ON MATHEMATICAL REASONING

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ABSTRACT

Mathematics and logic are subjects of a special kind, which stems from their formal character. Basically, mathematics and logic are often viewed as empty formal manipulation of symbols. However, this opinion hides the constructive character of the topics. The constructive steps in mathematical and logical reasoning bring new information into the reasoning process which allows us to see mathematical and logical reasoning as dialog. Connecting this dialogic character to Platonic approach we can understand the dialog as foundation of education of mathematics. The approach of the paper is basically philosophical. However, we connect philosophical discussion to present-day pedagogical discussion in which dialog is taken seriously. We will show that there are interesting interconnection between (formal) mathematical reasoning and proper dialog.

1 Introduction

According to Popper (1979, p. 133), we may be interested in mathematics either by being interested in theorems or being interested in proofs. The first viewpoint emphasizes the truth or falsity of mathematical statement, and the latter emphasize the existence of proofs. Mathematical statements, as with all statements, are either true or false, and this in principle, can be listed.[1] The listing of mathematical truths, or given truths, makes mathematics static and, as such, does not teach mathematical reasoning. Reasoning is a factual process in time and space. Proofs express the reasoning process, and the search for proofs exemplifies mathematical reasoning. Hence, there might be some truth in a common opinion that says that we learn mathematical reasoning while learning mathematics.

In logic, mathematical proofs are defined as sequences of statements in which each statement is an axiom (or a premise) or is achieved from earlier statements by the application of an inference rule.[2] This kind of "statement view" of mathematical proofs, therefore, entails an opinion that mathematics is about manipulation of symbols. This opinion is supported by textbooks of mathematics in high school and in elementary school. At the same, this opinion hides that mathematical argumentation can be not only linguistic, but also visual (or pictorial). Therefore, there is a need to study mathematical reasoning more closely.

Mathematical reasoning is a more general kind of human reasoning, which is a kind of human mental process that takes place in someone's mind or brain. Unfortunately (or fortunately), we do not have access to what happens inside someone's mind or in someone's brain.[3] Haack (1978, pp. 240–241) specifies the focus of logic as follows:

[1] In principle, a computer can list all the mathematical truths. However, it would take an infinite amount of time to do so. Mathematical truth has a complexity of Δ^1_1, which is the logical complexity of the mathematical truth that cannot be avoided. For details, see, for example, Mutanen 2004.
[2] See any textbook of logic to confirm this.
[3] Of course, present-day study of brain processes has access, but we will not consider this question here.

Thoughts that are in someone's mind are something subjective, and we have no access to them. The second possibility is that logic would be concerned with propositions that are objective but, as the philosophy of logic shows, they are not accessible and hence cannot be the topic of logic. Therefore, the only possibility in logic is to focus on sentences, i.e., syntactical expressions that are both objective and accessible. This separates logic from psychologism, in which logic was understood as a theory of how human reasoning works.

The discussion of how logic is related to human reasoning is still reasonable, even if psychologism is not a serious possibility. In the philosophy of logic, the relationship between logic and reasoning (or thinking) is basically understood in three different ways: (i) logic describes mental processes (strong psychologism); (ii) logic prescribes mental processes (weak psychologism); or (iii) logic has nothing to do with mental processes (anti-psychologism) (Haack 1978, p.238). According to Haack, Kant represents opinion (i), Peirce represents (ii), and Frege represents (iii). In fact, in the late nineteenth century, especially in Germany, there was strong debate concerning whether logic is descriptive, i.e., whether strong psychologism was true. Echoes of the debate can still be recognized in our understanding of mathematics and logic. Anti-psychologism can be understood as a reaction against strong psychologism. The idea was to develop logic as a formal science. For example, Frege understood logic as *lingua characteristica* and, hence, not formal science in a present-day sense. However, this Fregean understanding is a version of anti-psychologism (Haaparanta 1985). The distinction between strong psychologism and anti-psychologism is not very fruitful, since it seems to be obvious that strong psychologism is not true. However, all the same, it seems evident that logic has something to do with human mental processes (reasoning), and hence anti-psychologism also seems to be wrong (Haack 1978). The discussion of the nature of logic and mathematics has not been restricted to discussion of psychologism and its alternatives; it also includes the relationship between logic (and mathematics) and the sciences. For example, Russell (1903) said that mathematics is a science like other (experimental) sciences, such as zoology; of course, mathematics is more abstract than the other sciences.

