PHILOSOPHICAL AND DIDACTIC PRACTICE IN THE UNIVERSE OF FRACTIONS

Trace and Icon

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ABSTRACT

The reflection on the long persistence of unsatisfactory results, has led us to upset the most common idea of fraction: fraction-of-something. To this, we have: (a) corrected the "primitive intuition" of fraction by constructing an "intuitive representation", (b) set the familiarization of children with fractions as a goal, and (c) practiced the idea of fraction as megaconcept. In our enquiring we have focused on the Pythagorean statement: the comparison is a logos. This latter is like a modern act of mathematisation of the comparison and it is the starting point of our didactic practice. It has the following features: (a) it is imposed, (b) it is a leap, (c) it is elementary, (d) it is an axios, (e) it has the characteristic of presence/absence along the path towards the megaconcept.

KEY WORDS: Familiarization – Megaconcept – Trace – Icon – Plurality of truth.

1 Introduction

This reflection concerns the didactics of fractions in primary school. Our attention to this topic has been attracted by the observation of the long persistence of unsatisfactory results in the teaching and learning fractions; persistence that is widely recalled in the scientific literature and which continues, notwithstanding the efforts over decades both in research and in practice.[1]

"One reaction to the prolonged history of poor results in rational number instruction is that … instruction in rational-numbers should be postponed ..." [Kieren, 1980]. In addition to the question of postponing, the long persistence of unsatisfactory results also generates social considerations: although Western knowledge on the didactics of fractions is wide and deep, teaching is ineffective; a discrepancy between knowledge and effectiveness that raises ethical and democratic issues.

These considerations constituted the foundation on which our activity related to the fractions arose. Reflecting on them, we have obtained the following indications: (a) It is important to start teaching fractions already in primary school; this requires adjustments both in some didactic principles and in structuring of the content. (b) We have abandoned the common teaching and learning practice, pursuing a higher level of mathematisation without introducing a higher level of difficulty and opening a wider horizon of potentiality toward the formation of critical citizenship.

[1] "The concept of fraction has manifested itself in education as a refractory one." [Streefland, 1978]. "As investigated in a variety of research and demonstrated with standardized tests, learning fractions is difficult across countries" [Tunç-Pekkan, 2015].

2.1 Two distinct groups of practice

Our teaching practice has had the particular characteristic of proceeding through the interaction of two distinct groups of practice: a group of teaching practice and a group of “reflective philosophical practice”. The first group has carried out an enquiring activity, practicing directly in the classes, from the third to the fifth of the primary school. The second group has carried out both a process of exploration and a process of reflection on the progress of the dialogical interaction[2] that involves teacher, children and the proposed idea of fraction.

The interaction between the two groups of practice involves a double aspect of the report on our activity: one concerns the way of designing the curriculum and practicing it in the classroom[3]; the other concerns the historical and philosophical reflections and discussions that have accompanied its planning and practice. In this conference we give priority to historical and philosophical considerations.

1.2 Fraction as fraction-of-something

Our practice and considerations upset the most common idea of fraction that pervades the ordinary curricula for primary school: the fraction as fraction-of-something. This idea is excellently summarized in Bobos and Sierpinska. “Fraction of a quantity is a mathematical theorization of the visual and intuitive idea of fraction of something. … The idea of fraction-of-something stays in its <u>primitive, intuitive state</u> and functions as an <u>obstacle</u> to the construction of a <u>systemically connected knowledge</u> about fractions.” The fraction-of-something remains the primitive and intuitive representation on which the knowledge of fractions of nearly all people is built.

The idea of fraction as fraction-of-something constrains the way of thinking fractions and practicing with them, favoring situations of division associated with the part-whole subconstruct [Kieren, 1980].[4] In this way, the subconstruct part-whole becomes the only scheme of action that innervates the entire teaching process. Scientific literature has now largely confirmed that these situations have limited teaching effectiveness.

Our alternative proposal to the idea of fraction-of-something is based on two basic believes that we here present as reaction to the statement of Bobos and Sierpinska : (a) we believe that the “<u>primitive, intuitive state</u>” of fraction can be corrected by constructing an “<u>intuitive representation</u>” that <u>does not function as an obstacle</u>; (b) the construction of a <u>systemically connected knowledge</u> about fractions is not a primary school goal; the goal is rather to come up with a process of <u>familiarization</u> aimed to the cohesion of the different subconstructs.

