DIGITAL TECHNOLOGIES AS A WAY OF MAKING ORIGINAL SOURCES ACCESSIBLE TO STUDENTS

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ABSTRACT

In this paper, we argue for the use of digital technologies in making original sources more accessible to students. We present a teaching scenario outlining a use of GeoGebra to 'unpack' a selected proposition from Euclid's Elements. We discuss potential benefits of applying digital technologies through Duval's (2006) framework of semiotic registers, through Trouche's (2005) framework of instrumental genesis, and a use of Barnett and colleagues' (2014) approach of guided reading. The combination of original sources and use of digital technologies appears to be a somewhat overlooked area in the HPM research, not least in terms of empirical investigations. Yet, in this paper we lay down the theoretical bricks for such further investigations.

Keywords: Original sources, digital technologies, semiotic registers, instrumental genesis, guided reading.

1 Introduction

As often pointed out in the HPM literature,[1] use of original sources is one of the most rewarding but also one of the most challenging endeavours in the teaching and learning of mathematics (e.g. Jahnke et al., 2000). One challenging aspect for the students of course concerns situating the source in a historical context often rather different from that of the present. Another challenging aspect concerns the fact that original sources often are quite inaccessible to the students, e.g. because the language of the source is different from that in their usual textbooks, the mathematical notation is different, mathematical concepts are defined differently or even called something different. Hence, in the HPM literature it is often debated how to make the original sources more accessible to the students (e.g. see Jankvist, 2014). Several approaches have been suggested, developed and refined over the years, e.g. the hermeneutic approach (Jahnke et al., 2000; Glaubitz, 2010), guided reading (Barnett, Lodder & Pengelley, 2014), comparative readings (e.g. Siu, 2011). (For further discussion of these and other approaches, see Jankvist, 2014).

In this paper, we argue for, if not an actual approach, then yet a different way of making original sources more accessible to students, namely by having students use digital technologies, e.g. Dynamic Geometry Software (DGS) or Computer Algebra Systems (CAS) or other digital technologies, to 'open up' the original source material. The free dynamic geometry tool GeoGebra (www.geogebra.org) combines geometry and algebra as well as spreadsheets on one interface and can offer dynamic visualisations of mathematical concepts, facts, statements, axioms, etc., and potentially motivate students in accessing original sources and learning mathematics. We are not the first to suggest a combined use of historical sources and digital technologies. Oftentimes students are quite

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familiar with one or several such digital technologies – e.g. in Danish upper secondary school CAS is mandatory in the mathematics program – and may thus be able to use their competencies within these as a way of 'unpacking' an original source. In the terminology of Jahnke (2019), we could consider the digital technologies as making up another hermeneutic circle for the students' work. If relying on Sfard's (2008) notion of discourses in relation to mathematics, we might argue that the digital technologies makes up a familiar discourse for the students, and that this familiarity can be profited from in relation to using original sources. Or if applying Duval's (2006) notion of register, we may argue that the digital technologies offer a more familiar 'register' to the students. But whether choosing one or the other theoretical basis, the fact remains that there seems to be an unresolved potential in relation to using digital technologies when teaching with original sources in mathematics – a potential which the HPM literature largely seems to have missed out on – with a few exceptions[2] – and this despite the fact that attention was already drawn to such potentials in the ICMI Study (Fauvel & van Maanen, 2000).

Through a number of examples, Isoda (2000a) argued how digital technologies – e.g. DGS, CAS, spreadsheets – could contribute to students' mathematical inquiry and reflective thinking by providing multiple representations. For example, in relation to a use of DGS in the reading of Descartes' *Geométrie*: "One of the major pedagogical concerns for many years has been that students have lost the opportunity to experience classical geometrical intuitions, which are not replaced by a haze of algebraic symbols; DGS begins to offer a chance to re-experience some age-old intuitions." (Isoda, 2000a, p. 354). Further argumentation and illustration are to be found in the HPM proceedings (Isoda, 2000b; 2004), the ESU proceedings (Aguilar & Zavaleta, 2015; Bruneau, 2011; Chorlay, 2015; Hong & Wang, 2015; Jankvist, Misfeldt & Aguilar, 2019), and in miscellaneous other channels (Baki & Guven, 2009; Burke & Burroughs, 2009; Caglayan, 2016; Erbas, 2009; Kidron, 2004; Olsen & Thomsen, 2019; Papadopoulos, 2014; Zengin, 2018). Yet, only about a handful of these may be considered as actual *empirical research studies*.[3] And equally important, only very few of these studies make use of the extensive mathematics education literature on digital technologies in the teaching and learning of mathematics.