Mathematics is very special kind of science, and hence, mathematical reasoning is not easy to characterize. To start, let us take a look at, for example, axioms of Peano arithmetic, which contains four sentences and one sentence schema. According to the basic idea of axiomatization, they say everything about the topic: Let Ω be a set of axioms of Peano arithmetic. Then, the Peano arithmetic (PA) is the set of theorems of the set Ω of axioms that can be expressed $PA = \{\varphi : \Omega \vdash \varphi\}$. However, this does not tell us anything substantial about mathematical reasoning. Mathematical reasoning is coded into the formula "$\Omega \vdash \varphi$" which means that for a given φ, there is a finite sequence $\varphi_1, \ldots, \varphi_n$ such that $\varphi_{\mathrm{n}} = \varphi$ and for all $i < n$, $\varphi_i \in \Omega$ or is achieved from $\varphi_1, \ldots, \varphi_{i\text{-}1}$ by application of an inference rule. This shows the formal character of logical and mathematical inference.

When Hermann Weyl (1956, p. 1832) says that "mathematicians are no Ku Klux Klan with a secret ritual of thinking," he is intending to denote the fact that mathematical reasoning is something public and "objectively" recognizable that can be achieved if attention is focused on linguistic expressions. According to Haack (1978, p. 239) "logic is primarily concerned with *arguments*." Arguments are linguistically expressed formal structures whose strength is of logical interest. Arguments have a dual structure: a set of premises and conclusions inferred, according to inference rules, from the premises as expressed above. In logic, argumentation is analyzed and evaluated; the strongest argument is one in which the relationship between premises and conclusion is deductive.

Arguments in mathematics and in logic are formal and well-structured, which makes them explicit and transparent; from the argument, anyone can see all the information used in the argument. However, as formal structures or deductions do not constitute reasoning, the explicitness and transparency of argument does not make mathematical reasoning similarly explicit and transparent. As Herman Weyl (1956, p. 1832) says, nobody should expect him “to describe the mathematical way of thinking much more clearly than one can describe, say, the democratic way of life.” We may not mystify mathematical reasoning – or the mathematical way of thinking.

Mathematics and logic must not be confused with empirical research into human reasoning, even if there is some (external) connection between the two. Rather, in mathematics and logic, the question “What is mathematical reasoning?” seeks a normative answer. In the study of mathematics education, the focus is on learning mathematics: Questions like “How can one learn mathematics?”, “What kind of learning strategies do students have?”, and “How can we teach mathematics effectively?” are important. Hence, central problems in mathematics education consider the relationship, for example, between mathematical concepts and psychology (Ben-Hur, 2006), or between reasoning and communication (Berinderjeet & Toh, 2012). We are here, basically interested in mathematical reasoning as part of mathematics and logic itself which has interesting consequences to the education of mathematics.

2 About Logic and Mathematics

To understand mathematical reasoning better, we will consider more closely some aspects of logic. Basically, logic (and mathematics) can be seen from two different points of view: Logic and mathematics consist in some factual inferences and calculations, which are the everyday practice of mathematicians and logicians. Often, exercises in school mathematics are focused on this area. Let us call this “micrologic.” On the other hand, the focus might be on the consideration of mathematical and logical reasoning from an “external” perspective. Questions like whether mathematics or logic are decidable, i.e., whether mathematics or logic have a decision method. It is well known that mathematics (any system that contains elementary arithmetic) is not decidable, and it is well known that, for example, sentence calculus and elementary geometry are decidable. Another example of such an “external” perspective is to consider the kinds of model that theories (i.e., sets of sentences) have. The well-known *Löwenheim–Skolem theorem* says that each theory that has an infinite model also has a denumerable model. So, the theory of the reals, which is known to be uncountable, has a denumerable model. These are metatheorems that characterize mathematics and logic from the “macro level.”