1.3 Intuitive representations versus primitive intuition

We have mentioned the possibility, according to some authors, of postponing the instruction concerning the fractions to when the students have reached the stage of formal operations. We have instead tried to find alternatives to the hypothesis of

[2] We have derived the term "dialogical interaction" from the presentation by D. Guillemette at ESU-8 (ch. 1-8 in this volume).

[3] The curriculum has been presented and discussed at ICMI Study 24.

[4] Kieren identifies five “subconstructs of the rational number construct”: part-whole, quotients, measure, ratios, and operators; but there are other possible subconstructs: proportionality, point on the number line, decimal number and so on.

postponement, involving the idea itself of intuition. If primitive intuition works as an obstacle, it is necessary to proceed already in primary school to the construction of an intuitive representation that elaborates a larger and more effective idea of fraction. The distinction between primitive intuition, understood as "commonsense representation", and intuitive representation, has been proposed by Fishbein, who contrasts to the intuitions firmly correlated with the primitive feeling, with the idea that "intuitions are deeply rooted in our previous, practical and mental, experience". Proper practice with fractions must start in primary school, in order to construct intuitive representations, and to avoid the formation of those obstacles that accompany the idea of fraction-of-something.

1.4 Familiarization

Our teaching proposal is characterized by the choice to avoid, in primary school, to set the aim of constructing a systemically connected knowledge about fractions; we have instead set the familiarization of children with fractions as a goal. The term familiarization was derived from Davydov, and requires an explicit specification when referring to the teaching fractions in primary school.

Practices. To clarify the main features of the practice of familiarization with fractions in primary school, we refer to the following classification of practices: conceptual, algorithmic or executive, strategic or resolutive, semiotic, communicative [D'Amore & Radford, 2017]. The process of familiarization with fractions differs from the usual process of teaching and learning with regard to some of these.

(a) *Conceptual practices*. Familiarization is not directly aimed at the cognitive construction of mathematical concepts; it is rather aimed at shaping the learning environment; cognitive construction will start in the following years.

(b) A*lgorithmic or executive practices*. Speaking of familiarization means "freeing children from reliance on schooled algorithms" [Erlwangers, 1973], and carrying on the practice "without using pre-established formal rules" [Pitkethly & Hunting, 1996].

(c) *Semiotic practices*. The practice of familiarization with fractions is aimed at the construction of the "forms", that is, of those mental structures that precede the concept and the formula.

(d) The other practices, *strategic or resolutive*, and *communicative*, develop in the dialogical interaction among teacher, children and the proposed idea of fraction.

Teaching principles. The difference between the familiarization with fractions in primary school and the usual process of teaching and learning also manifests itself in innovations with respect to the indications contained in the usual teaching principles.

Here are some examples. (a) The process of familiarization proposed by us, introduces a fracture in the historical process of development of the concept of fraction, rediscovering, thanks to a historical/philosophical reflection, an initial act of mathematisation that had been put aside by science; this fact suggests to critically take into account the *principle of scientificity* [Blažková, 2013], according to which a school subject is based on scientific math. (b) Moreover, the initial act of mathematisation constitutes a leap into the didactic process, interrupting systematicity and gradualness and introducing "a form of reflection qualitatively different from the previous one" [Davydov, 1990]. This does not entirely agree with the *principles of systematicity* (the Math curriculum is organized systematically in a logical succession which is necessary to respect), and *gradualness* (“step by step”, Skinner; “natura non facit saltum, gradatim procedit”, Comenio). (c) The initial act of mathematisation is not obtained by generalization and abstraction but it is imposed. This does not tune with the *principle of the concrete operativeness*, according to which all mathematical concepts arise from problems, so ... the rules, the formulas ..., are not imposed by the teacher but naturally conquered by the students [Faggiano, 2008]. (d) The idea of a trace, which, with its presence/absence, is central to our teaching proposal, partially puts into question the *principle of purposefulness*. According to this principle “the aims of each lesson should be formulated in the way to be concrete, achievable, and checkable; in our teaching proposal, the focus is instead on the centrality of dialogical interaction. (e) Finally, the very fact that the aim of our teaching proposal is the familiarization, redefines the content of the *principles of adequacy and verification*, enhancing, in particular, the role of lightness and flexibility. The dialogical interaction, that involves the teacher, the children and the contents, is finalized to guide the children in practicing with “leggerezza” (lightness); that is with serenity, quiet and confidence, achieving adequate results despite the complexity of the theme. Flexibility in practicing has to be favored: in choosing the most appropriate manipulative; in varying the conditions under which the practice is developed; in making stronger the link with real demands; in outlining how some concepts find different realizations in different contexts.

