HPM LESSON STUDY IN THE CONTEXT OF AN HPM LEARNING COMMUNITY

A Case Study in Chinese Senior High School

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ABSTRACT

Considerable work has been done over the last 15 years on how to integrate History of Mathematics (HM) into Mathematic Education (ME). However, integrating HM into ME from the perspective of teaching research system has been less researched. In Mainland China, a teaching research system has been practiced nationally since the 1950s. China has a long history of collaboration among teachers. This paper deals with the teaching research system that integrates HM into ME at Chinese education system. The conceptual framework in this research including HPM Learning Community (HPMLC) and HPM Lesson Study (HPMLS). We used the research method of case study. Based on the conceptual framework above, we selected one case of HPM lesson that integrates the history of conic section into teaching practice. Through case analysis, the conceptual framework is modified to obtain the result of this study that how the HPMLS work under the context of HPMLC.

1 Introduction

Considerable work has been done over the last 15 years on how to integrate History of Mathematics (HM) into Mathematic Education (ME) (Clark et al., 2016), and there are various classifications of approach (Tzanakis et al., 2000; Jankvist, 2009). However, integrating HM into ME from the perspective of teaching research system has been less researched.

In Mainland China, a teaching research system has been practiced nationally since the 1950s, which refers to various activities of professional development institutionalized by four hierarchical organizations: province/city, district/county, school, and lesson plan group. The system focuses on "guiding teaching research, overseeing teaching administration in schools on behalf of educational bureaus, providing consultation for educational authorities, mentoring the implementation and revision of new curricula, building the bridge between modern educational theories and teaching experiences, and promoting high-quality classroom instruction" (Huang, Ye & Prince, 2017).

The core components of Chinese teaching research activities is studying lessons which include Keli (Exemplary Lesson Development) (pronounced: Ker-Lee) and public lessons, Both Keli and public lessons are known as Chinese Lesson Study (LS) (Huang & Shimizu, 2016). Chinese LS is a form of school-based professional development that aims to update ideas of teaching and learning, to design new learning situations, and to improve

classroom practice through Keli, which is a community-mediated process of developing an exemplary lesson, including the planning, delivery, debriefing, revision and re-teaching of the lesson, and this form develops a model known as action education (Gu & Gu, 2016).

Unlike the West, China has a long history of collaboration among teachers (Wong, 2010). Recently, concepts like "Profession Learning Communities" (PLC) are flourishing (Cheng & Wu, 2016). Hord define PLC as defined it as ''teachers in a school and its administrators continuously seek and share learning and then act on what they learn. The goal of their actions is to enhance their effectiveness as professionals so that students benefit (Hord, 1997). So, it is meaningful to research LS in the context of PCL in China.

This paper deals with the teaching research system that integrates HM into ME at Chinese education system. This paper discusses the following questions:

1. What is the structure and characteristics of PLC that integrates HM into ME?
2. How does Chinese Lesson Study integrate HM into ME work in the context of PCL?

2 The conceptual framework

2.1 HPM Learning Community

The whole process was done by two teams (see fig. 1), HPM Research Team and School Teacher Team, and each team has its own expertise. The research team includes HPM expert, Ph.D. and M.A. students in the direction of HPM, whose team was responsible for theoretical guidance and providing historical materials, meanwhile, the school teacher team was mainly responsible for designing discussions and teaching practice. They related each other by HPM seminar.

