzylinder TG, der, während er auf AT von A fortgeführt wird, C anzieht. Der beschreibende Punkt oder Stift C schiebe nun HR vor sich her, einen in derselben Horizontalebene senkrecht auf AH (dem andern Schenkel des unbeweglichen rechten Winkels TAH) nach A hin fortschreitenden Stab.

Fig. 8.

Figure 3.2: The development of calculus based on curves as empirical objects, constructed and drawn on paper

The development of modern views of mathematics can be illustrated particularly well using the example of geometry (cf. Witzke, Struve, Clark & Stoffels, 2016). The first axiomatic structure of geometry can be found in Euclid's elements in 300 B.C. The justification of the axioms occured by evidence (cf. Garbe, 2001). Thus they had a clear relation to the objects of the real world, e.g. a line drawn on a sheet. The further development of the Euclidean geometry took place in the 18th and 19th centuries. The famous mathematician Moritz Pasch wrote in 1882: "The geometric terms [...] serve to describe the world around us [...]. Geometry is nothing more than a part of the natural sciences". One goal of geometry is the description of the physical world, although there is an increasing axiomatization. The relationship of geometry to the real world changed dramatically with the development of non-Euclidean geometries in the 19th and 20th centuries. These internal consistent theories are based on axioms that are initially independent from the surrounding world. However, by striving to find the geometry that describes the physical space, there is still a connection to reality. The underlying axiom system and the physical world were then disconnected consciously by the development of Hilbert's foundations of geometry in 1899. The axioms no longer need any connection to reality. It is a pure inner-mathematical theory.

The previous explications can be described simplified in a bipolar model of belief systems. On the one hand there is the empirical-concrete mathematical belief system. It can be found in the history of mathematics as well as in school mathematics and is based on didactical (learning theory, e.g. Gopnik et al. , 2007; educational theory, e.g. Winter, 1969; empirical reasons, e.g. Schoenfeld, 2011, Struve, 1990) and epistemological reasons (parallels with natural science, e.g. Einstein, 1921; historical reconstructions, Witzke, 2009; structuralistic reconstructions, Balzer, Moulines & Sneed, 1987). On the other hand, there is a formal-abstract mathematical belief system that can be found in mathematics courses at universities and in the history of mathematics since Hilbert.

4 Epistemological beliefs: Back to the future

The questions remain what we can learn from these perspectives and how history can inform modern mathematical education. One possible answer can be provided by looking at the use of digital media in mathematics classroom. The use of digital media is usually connected with an emphasis on qualitative and empirical working methods. The potential of digital media can be illustrated using the example of calculus. Textbooks at school contain a large number of graphical representations. Often, they form the basis for argumentations; questions of existence such as continuity or differentiability become less relevant (cf. Witzke, 2014). Graphic calculators and function graphing software enable the dynamic investigation of curves. In this way, concepts can be developed qualitatively in a first step, so that the students can develop sustainable ideas (e.g. function microscope by Elschenbroich, 2015). On epistemological grounds, these objects represented in an iconic way constitute parallels to the construction of curves at the time of Leibniz.

Another example results from graphical differentiation and integration. This means the qualitative drawing of the graph of a primitive integral or the derivative by graphical determination of the integral or the derivative at single points which is somewhat problematic because of the discreteness. The 3D printing technology offers the possibility to develop a so-called integraph (cf. Witzke & Dilling, 2018). This is a device that continuously draws the graph of a primitive integral of a graphically given function in a mechanical way. It enables the students to justify the first part of the fundamental theorem of calculus visually. First concepts of an integraph reach back to Leibniz (1693).