1.5 A different “base of belief”: Fraction as megaconcept

In constructing an intuitive representation that allows avoiding the a priori formation of obstacles, we have explored and practiced the idea of fraction as megaconcept. This idea is suggested by Wagner (1976): “... for the person rational numbers should be a megaconcept involving many interwoven strands” [in Kieren, 1980]. It contrasts with the idea of fraction-of-something exposed by Bobos and Sierpinska and corresponds to a different “base of belief” [Bell in Fischbein, 1982].

Megaconcept demands that all strands/subconstructs contribute to the determination of its nature, and constitute its structural elements. In the megaconcept, the subconstructs find cohesion: they tune in to each other, so that the practice can naturally switch among them.

Developing the path towards the cohesion of the megaconcept, we have started from a fundamental strand and then we proceeded at an appropriate “interweaving rhythm”, assuming the times and the ways of involving the different strands: (a) The subconstruct ratio is the first to be practiced, taking advantage of a meaning of the fraction hidden in Pythagorean mathematics; this meaning is brought to light in the form of an initial act of mathematisation. (b) The appropriate choice of the manipulative allows developing the subconstruct ratio in the subconstruct measure. (c) The practice with appropriate manipulative also allows to support children in discovering the link with the division by themselves; the teacher reinforces this link and proposes activities that lead pupils to practice the Euclidean division. The interweaving of the subconstructs ratio, measure and division is so achieved; these subconstructs attune in the Euclidean division. (d) In our practice, the part-whole scheme is no longer a subconstruct but rather it is an important instance; moreover, the choice of familiarization minimizes the role of the subconstruct operator. (e) The Euclidean division becomes the core of the subsequent interweaving process that involves other subconstructs: point on the number line, decimal number and so on. These practices have taken into account the specificity of the enacting process and the peculiarities of the context.

1 Historical and philosophical considerations

The central part of our presentation concerns the historical and philosophical reflections and the discussions that have accompanied the planning and the practice of our proposal. In the planning, two have been the main authors of reference: Davydov and Toth.

2.1 Davydov

Our exploration of fractions is started by practicing in classroom the situations proposed by Davydov; many of the activities enacted by us, resume the activities indicated by him. Two fundamental ideas have been taken from Davydov and have become the object of our reflection: the ideas of familiarization and of essence of the concept of fraction.

The research of the essence of a concept, central to Davydov's philosophy, and in particular, his research for the "objective content of the concept of fraction" deserve separate reflection. This research suggests two considerations: a) To speak of "essence" obliges to deal also with the "violence of the ontology", treated by Levinas, and to ask ourselves in what form and to what extent the concept of essence has a role in contemporary culture. In the continuation of the presentation we will discuss some considerations on this point. B) Davydov's direction of research raises the question "What to teach?" before the question "How to teach?". This has strongly influenced our investigation and has directed us towards the search for the "originary"[5] content of the concept of fraction.

2.2 Toth

In the course of our investigation we met Imre Toth and we were fascinated by his suggestion to listen to hidden meanings that could still be kept in the Pythagorean mathematics. This indication harmonizes with his basic conviction of the plurality of truth; an idea strongly linked to his discovery of non-Euclidean affirmations in the text of Aristotle. In the pre-Euclidean debate, the Euclidean hypothesis has been successful, certainly due to its effectiveness, but possibly due to philosophical reasons, too: the fourth postulate of Euclid, "all the right angles are equal", is justified, according to some authors, if completed in the form "all the right angles are equal, while the acute and obtuse angles are multiple"; it's the One of Parmenides that has imposed its primacy. Koiré's statement that the birth of science took place with an act of purely philosophical-metaphysical foundation, finds in Euclidean truth a place of implementation. The birth and development of Euclidean geometry have kept hidden non-Euclidean hypotheses for thousands of years. Only during the nineteenth century these hypotheses were unveiled and became "truths", alongside the Euclidean truth; here is the plurality of truth. It is therefore possible to hypothesize similar occurrences for other truths: there are still hidden truths that could be brought to light alongside