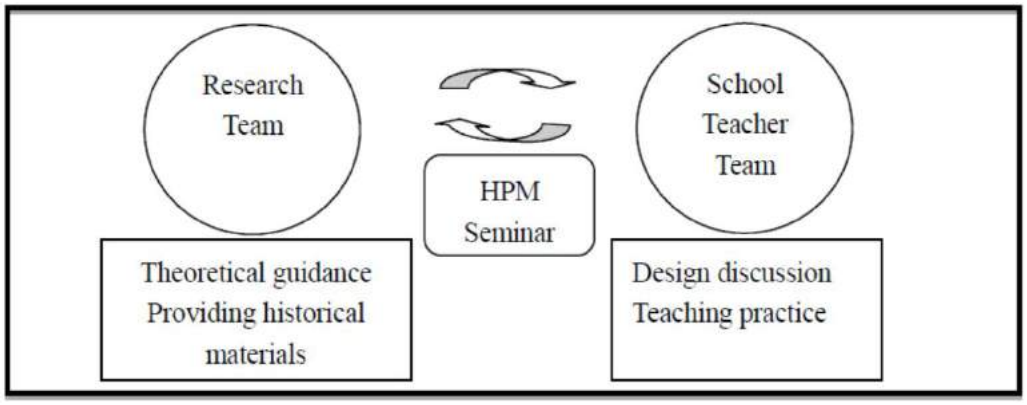


Figure 2.1: HPM-LC

These two teams form a community, and as shown in table 2.1, it has the characteristics of PLC (Huffman, Hipp & Hord, 2003), so we call it HPM learning community (HPMLC).

Table 2.1: Characteristics of PLC

Characteristics	connotation
Shared and Supportive Leadership	●Nurturing leadership among staff ●Shared power, authority and responsibility ●Board-based decision-making that reflects commitment and accountability
Shared Values and Vision	●Espoused values and norms ●Focus on student learning ●High expectations ●Shared vision guides teaching and learning
Collective Learning and Application	●Sharing information ●Seeking new knowledge, skills and strategies ●Working collaboratively to plan, solve problems and improve learning opportunities
Shared personal practice	●Peer observation to offer knowledge, skill and encouragement ●Feedback to improve instructional practices ●Sharing outcomes of instructional practices ●Coaching and mentoring
Supportive Condition	●Relationships ●Structure

2.2 HPM Lesson Study

By using the concept of Chinese LS and PLC, we using the concept structure of the teaching research system that integrates HM into ME in this research. We call the lessons that integrate HM into teaching as HPM lesson, and we call The Chinese LS combining HPM as HPM Lesson Study (HPMLS). The development process of HPMLS includes four stages (see Fig 2) (Wang, 2017).

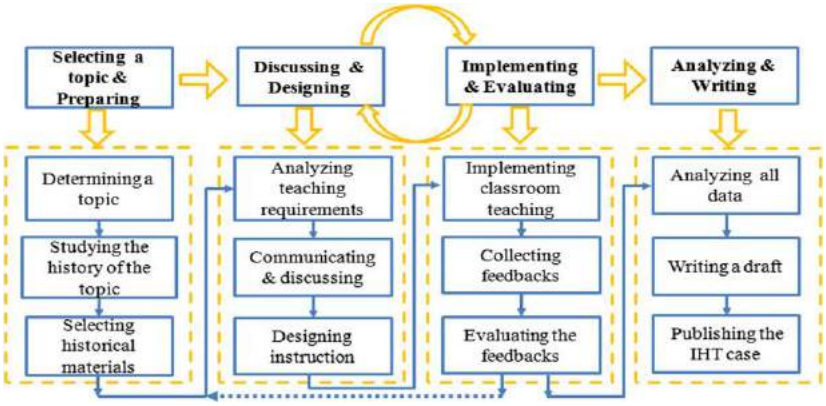


Figure 2.2: HPM-LS

All concepts mentioned above constitute the concept structure of this research.

3 Methodology

In this research, we used the research method of case study, and the research process is shown in the following figure 3.1.