The internal point of view of logical and mathematical reasoning shows how to do mathematics and logic, that is, how to prove mathematical and logical results, which is emphasized by Weyl (1956). In school mathematics and logic, this aspect is emphasized. For example, Usiskin (2015) shows that this kind of logic has several interesting aspects that are essential in understanding mathematics, and in teaching and learning mathematics. Unfortunately, formal theorems do not show how to find proofs or how to construct proofs. Maybe this is a reason why mathematics remains such a remote and difficult topic in schools.

Besides mastery of formulating proofs and calculations, we need some general understanding of what mathematics as a whole is, which is the subject of metamathematics and metalogic. Therefore, it is not enough that one can answer mathematical questions, but one has to understand what kinds of questions are mathematical. Unfortunately, as *Gödel's incompleteness theorem* (1931) shows[4], not all mathematical questions are answerable within mathematics.

Why it is not enough that one can just answer mathematical questions? It is obvious – based on school mathematics – that mathematics means answering given mathematical questions, and that all the questions have a correct and true answer. In the case of applications of mathematics, like physics in schools, the problem is not to understand mathematics but to understand physics; so, in applications, mathematics is just a tool that is used. There is no need to understand mathematics or mathematical reasoning.

The need for metatheoretical logic become evident when we speak about the character of mathematical reasoning. Of course, the practice of mathematical reasoning lies in proving theorems and single computations, but all this does not characterize the foundational character of mathematical reasoning. The metalogic is, by definition, a key to understanding the foundations of mathematics, as the famous metalogical results (like the theorems of Löwenheim and Skolem, of Gödel, or of Tarski) demonstrate. These metalogical results give information about mathematical reasoning, and about mathematics more generally.

Metalogical results, at the same, give important information about the methodology of science. In fact, this allows us to see the connection between logic and metalogic (Hendricks, 2007; Shapiro, 2002), which deepens our understanding of mathematics and mathematical reasoning (Usiskin, 2015). Metalogical knowledge also deepens our pedagogical understanding; it helps us to develop the teaching methods of mathematics, but also of the natural sciences (Sieg, 2002; Koponen & Kokkonen, 2014). Metalogic is an important branch of mathematical study that has great theoretical importance in understanding mathematics and methodology of science. For example, the metalogic allows us to analyze the reasonability of structuralism in the philosophy of science: in structuralism, the intention is to generate a metalogical framework without using explicit logic.

3 Mathematical Reasoning

Neither the "micrologic" nor "macrologic" characterized above give us a good understanding how to reason logically or mathematically. This can be seen if we consider more closely how to construct mathematical and logical arguments. Geometry is an excellent example in which mathematical reasoning becomes actual, as the presentation by "Capone, Del Sorbo, Ninni, Fiore & Adesso" at ESU-8 clearly demonstrated. The very idea of geometrical proof is its constructive character, which becomes evident via the pictorial nature of the proofs. In geometry, there is a long tradition of using pictures in the proofs. The pictures and the auxiliary constructions are essential parts of geometrical proofs. The proofs are demonstrative in the sense that the fact to be proved can be seen from the picture constructed by the proof. As the presentation referred to showed, there are

[4] Any consistent formal system *F* within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of *F* that can neither be proved nor disproved in *F* (Raatikainen, 2018).

several different constructions that can be created to prove even a very simple geometrical statement, like the Pythagorean theorem.