[5] The word "originary" is not an English word. Nevertheless, some authors [Roth & Radford, 2011] are beginning to use it. In this way the wealth of meaning possessed by the corresponding term "originario/originaire" that is used in continental philosophy, is recovered.

the known truth. It is this hypothesis that underlies Toth's suggestion to listen to hidden meanings.

The meeting between the searching for the essence of the concept (Davydov) and the listening to hidden meanings (Toth) has become the source of our proposal concerning fractions. History has become "the site" where our project has found its structure; in this site we have looked for the "originary" meaning of the concept of fraction and we have found some foundational aspects that allow rethinking its didactics. At the same time, the plurality of truth, generated by the disclosure of hidden meanings, responds to the need, according to Lévinas, to dissolve the violence associated with the search for the essence of a concept.

2.3 Anthyphairesis

After hearing Toth at the conference held in Bergamo,[6] while he was exhorting to listen to hidden meanings, we discovered, thanks to the reading of the *Menone* proposed by him, the anthyphairesis, the Pythagorean procedure of comparison. The practice with anthyphairesis provided us the "ladder" (Wittgenstein) to climb to a higher level and from here to look at fractions. However, we do not present here the anthyphairesis, due to the fact that there is no need for teachers and children to know and practice it.[7]

Walking step by step the concrete actions of the anthyphairetic process, we have met some indications stored in it and still potentially significant: a) indications of historical type, as this walking allows to listen again to not secondary aspects of Greek philosophy; b) indications of mathematical type, as it highlights a "physical" language for rational numbers and it enables an unusual outlook on the intrinsic reciprocity of a primitive concept of measure; c) indications of pedagogical type, which have moved our project.

2.4 The mathematisation of the comparison

The reflection of historical type and the practice of anthyphairesis have led us to focus on the Pythagorean statement: "The comparison is a logos (ratio)". This statement, in its flatter formulation: "The comparison is a pair of natural numbers", turns out to be extraordinarily modern. If we compare it with Eddigton's statement that a relativistic event is a quadruplet of numbers,[8] it becomes an act of mathematisation that operates an universal synthesis concerning the concept of comparison.

The method of anthyphairetic comparison did not last long. Already at the time of Euclid it had been forgotten; its crisis came with the discovery of incommensurable quantities and its difficulties were contrasted by the effectiveness of the Euclidean algorithm. This latter overshadowed the anthyphairetic comparison, which was consequently forgotten. But with the anthyphairesis, even the act of mathematisation of the comparison has been forgotten: <u>scientific practice and teaching have put it aside</u>.

Our enquiring led us to conclude that the 'originary' content of the concept of fraction is ascribable to the comparison of quantities. So we have recovered the forgotten act of mathematisation: "the comparison between two quantities is a pair of numbers". This act

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[6] "Matematica, Storia e Filosofia: quale dialogo nella cultura e nella didattica?" Bergamo, 1999, May 19.

[7] We briefly presented the procedure of antyphairetic comparison at HPM 2016, in Montpellier.

[8] "An event in its customary meaning would be the physical happening which occurs at and identifies a particular place and time." (Eddington, p. 45)

has therefore become the starting point of our didactic practice. Two considerations: (a) Our choice corresponds to placing the subconstruct "ratio" as starting point of our classroom activity. But, while usually "Ratio is a complex concept, which demands a long-lasting learning process" [Streefland, 1984], historical reflection allowed us, instead, to make it elementary; elementary because turned to "the originary elements", but also by reason of the "lightness" with which the children have lived the proposed acts: their answer has been quiet, serene, with adequate results. (b) Furthermore, while the analysis and the study of comparison are central in psychology and in teaching, its mathematisation is not part of the common way of considering it. So the mathematisation of the comparison is a "new" way of looking at the world.