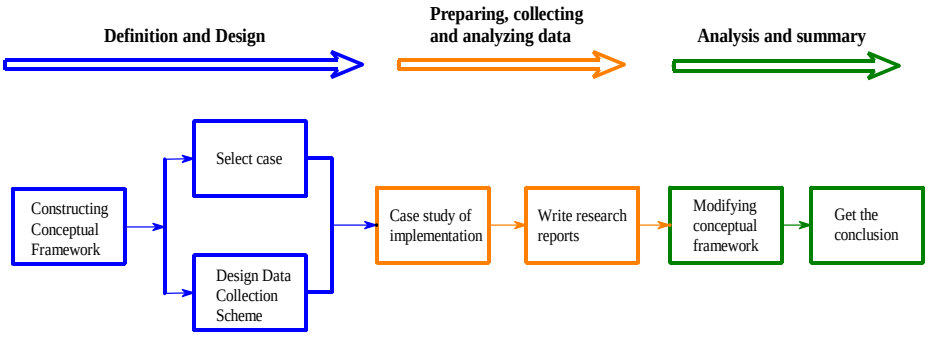


Figure 3.1: The research process

Based on the conceptual framework above, we selected one case of HPM lesson that integrates the history of conic section into teaching practice and was conducted by an HPM studio in a Chinese senior high school. Through case analysis, the conceptual framework is modified to obtain the result of this study.

3.1 The case school and participants

The research sample of this study is a HPM studio in a senior high school in China. The school was founded in 2005, and its students here are of the medium level. The participants include the research team and school teacher team, and are Research Team consists of University teachers and graduate students, while School Teacher Team includes three teachers and a teaching expert in senior high school. The basic information of the members in school teacher team is shown in the following table 3.1.

Table 3.1: Basic information of the members in school teacher team

members	Gender	Teaching age	title	Educational background	Position
A	Male	25	senior	master	Leader
B	Male	10	Intermediate	undergraduate	Teacher*
C	Male	11	senior	undergraduate	Teacher*
D	Male	13	senior	undergraduate	Teacher

* Teachers that responsible for implement the HPM lesson

3.2 Data collection and analysis

We use the modules approach in this case (Jankvist, 2009), and integrate the history into the unit of conic section, which lasts two-class periods. Through the planning, discussing, implementing and analyzing, two HPM lessons become two exemplary lessons. The whole process has undergone three discussions and two implementations. The schedule is

as following table 3.2.

Table 3.2: the schedule of HPMLS

time	activity	Data collection
2017.9.28	Discussion(before lesson)	instruction design and reflection list
2017.11.17	Implementation(1 round)	instruction design、tapes and videos
2017.11.17	discussion(after lesson)	tapes and videos
2017.11.20	Implementation(2 round)	instruction design、tapes and videos
2017.11.20	discussion(after lesson)	tapes and videos

We videotaped and recorded the whole process of the HPMLS, and collected the reflection list and instruction design of the teachers.

By analyzing the data of tapes, videos, instruction design and reflection list in the development process of this HPMLS, we can better explain what the structure and characteristics of the HPMLC are and how the HPMLS works in the context of HPMLC. Therefore, it can provide enlightenment to how to conduct HPM lessons from the perspective of teaching research system.

4 Implementation of framework

4.1 Selecting a topic and preparing

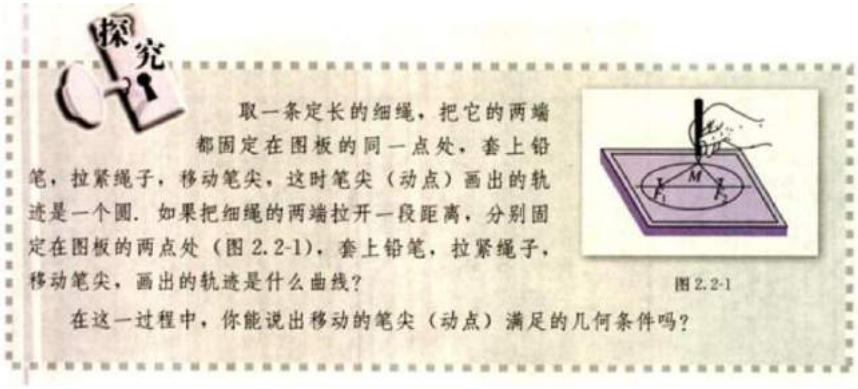
Firstly, the leader A confirms the topic of conic section, which in Chinese textbooks is the content of a chapter, including curves and equations, ellipses, hyperbolas and parabolas. The first and second class periods which are about curves, equations and ellipses in this chapter were selected

The reason of choosing the curves and equations is that some of the curves in the textbook have been studied by mathematicians in history.