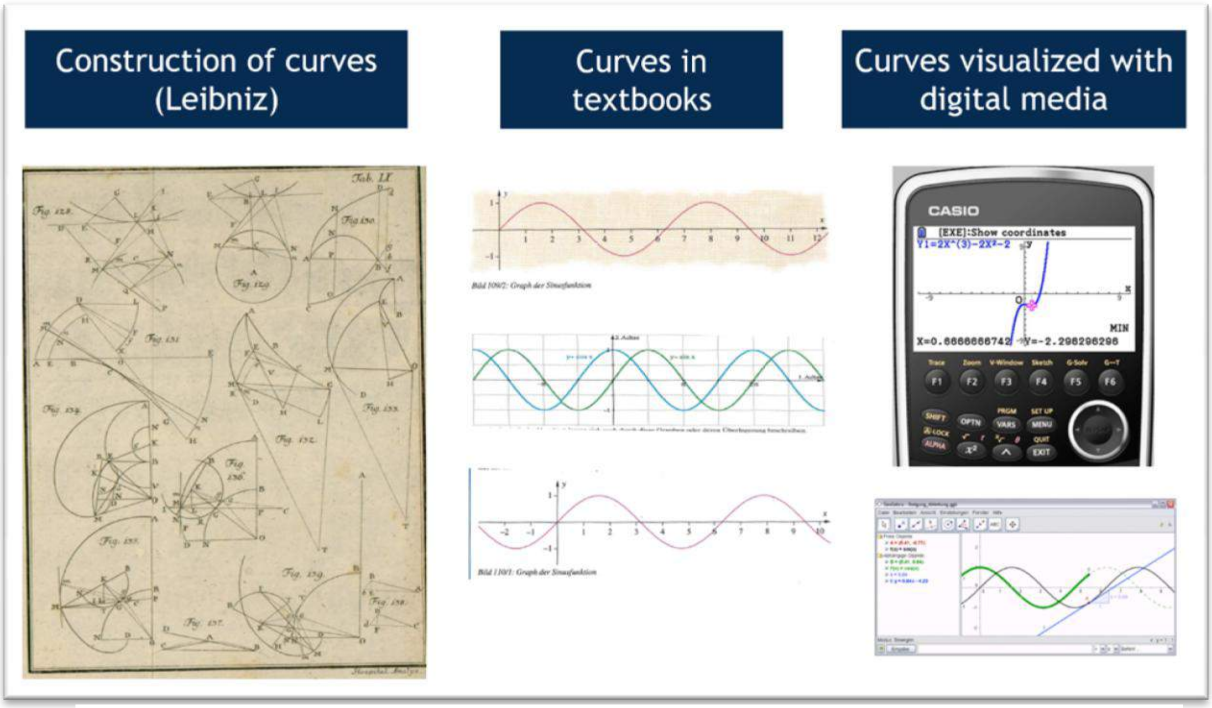


Figure 4.1: Curves in the history of mathematics, in the mathematics textbook and visualized with digital media

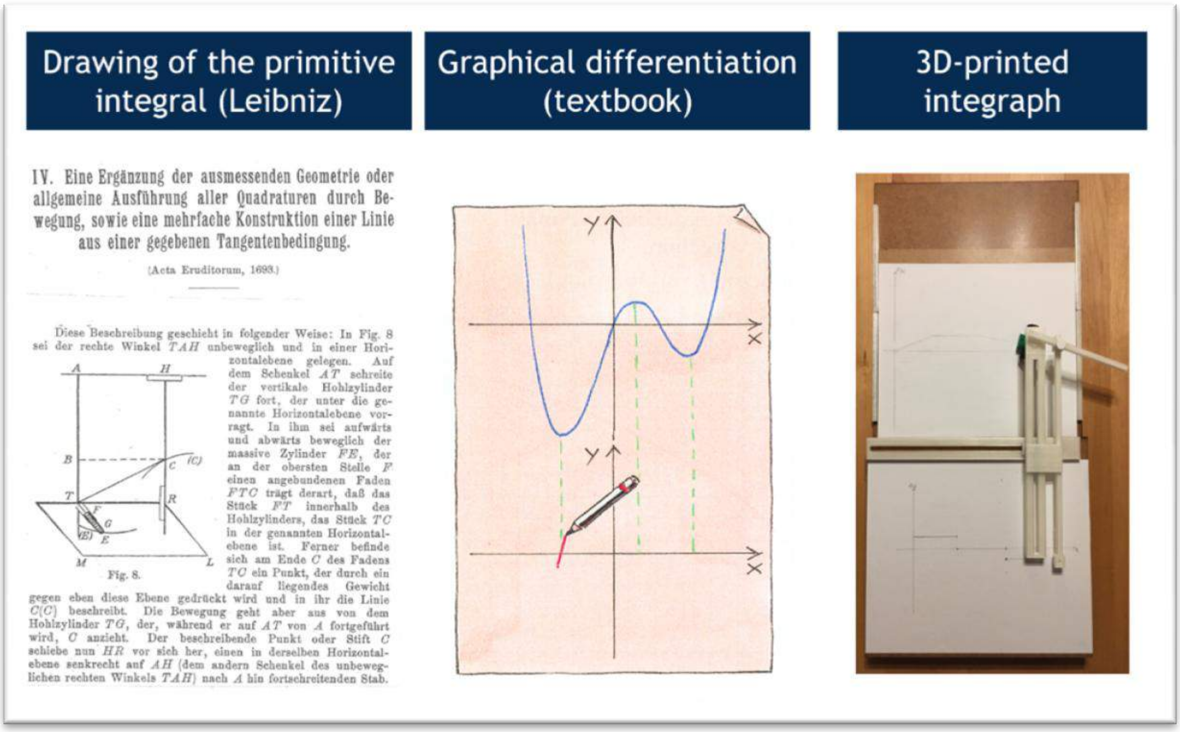


Figure 4.2: Graphical determination of the derivative or primitive integral in the history of mathematics, in the mathematics textbook and by the use of an integraph

The examples show that working with digital media promotes an empirical view of mathematics that was established in the history as well. However, these empirical approaches are not to be equated with pure empiricism, since concepts are consciously emphasized and systematically developed. The aim of the authors is not the equalization of school and university mathematics. Instead, the differences and the resulting obstacles for the transition from school to university should be specifically addressed. This is connected with the hope that in this way more students bridge the gap and develop an adequate perspective regarding the nature of mathematics in different contexts.

Acknowledgement: I am particularly grateful to Frederik Dilling for his comments on the manuscript and the support in the development of the paper.

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