Geometrical constructions bring new information into the reasoning process. The information is formulated in pictorial form. In logic the proofs are constructed symbolically (or linguistically). However, the similar increase of information as new geometrical constructions do in geometry can be achieved by instantiation of new individuals. As Hintikka (1973, pp.188-190) shows there is precise measure for the information which can be used to characterize the depth of the given proof. This is connected to aesthetic value of the mathematics (Sinclair 2011). Mathematics uses both pictorial and symbolic argumentation (De Toffioli, 2017; Hintikka & Remes, 1974), but also even bodily argumentation as Sinclair (2011) refers. This shows the importance of understanding of the character of mathematical reasoning. To develop education of mathematics one need to understand the multiplicity of mathematical reasoning.

Unfortunately, there are some restrictions in generating the proofs. For example, there is no effective procedure to find out the best construction for a given proof. However, as Michie (1961) shows, there can also be syntactical proofs for geometrical theorems. Michie refers to the fact that a computer discovered a new proof for a simple geometrical statement.[5] Now we know that geometry can be expressed as an axiomatic syntactical theory that does not need pictorial arguments (Tarski 1968).

Frege explicitly separated axioms from rules of inferences, which is one of the first explicit formulations of the present-day understanding of logic. The formal way to explicate logic (and mathematics) is further developed, for example, by Hilbert. “Hilbert’s program” is an overall metalogical approach in which Hilbert tried to explicate the very character of mathematical and logical reasoning. The use of formal methods made it possible to generate explicit metalogic that studies logic within logic itself, as Gödel’s proof (1931) demonstrates. However. metalogic is not only a specific area in mathematics and logic; several very common results are, in fact, metalogical. (See Quine, 1981; Kneale & Kneale, 1962; Mancosu, 2010.)

Hilbert distinguishes the following three levels of mathematics: (i) ordinary mathematics; (ii) proper mathematics (mathematics in strict sense); and (iii) metamathematics. The intention was not to generate different kinds of reasoning, but these are, as Hilbert says, “the familiar modes of logical inferences” (Mancosu, 2010). This may be a reason why in school mathematics, there is only one kind of reasoning. However, metamathematical results tell us about mathematics – what can be done and what cannot be done. For example, Gödel’s incompleteness theorem tells us that there cannot be a general method to check computer programs (if the programming language is as complex as elementary arithmetic).

4 Argumentation as Computation

Arguments are basically sequences of sentences. The end point of an argument is called its conclusion, and the foundational sentences are called premises. The evaluation is to consider the logical relationship between the conclusion and premises. If the conclusion logically follows, or if the conclusion can be deduced, from the premises, then the

[5] Michie (1961) said that even if the proof was a new one for them, it had already been generated by a human mathematician.

argument is valid. Deduction is a syntactical process, and validity, instead, is a semantical notion. According to Gödel's completeness theorem (1930), a given sentence is deducible if and only if it is valid.

To say an argument is valid means that if the premises are true, then the conclusion must be true. Therefore, logic preserves truth. The sequence of sentences does not presuppose any agent who argues or infers them. Hence, logic is non-personal or objective. Therefore, it is interesting that Hodges (1977, p36) gives the following personal characterization:

> An *argument*, in the sense that concerns us here, is what a person produces when he or she makes a statement and gives reasons for believing the statement. The statement itself is called the *conclusion* of the argument (through it can perfectly well come at the beginning); the stated reasons for believing the conclusion are called *premises*. A person who presents or accepts an argument is said to *deduce* or *infer* its conclusion from its premises.

According to Hodges, argument is related to argumentation. However, in logic, the study of argument is a study of the logical relationship between premises and conclusion. Syntactically, a central problem is to make a deduction from the premises of the conclusion. If one can find a deduction, then the deductive relationship is demonstrated, but if one does not find a deduction, this does not show that there is no such deduction. In fact, Gödel's (and Turing's, among others) achievement was to formalize logical inference such that it is possible to prove that it is not possible to find a deduction (incompleteness theorem). A foundational question behind Turing's, Gödel's, and Church's work was "What is an effectively calculable function?" The history of mathematics provides excellent examples of algorithms that show how to compute certain functions. Moreover, there was a certain consensus about what computability means – the consensus gave an "intuitive" notion of computability, which was not well-specified. However, the need for a formal definition becomes evident, at least partly because of the search for an answer to the famous open problems in mathematics formulated by Hilbert.