"In the section of curve and equation, whether it is an example or a exercise, it is found that these are some of the curves that mathematicians have studied in history. So, why don't you directly tell the students that mathematicians have studied these curves, now it is just to let us restudy it, which can increase the historical sense of mathematics and cultural charm, and let the students understand the inheritance and development of mathematical content."(in reflection list from teacher A)

The reason of choosing the ellipses is that the introduction of ellipse in textbooks is not very appropriate (see fig. 4.1), and using trajectory to define ellipse, students will not know why ellipses are called conic curves.

Based on the reasons above, the research team provided the school teacher team with published papers about "Generation of plane analytic geometry", "Origin and Development of Conic Curves" and "Travel of Elliptic Equation". Those papers were written by the HPM expert in the research team through reading the original historical sources and secondary literature.


探究

取一条定长的细绳，把它的两端都固定在图板的同一点处，套上铅笔，拉紧绳子，移动笔尖，这时笔尖（动点）画出的轨迹是一个圆．如果把细绳的两端拉开一段距离，分别固定在图板的两点处（图 2.2-1），套上铅笔，拉紧绳子，移动笔尖，画出的轨迹是什么曲线？

图 2.2-1

在这一过程中，你能说出移动的笔尖（动点）满足的几何条件吗？

Figure 4.1: The introduction of the ellipse in the textbook

After studying the history of the conic section, School Teacher Team has selected the following historical materials.

Three classic geometry problems in ancient Greek, the Greeks very early found themselves confronted by three problems which they could not solve, at least by the use of the unmarked ruler and the compasses alone. The first was the trisection of any angle, the second problem was the quadrature of the circle and the third problem was the duplication of a cube. (Smith, 1925, pp. 297-298)

Hippocrates showed that the problem of duplicating the cube resolves itself into the finding of two mean proportionals between two given lines. If $a:x=x:y=y:b$, then $x^2=ay$, $y^2=bx$, and hence $x^4=a^2y^2=a^2bx$, or $x^3=a^2b$. If $b=2a$, then $x^3=2a^3$, that is, the cube of edge x will then have double the volume of a given cube with edge a. since we have the three equations $x^2=ay$ (parabola), $y^2=bx$ (parabola), and $ab=xy$ (hyperbola), we can evidently solve the problem by finding the intersection of two parabolas or of a parabola and hyperbola. These methods are credited to Menaechmus (Smith, 1925, p. 313).

Menaechmus may have used that property of the parabola expressed by the equation $y^2=px$, and also that property of the rectangular hyperbola expressed by the equation $xy=c^2$. Archimedes used the same relation for the parabola, Apollonius carried the method much farther, the names “ellipse”, “parabola” and “hyperbola” are probably due to Apollonnius (Smith, 1925, p. 319).

We have seen that Menaechmus solved the problem of the two mean proportionals by means of conic sections. Menaechmus came to think of obtaining curves by cutting a cone. Aristaeus used the title ‘solid loci’ instead of ‘conics’ to indicate that the main devote to conics regarded as loci. He must have discussed the locus of three-line and four-line problem. Apollonius also studied this problem in Book III of *The Conics*. Pappus studied this problem and distinguished three loci: ‘plane loci’, meaning of straight line and circle; solid loci’, meaning conic sections; and ‘linear loci’, meaning curves with a more complicated and indeed a forced or unnatural origin such as spirals, quadratrices, conchoids and cissoids (Heath, 1921).