2.5 The new universe of fractions

Our approach to fractions is diversified from that of Davydov: Davydov considers the measure as a juxtaposition and "The relationship between one quantity and any other that is taken as a measure is recorded in the form of a number"; the historical - philosophical practice instead leads us to consider measure as comparison; then measure is an ordered pair of numbers.

This implies a substantial difference in the didactics of numbers in primary school. While in Davydov the teaching of numbers proceeds through "extension, intensification, and expansion" starting from natural numbers, in our approach there is a split: the initial acts concerning the concept of number are two: a) counting is a number, b) comparing is a pair of numbers. This splitting breaks the categorical framing in which acts concerning the number teaching are usually blocked.

In our approach, the universe of fractions is not obtained as an extension of the universe of natural numbers; it is new universe, with its own rules and properties.

2.6 Reflections about the act of mathematisation

In this presentation we do not describe the classroom practice to introduce the act of mathematisation of the comparison. We show instead some features of this act.

a) The mathematisation of the comparison is imposed. This act of mathematisation is not present in Western culture, and in some workshops we have kept (in Milan Bicocca and at CIEAEM 66, Lyon), no one spontaneously has hinted at it. Despite its simplicity it is not a spontaneous consequence of the common observing and acting. Then it cannot be discovered by the children.
b) The mathematisation of the comparison is a leap. Another form of recording of activities in the exercise book, the representation by segments, is a form of mathematisation obtained through generalization and abstraction from concrete experiences. The representation of the comparison by a pair of numbers is instead a leap: the special symbolic representation A;B = 9;5 ("the comparison between A and B is the pair of numbers 9;5) is non-spontaneous; it consists "in an "interruption in the gradualness," in a "leap," in the appearance of a new form of reflection that is qualitatively different from the preceding stage in knowing. …" (Davydov).
c) The mathematisation of the comparison is elementary. There are no difficulties for children. Children have lived these activities with "lightness – leggerezza" (Calvino).
d) The mathematisation of the comparison is an "axios", "dignum", worthy, because it keeps and opens the trace of the didactic process.

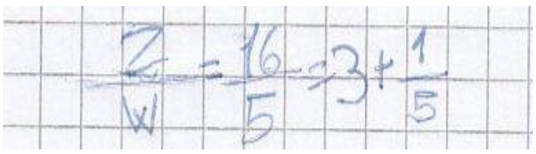
e) **Trace**: the mathematisation of the comparison is “pregnant with significance”. To affirm that the act of mathematisation of the comparison keeps and opens the trace of the didactic process, means two things: first, the process of interweaving that articulates the different strands of the megaconcept into a cohesive and effective structure, begins from this act; second, this act enacts a presence that directs the subsequent didactic path; this presence is due to inescapable indications that this act keeps; indications that orientates in assuming the times and the ways of involving the different strands. However, the fact that the indications find different realities in specific actualization, causes an absence to be breathed, while seeking the most effective enacting in the particular context. Teacher is forced to “suspend” [Bobos & Sierpinska, 2017] his knowledge on fractions and to carry on a process of dialogical interaction with children, in order to unveil the path that the initial act indicates. This presence/absence has prompted us to make use of the name “trace”, echoing the philosopher Lévinas.

2.7 Inescapable indications

In order to highlight inescapable indications imposed by the initial act, let us briefly retrace the whole path of mathematisation we have practiced. (a) The act of mathematisation of the comparison leads the children to practice the concept of common unit. (b) Practicing with common unit, children arrive to measure as ordered pair of numbers, and to fraction as measure. (c) The idea of common unit guides towards the rethinking of the usual language related to the concept of measure. This compels a reconceptualisation that involves the names “whole”, “unit”, and “quantity” in a linguistic mathematisation. Their corresponding formulas are a safe reference for didactic activities but are not directly used; children do not know them but they grasp the corresponding “forms”. (d) Practice with appropriate manipulative brings children to discover the link with the division by themselves. (e) The teacher reinforces this link and proposes activities that lead children to practice the Euclidean division.

2.8 Euclidean division: an icon

The mathematisation process comes realized in the Euclidean division.



This latter is the historical milestone that has created confidence in our approach.