The computational approach and logical approach have different roots. In logic, the historic roots are connected to the tradition of a universal language, which had historical advocates like Raymond Lull, Leibniz, and Frege (Kneale & Kneale, 1962). The computational tradition is connected to the algorithmic tradition, which also has its roots in the history of mathematics, Rogers (1967, p. 29) gives examples of well-specified algorithms from Ancient Greece.[6] A specific mathematical study of algorithms was started in the ninth century by the Persian mathematician al-Khowrazmi (Russell & Norvig, 1995, p. 8). Rogers (1967, p. 1) gives a good intuitive characterization of an algorithm by saying that it is "a clerical (i.e., deterministic, book-keeping) procedure".

The idea of formalizing the notion of computation was to formalize deduction such that it becomes a mechanical process that does not presuppose intellect. Turing, in particular, used very attractive language. For example, he suggested to "compare a man in the process of computing a real number to a machine." Turing's intention was to formulate a mechanical computing machine that computes essentially the same way as a human using a paper and pen. The resulting notion of computation was of epistemic character. Being mechanical, it was also independent of the formalism chosen. This was recognized by

[6] Rogers's examples are Eratosthenes's method and the Euclidean algorithm.

Gödel (1946) when he said that the importance of the explication of the notion was that it had "for the first time succeeded in giving an absolute definition of an interesting epistemological notion" (Davis 1965, p. 84).

The identification of the notion of computability with Turing machine computability is known as the *Church–Turing thesis* (Copeland, 2017). There are several formulations of the notion of computation, but they have all been proved to capture the same class of functions. Of course, there are formulations of the notion of computation that extend the computing power of Turing machines, which are both logically and philosophically interesting (Hintikka & Mutanen, 1998; Syropoulos, 2018).

The theory of computation is very abstract and extremely complex field. Still it has deep pedagogical significance. Computational approach emphasizes both agenthood of a learner and process of learning (Hendricks, 2007; Hendricks & Symmons, 2015; Mutanen, 2004). So, there are interesting connections between the computational approach and different kinds of constructive approaches.

In logical reasoning, the intention is to explicate and to make transparent the reasoning. The syntactical formulations, as Frege says, bring "to light every axiom, assumption, hypothesis or whatever else you want to call it on which a proof rests; in this way we obtain a basis for judging the epistemological nature of the theorem" (as quoted in Sieg, 2002, p. 228). The computational approach emphasizes more explicitly the methodical and epistemological aspects of mathematical and logical reasoning that were emphasized by Gödel, as the quotation above shows.

5 Mathematical and Logical Reasoning

Formal arguments are static; hence they do not give an adequate characterization of reasoning. A computational approach brings dynamics into the picture. However, the agent is still missing. Computations are nonpersonal, formal algorithmic processes, even if, as Turing's example shows, computations allow us to consider agenthood. We have seen that geometrical reasoning is a paradigmatic example of mathematical and logical reasoning. First, its strict logical structure is clear. Second, its pictorial character makes the reasoning process informative.

The logical, or arithmetic, approach has a more formal character. It appears as formal manipulation of symbols and formulas. However, this is not the whole truth. As we have seen, geometry can be seen as a formal syntactical theory, just like any other formal logical theory. On the other hand, logical and arithmetical reasoning can be interpreted similarly to geometrical reasoning. Kant speaks of intuition in mathematics, referring to the use of individuals. According to him, in arithmetical reasoning, the intuitive step is to use singular numbers, which is usual in arithmetic (Hintikka, 1973).