Apollonius proved that ‘in an ellipse the sum of the focal distance of any point is equal to the long axis’ (Apollonius, 1896). The French mathematician and astronomer Lahire (1640-1719) gave the definition of the focal radius of the ellipse in his work (Lahire, 1679).

The definition of the focal radius of the ellipse is widely used. The French

mathematician L'Hospital in 18th century (L'Hospital, 1720) and the British mathematician Wright in the 19th century (Wright, 1836) gave the method of deriving the the equation of an ellipse.

Dandelin's spheres were discovered in 1822. They are named in honor of the Belgian mathematician Germinal Pierre Dandelin. That the locus of points such that the sum of the distances to two fixed points (the foci) is constant is an ellipse was known to ancient Greek mathematicians (like Apollonius of Perga), but Dandelin's spheres facilitate the proof (Boag, 2010).

4.2 Discussing and Designing（First round）

After the historical materials being selected, the leader A designed three continuous lessons. They are "curves and equations" (responsible for teacher B); "solving the equation of a curve" (responsible for teacher C); "the ellipse and its standard equation" (responsible for teacher D). Each teacher carried out a preliminary teaching design. After communicating within the school teacher team, they present their teaching designs to Research team in the HPM Seminar. The designs of the three teachers are as follows.

Curves and equations: (1) To introduce the three classic geometry problem in ancient Greek to students, analyze the method of duplication of a cube by Hippocrates and tell students who the ancient Greek discovered the conic curve while solving the problem. (2) Use a plane to cut the cone and get the oval. (3) Some trajectory problems of ancient Greek research are given. (4) Teach the "purity" and "completeness" of curves and equations around the trajectory problem. (5) Solve the trajectory problem, and then summarize several steps to track equation, and then do some exercises.

Solving the equation of a curve: (1) Take the contents of the lesson above and review the steps of solving the trajectory equation. (2) Continue by finding the equation of the trajectory problem given in the first lesson. (3) Give an example: the known curvilinear equation is a circle, a moving point is on the circle, and the vertical axis of the x-axis is passes through the moving point. To find the locus of the midpoint of the perpendicular line (the locus is ellipse). (4) Continue to study the three line problem (if the ratio is not 1, the trajectory is an ellipse).

Ellipse and its standard equation: (1) Along with the content of the first and second lessons, the students knew that the ellipse can be cut by the cone, and then the geometric method of the second lesson is used to lead to the ellipse, and it is explored by the Dandelin spheres. (2) The standard equation of an ellipse is derived from the analytic method, and it is prepared to be deduced in three or four ways. (3) More examples and exercises about ellipses are given.

After their presentation, the research team discussed these three teaching designs with the teachers' team, and the teachers' team modified their designs afterwards.

4.3 Implementing and Evaluating (First round)

Because teacher D have another task, so he is no longer in charge of the lesson of Ellipse and its standard equation, as the result, teacher B was still in charge of the lesson of curve and equation, teacher C become in charge of the lesson of Ellipse and its standard

equation. So the three lesson planed in the first place became two lessons when implementing. Those two lessons that implemented are as following. Research team and teachers' team observe those two lessons together.

The first lesson started by introducing three classical geometric problems in ancient Greek, and used a plane to cut the cone and get the oval. Then began to study the six trajectory problems in ancient Greek:

(1) The distance to a fixed-point is equal to a constant;

(2) The sum of the distances to the two fixed-point is equal to a constant;

(3) The absolute value of the difference between two fixed-points is equal to a constant;

(4) The ratio of the distances to two fixed-point is equal to a constant;

(5) The ratio of the distances to two straight lines (parallel or intersecting) is equal to a constant (two-line problem)

(6) Given three straight lines, the ratio of the product of the distances of a moving point to the two lines, to the square of the distance to the third straight line is equal to a constant (three-line problem).