In our didactics, the Euclidean division is not lived by children as a formula to memorize; it is rather the "icon" of their active process of learning. The evocative value we ascribe to the word "icon", refers to art history, to which we have recourse to highlight and enhance deeper meanings that this word has in the context of our cultural training.

So the word "icon" houses many "indications":

a) icon as “memorative (mnemonic) synthesis” of own history of active learning;
b) icon as target towards which the enacting steps attune;
c) icon as opening meanings and then making the “trace” to the future activity;
d) icon as medium between teacher and pupil, as it allows the understanding between the two.

3 Concluding observations and perspectives

We conclude our presentation with two types of considerations: about the role of history in our enquiring on fractions, and about some key words.

3.1 Role of history

In this conference we have given priority to historical and philosophical considerations. In particular, the reference to history is characterized by three features:

a) *Break in the historical process of development*. The most widespread idea of fraction is that of "fraction-of-something". This idea is the result of the "historical process of development" of the concept of fraction. Reflecting on history has allowed us to create a break in the historical process of development, rediscovering an act of mathematisation put aside.
b) *Wittgenstein's ladder*. "... He must, so to speak, throw away the ladder after he has climbed up it." Practice with anthyphairesis provided us the ladder to rise to a higher level and from here to design our teaching proposal.
c) *Clues of history*. Different actualizations of the teaching path regain unity in the Euclidean division. So the Euclidean division is the clue left behind by history, and the milestone that has created confidence in our approach.

3.2 Key words

We now recall some key words, trying to briefly indicate any prospects.

Long persistence of unsatisfactory results. Our enquiring try to react to the long persistence of unsatisfactory results in teaching and learning fractions: although Western knowledge on the didactics of fractions is wide and deep, teaching is ineffective. The discrepancy between knowledge and effectiveness requires reflections on the responsibility of knowledge in today's society.

Familiarization. In our teaching proposal about fractions in primary school, the familiarization is opposed to the usual teaching/learning process: familiarization as shaping the learning environment precedes the proper cognitive construction. In our classroom practice we have tried to clarify how the familiarization with the fractions in primary school works. This opens topics of investigation and discussion: (a) how do the ordinary teaching principles change in the presence of a process of familiarization? (b) Does the familiarization work with other subjects in primary school?

Megaconcept – Cohesion. The idea of fraction as a megaconcept that we have practiced in our classroom activities, requires the search for cohesion between the different subconstructs. Cohesion acquires a fundamental role because it gives unity to the interweaving of the strands that integrate into the megaconcept. It depends on the dialogical interaction between teacher, pupil and content, and the nature and quality of the teaching proposal, rely on it. Therefore cohesion must be subject to a constant process of investigation.

Trace. To bring the construction of the new didactic universe of fractions to the initial act of mathematisation of the comparison reveals an attitude to think of a principle pervading that universe. Fragmentation[9] and singularity[10] prevail today in enacting

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[9] "Modeling is the application of a fragment of mathematics to a fragment of reality" [Israel, 2002]
[10] "The singular as knowledge actualized in activity" [Radford & Sabena, 2015]

mathematics. However, there is also something else that deserves to be investigated. To talk about the trace, about of the presence/absence that it brings within itself, means believing in the presence of a principle that indicates, and comes back continuously throughout the path of enacting.

Icon. We resort to the word “icon” to denote the Euclidean division. This choice finds justification in the reference we make to art history and to deeper meanings that this word has in the context of our cultural training. The word icon highlights the roles of evocation (memorative synthesis), indication (trace), and mediation (medium between teacher and children) that the Euclidean division possesses. These roles make Euclidean division the core of the didactic process of fractions. This interpretation of the word icon raises the question of the possibility that icons are also identified for didactic processes related to other topics.

Plurality of truth. Following the indications of Toth, we have discovered a hidden meaning in Pythagorean mathematics, in the form of a modern principle of mathematisation: the mathematisation of the comparison. This allowed us to highlight an "other" concept of measure, more primitive, with reciprocity, different from the usual one. The bringing to light another definition of measure, recalls the plurality of truth, so dear to Toth, and echoes the freedom (Cantor) of mathematics: mathematics can progress not only by “extension, intensification, and expansion”, but also by working up an unveiled, originary meaning. This make sure that universality replaces totality, plurality of truth replaces uniqueness, the violence of ontology fades away, while the power of truth is preserved. This makes mathematics the paradigm on which to rebuild critical citizenship.