This can be generalized such that the use of the existential instantiation rule is a Kantian intuitive step in reasoning. We can further analyses this as separating instantiation of the "dummy name" and real name, in which the first is a formal stem and the second is a substantial step in the reasoning process that brings ne substantial information into the reasoning process (Hintikka, 2007). A similar thing can be seen in mathematics, where we have ε-γ-argumentation, which is usually read as "for a given $\varepsilon > 0$, a $\gamma > 0$ can be found such that …" There, the constructive or substantial step is "can be found," which shows how mathematical reasoning brings new information into the reasoning process by using Kantian intuition.

This shows that pictures and substantial intuition are part and parcel of mathematical and logical reasoning (De Toffoli, 2017; Bråting, 2012). This is an important observation. Pictures and other intuitive steps are substantial steps in mathematical reasoning that convey the information needed to make the inferences needed to complete the intended proof. This is an important observation both because of logic (Hintikka & Remes, 1974) and because of pedagogy (Hintikka, 1982; Plato). This shows how important it is to analyze logical and mathematical reasoning.

In *Meno*, Plato shows how to have a logico-pedagogical dialog. In *Meno*, the dialog is logically strict and pedagogically motivated. The dialog shows the power of logico-pedagogical dialog. Hintikka has generated this approach in such a way that it can be applied to scientific reasoning (Hintikka, 2007), to pedagogy (Hintikka, 1982), and to general human reasoning (Hintikka, Halonen, & Mutanen, 2002). The approach has a firm logical basis (Hintikka & Remes, 1974; De Toffoli, 2017).

The logico-pedagogical dialog belongs to a more general dialogical tradition in science, which can be contrasted with the formal-scientific tradition in science. These two traditions have different roots: the first is connected to the Platonic tradition, and the second is connected to the Parmenidean tradition (Mutanen, 2018). The present-day science and pedagogy are basically seen as separate approaches, which is connected to the Parmenidean tradition. In the Platonic tradition, science and pedagogy are essentially connected, which entails that scientific research, as such, is a dialogical process, which is exemplified in *Meno*. The Parmenidean tradition emphasizes that truth, as such, is the goal of scientific research. The pedagogical understanding is something external to the scientific research.

Dialog has been used more generally in pedagogical literature, which enriches pedagogical and scientific understanding. Dialog may be connected to science and pedagogy in different ways. On one hand, dialog has been connected to the methodology of science and mathematics, as in the works of Hintikka and De Toffioli referred to above. On the other hand, dialog has been understood more generally and the dialog has been connected to dialog within a classroom (Bråting, 2012) or to more general dialog (Radford, 2011). Both enrich our understanding of science and the pedagogy of science.

We have considered dialog from two different points of views. First, we looked at the language and recognized that in mathematics argumentation is based on different kinds of notions (symbolic (or linguistic), pictorial, and bodily). Dialog is based on these different kinds of notions. Proofs are formal expressions of the dialog in this sense which De Toffoli (2017) shows. Second, dialog can be understood as part of general narration as, for example, Burton (2012) and Radford (2011) show. However, to develop education of mathematics these two must be unified which can be done using different kinds of approaches. The notions of information and especially, understanding play central role. The most clear-cut example of the unification is Plato's *Meno*. However, this can be done also within context of modern (formal) logic (Hintikka, 1982).

6 Closing Words

We have seen that mathematics and logic can be, and usually have been, understood as formal sciences, which is well justified: mathematics and logic are formal sciences. Metaresults give some formal restrictions that must be recognized. However, the formal character of mathematics and logic does not entail that there is nothing to be understood in mathematics and logic. The formal character entails that the content is “thin.” However, mathematics and logic are constructive sciences in a proper sense, which entails that the proofs, as such, provide the keys to understanding. However, we must emphasize the pedagogical role of the construction of proofs. This can be done by dialogical methods, which has been increasing in the present-day study of pedagogy of mathematics and logic. This pedagogical approach opens new ways to understand mathematics and logic, and their pedagogy.

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