The teacher said that Apollonius, the Greek mathematician, used geometry to solve the problem of "three lines". However, the process was very complicated, and the "four line" trajectory problem cannot be completely solved by geometric methods. In the early 17th century, Descartes invented the coordinate system, so he, together with Fermat, began to study the trajectory problem using algebra on the basis of the coordinate system – "analytic method".

Let the students find the conic section through the problem by three-line and four-line problem. Then the teacher explained the relationship between the curve and the equation, and then let the students get the trajectory equation of the problem 1, 5 and 6. The ellipse is obtained from question 6. Then the teacher posed the four-line problem and introduced the conic sections.

The second lesson introduced the 'ellipse' from our life. Let the student recall that by using a plane to cut the cone and get the oval in the first lesson, and use of Dandelin's spheres to get the property that 'the locus of points such that the sum of the distances to two fixed points (the foci) is constant in the ellipse', could also solve problem 2 of the first lesson. After that define the ellipse by this property.

Then the teacher let the students explore the derivation methods of equation of an ellipse. Students are given a variety of methods and teacher complement the method in the history. After that, students do some classroom exercises, and then the teacher explains some extension to the ellipse. Finally, let's students take a look at what problem of trajectory had been solved after the two lessons. Problem 3 and 4 would not be solved until the following lesson.

After those two lessons, the teachers collect feedback from students and do some preliminary analysis. Then the research team and teachers' team conducted the evaluation activity.

4.4 Discussing and Implementing (Second round)

The evaluation activity found that (1) It is difficult for students to understand the three

classical geometric problems and it takes too long to introduce them. (2) The order of the trajectory problem does not conform to the cognitive order of the students. (3) Too many derivation methods are unacceptable to students. (4) Students need more explanation to the extension. So the teachers' team modified their teaching design as following.

Curve and equation: (1) Shorten the introduction of three classic geometry problems in ancient Greek. (2) Adjust the order of six trajectory problems in ancient Greek. Ellipse and its standard equation: (1) Cut down some derivation methods of equation of ellipse, and increase the time for students to explore. (2) Reduced the number of classroom exercises and add explanation to the extension.

Then those two HPM lesson become two exemplary lessons and implemented. Research team, teachers' team and teachers in this region observe these two lessons together. The following content will be focused on presenting the differences between the two rounds of implementation.

The first lesson is introduced from three classic geometry problems in ancient Greek and began to study the six trajectory problems in ancient Greek:

(1) The ratio of distance to two straight lines (parallel or intersecting) is equal to constant(two-line problem)

(2) The distance to a fixed-point is equal to a constant

(3) The ratio of the distance to the two fixed-point is equal to the constant

(4) Given three straight lines, the ratio of distance product from the moving point to the two lines the square of the distance to third straight lines equal to the constant(three-line problem)

(5) The sum of the distance to the two fixed-point is equal to the constant

(6) The absolute value of the difference between the two fixed-point is equal to the constant

The teacher said the "four line" trajectory problem cannot be completely solved by geometry method and introduced the "analytic method".

Teacher explained the relationship between the curve and the equation. And then let the students get the trajectory equation of the problem1, 2 and 3. The ellipse is obtained from question 4. Then the teacher posed the four-line problem and introduced the conic sections.

The second lesson introduce the 'ellipse' from our life and use Dandelin spheres to get the property, it is also solve the problem 5 in the first lesson, then define the ellipse by this property.

Then the teacher let the students explore the derivation methods of the equation of an ellipse. The students were given a variety of methods and the teacher focused on two methods in history. The students did some classroom exercises, and then the teacher explained some extension. Finally, it is found that problem 6 remained unsolved and that it would be solved in the following lesson.

After those two lessons, teachers collected feedback of students and do some preliminary analysis. Then the research team, the teachers' team and teachers in this region conducted the activity of the lesson's evaluation.

4.5 Analyzing and Writing

The last step of HPMLS is to analyze students' feedback and write the paper of this HPM lesson. Following present some preliminary result of student's feedback.