REFERENCES

Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio and proportion. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 296-333). New York: Macmillan Publishing.

Blažková, R. (2013) *Didactics of mathematics*. www.ped.muni.cz/wmath/interma/blazkova_en.doc

Bobos, G., & Sierpinska, A. (2017). Measurement Approach to Teaching Fractions: A Design Experiment in a Pre-service Course for Elementary Teachers. *International Journal for Mathematics Teaching and Learning*, *18*(2), 203 – 239.

D’Amore, B., & Radford L. (2017) *Enseñanza y aprendizaje de las matemáticas: problemas semióticos, epistemológicos y prácticos*. Bogotá: Universidad Distrital Francisco José de Caldas.

Davydov, V. V. (1990) *Types of Generalization in Instruction: Logical and Psychological Problems in the Structuring of School Curricula*. Soviet Studies in Mathematics Education, vol. 2. Reston Virginia: National Council of Teachers of Mathematics,.

Davydov, V. & Tsvetkovich, Z. H. (1991). The object sources of the concept of fractions. *Focus on Learning Problems in Mathematics*, *13*(1), 13-64.

Eddington, A. (1920). *Space, Time and Gravitation. An outline of the General Relativity Theory*. Cambridge: Cambridge University Press.

Erlwangers, S. H. (1973). *Benny's Conception of Rules and Answers in IPI Mathematics*. http://www.wou.edu/~girodm/library/benny.pdf

Faggiano, E. (2008). *Elementi di Didattica della Matematica*. https://www.dm.uniba.it/Members/ faggiano/didattica/introduzione.pdf

Fishbein, E. (2005) *Intuition-in-Science-and-Mathematics*. https://www.scribd.com/doc/163196879/ Intuition-in-Science-and-Mathematics-E-Fischbein-2005-WW

Fowler, D. H. (1979). Ratio in Early Greek Mathematics. *Bulletin (New Series) of the American Mathematical Society*, *1*(6), 807-846.

Gabriel, F. C., Szucs D., & Content, A. (2013). The Development of the Mental Representations of the Magnitude of Fractions. *PLOS ONE.* Volume 8 (Issue 11) e80016.

Kerslake, D. (1986). *Fractions: Children's Strategies and Errors. A Report of the Strategies and Errors in Secondary Mathematics Project* (pp.1-139). Chelsea College, University of London: The NFER-NELSON Publishing Company Ltd.

Kieren, T. E. (1980). *Recent Research on Number Learning*. Edited by Kieren T. E. (pp. 1-175) University of Alberta, ERIC Clearing house for Science, Mathematics, and Environmental Education, Columbus, Ohio. Washington, DC: National Institute of Education (ED).

Longoni, P., Riva, G., & Rottoli, E. (2016). Anthyphairesis, the "Originary" Core of the Concept of Fraction. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016 ICME Satellite Meeting – HPM 2016* (pp. 477-487). Montpellier: IREM de Montpellier.

Pitkethly, A., & Hunting R. (1996). A review of recent research in the area of initial fraction concepts. *Educational Studies in Mathematics 30*, 5-38.

Radford L., & Sabena C. (2015). *The Question of Method in a Vygotskian Semiotic Approach*. http://luisradford.ca/publications/

Streefland, L. (1978). Some observational results concerning the mental constitution of the concept of fraction. *Educational Studies in Mathematics*, *9*(1), 51-73.

Streefland, L. (1984). Search for the roots of ratio: Some thoughts on the long term learning process (towards ... a theory). *Educational Studies in Mathematics*, *15*(4), 327-348.

Tenuta U. (1996). *I principi metodologico-didattici dell'educazione matematica*. www.edscuola.it/ archivio/didattica/emprin.pdf

Toth, I. (1998). *Lo schiavo di Menone*. Editore: Vita e Pensiero. Milano.

Tunç-Pekkan, Z. (2015) An analysis of elementary school children's fractional knowledge depicted with circle, rectangle, and number line representations. *Educational Studies in Mathematics*, *8* (3), 419-441.

Wagner, H. (1976) *An analysis of the fraction concept.* An unpublished report, Chicago.