About the question of 'What kind of teaching method do you prefer? Teaching methods integrating the history of mathematics, or traditional ways? Why?'. The result is that there were 33 students (40 in total) who liked to integrate the history of mathematics, accounting for 82, 5%, and 7 students who liked traditional ways of teaching, accounting for 17.5%. There are several reasons in favor of integrating history of mathematics: (1) Classroom is more interesting, not boring (25%). (2) It is easier to understand the mathematical content (15%). (3) Understand the development of the content (12.5%). (4) Acquired much extracurricular knowledge and enhanced interest in research questions (12.5%). (5) The classroom is vivid and impressive (12.5%). (6) A study spirit that can learn from the ancient people (5%).

About the question 'Please explain why we need to establish a coordinate system and use algebraic methods to find out the equation'. In the pretest stage, the results are (1) Because of convenience (70.3%); (2) easy to find the law (10.8%); (3) do not know (13.5%). In the post test stage the results are: (1) Because of convenience (62.5%); (2) it is difficult to solve some complicated problems by using a purely geometric method (20%); (3) do not know (10%).

About the question of 'Please tell me why the curve like a circle and an ellipse is called a conic section'. In the pretest, the result are (1) do not know (83.8%); (2) it is because the mathematicians took the name in the first place (10.8%). In the post test the result are: (1) Because it is a curve obtained by the plane when it cuts a cone (82.5%); (2) do not know (10%).

After students' feedback, the teachers reflect on teaching and write articles.

5 Findings and discussion

After the analysis of those cases, we can modify the concept framework and get the question of what the structure and characteristics of PLC and how Chinese Lesson Study that integrates HM into ME work in the context of PCL.

5.1 The structure and character of HPMLC

From the above case, we can see that the whole process was done by two teams, HPM research team and School teachers' team, and each team has its own expertise. There was a leader in each team; the leader in school teachers' team is responsible for selecting a topic and planning the lessons as a whole. The research team is mainly responsible for theoretical guidance and providing historical materials, while the school teacher team is mainly responsible for teaching design and teaching practice. They became related to each other via an HPM seminar. This seminar included the discussion in the stage of teaching design and the lesson's evaluation after the implementation of the lessons.

From the above case, we can see HPMLC have the common characteristics of the PLC,

they are Shared and Supportive Leadership, Shared Values and Vision, Collective Learning and Application, Shared personal practice and Supportive Condition, but there are also some special points. The revised model is shown as follows (see Table 5.1). From the perspective of Shared and Supportive Leadership, we can see that it nurtured leadership in the school teacher team and all teachers in the team shared responsibility. On the point of Shared Values and Vision, we can see that teachers have common values and norms, focus on student learning and shared vision of HPM guides teacher's teaching and learning. About the collective learning and application, school teachers work collaboratively to plan the HPM lesson and apply the new knowledge in the teaching practice. In relation to the shared personal practice, we can say that the teacher will modify his or her teaching based on the opinions which are shared by others during the lesson's evaluation stage. In relation to the Supportive Condition, we can say that the relation between teachers is very close and the school provides a lot of support.

Table 5.1: Characteristics of HPMLC

Characteristics	connotation
Shared and Supportive Leadership	● Nurturing leadership in the school team ● Shared power, authority and responsibility
Shared Values and Vision	● Common values and norms ● Focus on student learning ● Shared vision of HPM guides teaching and learning
Collective Learning and Application	● Working collaboratively to plan the HPM lesson ● Application the new knowledge in the teaching practice
Shared personal practice	● Peer observation in HPM lesson ● Sharing opinions in the lesson evaluating ● Feedback to modify lesson plan
Supportive Condition	● Relationships ● Structure

Compared with the previous characteristics, these characteristics have more HPM features.

5.2 The current model of HPMLS

From the above case, the current model of HPMLS in the context of HPMCL needs some modifications. The revised model is shown in figure 5.1.

The first stage is "Selecting a topic & Preparing". In this stage teachers and researchers need to determine a topic and select historical materials about this topic, then teachers need to complement the Preliminary teaching design by use the historical materials.

The second stage is "Discussing & Designing". In this stage, teachers will display teaching design to researchers and discuss their design with each other. Based on the discussion, the teachers will modify their preliminary teaching design.

The third stage is "Implementing & Evaluating". In this stage teachers will implement classroom teaching. After the implementation, the teachers will get students' feedback. Then the teachers' team and the researcher team will evaluate the lesson together. Based

on the lesson evaluation, the teachers modify their teaching design further. This stage is repeated; usually two to three times, until the teachers feel satisfied.

The final stage is "Analyzing & Writing". In this stage, the teachers analyze all data collected after the implementation. Based on these data, the teacher will reflect by themselves and publish a paper on the HPM lessons.

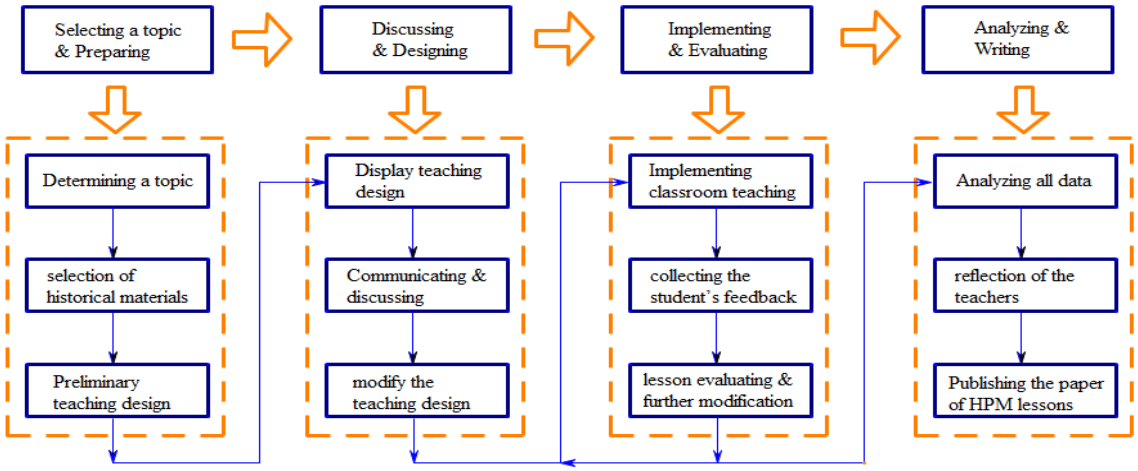


Figure 5.1: The running model of HPMLS

Compared with the previous model, the teaching design has advanced to the first stage. In the second stage, the display teaching design has been added, and in the third stage the cycle has been carried out, while it emphasizes the reflection of teachers in the fourth stage.

6 Conclusion

Integrating HM into ME from the perspective of teaching research system has been less researched. This research proposes a framework for integrating HM into ME in a teaching system. In view of the PLC, we present the HPMLC, based on the Chinese LS, and we propose the HPMLS. We portrayed how the HPMLS work in the context of HPMLC. We cannot say that this is the only framework to describe the teaching research system and it still requires further empirical studies to confirm its educational value.

The role of history of mathematics in general mathematics education research is still limited today. A possible reason for this may be that mathematics education researchers in the past decades had been striving towards a more theoretically founded discipline (Mosvold, Jakobsen and Jankvist, 2014). In this theoretical article, we aimed to theorizing the teaching research system of integrating HM into ME by the theory of PLC and LS. On the one hand, it enriches the related theories of HPM, like triangular pyramid IHT model and design-based IHT procedure (Wang et al., 2018). On the other hand, it also enriches the theories of community in mathematics education, and I hope more mathematics education researchers will pay attention to the study of HPM.

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