This paper aims to address the above claimed potentially fruitful interplay between the history of mathematics and the use of digital technologies. We do so by outlining mathematics education theoretical constructs, also related to digital technologies, which when combined may play a role in structuring such an interplay, and by providing an example of a teaching scenario on Euclid's proposition 22 from the *Elements* Book I and the use of GeoGebra.

[2] As part of a recent review in his master's thesis at the Danish School of Education, Balsløv (2018) identified only some fifteen incidents in decades of HPM-related literature that specifically address the combination of using history of mathematics in combination with digital technologies. All of them are published from year 2000 and on.

[3] Jankvist, Clark and Mosvold (in review) provided an empirical example of how a Danish School of Education graduate student on her own initiative used DGS to further her understanding of Vieté's geometrically-inspired method to solve two third degree equations. In addition, two master's theses from the Danish School of Education provide empirical evidence concerning a use of Fermat's method for evaluation of maxima and minima in relation to a use of CAS in upper secondary school (Balsløv, 2018), and a use of Euclid's proposition on the construction of equilateral triangles and a use of DGS in primary school (Olsen & Thomsen, 2017).

2 Representations and semiotic registers

As occasionally pointed out in the available HPM literature[4], digital technologies hold a large potential in terms of multiple representations. To this end, Duval's theory of semiotic registers seems suitable to articulate key aspects of this potential.

The outset for Duval (2006) is that in mathematics one cannot directly access the mathematical objects, as one for instance can in physics through various measuring instruments, etc. In principle, one can only access mathematical objects through semiotic representations, which makes students' work with semiotic representations all-important in their mathematical activities. In particular, it is the transformations between the different semiotic registers that are of importance rather than the representations in themselves. Duval points out that the role played by signs in this regard is not to be placeholders for the mathematical objects, but for other signs. This is to say, from this perspective, signs and transformations between different semiotic representations are the kernel of mathematical activity – as opposed to what is going on in other scientific disciplines. From a mathematics education point of view, one question of course becomes: If one can use different forms of semiotic representations for every mathematical object, how can students then recognize the same represented object through different semiotic representations, which are produced within different systems of representations?

The possibilities of substituting one semiotic representation with another depends on the semiotic system, and every system offers specific possibilities. The capacity of a given representation does not depend on the individual symbol (or sign), but on the semiotic system of which is a part. Natural distinctions are for example language (natural and symbolic) versus images (figures, graphs, etc.). But according to Duval such distinctions are too general and causes us to overlook an important point; namely that some semiotic systems may only be used to perform mathematical processes, while others possess a larger variety of functions. Duval (2006) suggests distinguishing between *monofunctional* and *multifunctional semiotic* registers: "Some semiotic systems can be used for only one cognitive function: mathematical processing… within a monofunctional semiotic system most processes take the form of algorithms" (p. 109). A multifunctional semiotic system "can fulfil a large range of cognitive functions: communication, information processing, awareness, imagination, etc." and "within a multifunctional semiotic system the processes can never be converted into algorithms" (p. 109).

Next, Duval distinguishes between a *treatment*, which takes place within one semiotic register, and *conversions* which happen between registers. An example of a conversion might be the mathematization of the equation story 'Aya is 3 years older than her brother Ali. Together they are 23 years old. How old are they?' into the equation $x+(x+3) = 23$, which takes place between a multifunctional (natural language) register and a monofunctional (symbolic system) register. Solving the resulting equation step by step, however, to reveal that $x=10$, is a treatment, since this takes place within the same register, i.e. the symbolic system. So, for conversions, source register and target register are different, whereas for treatments they are the same. Duval also notes that there are two different types of conversions. A *congruent* conversion is a straightforward translation – or coding – e.g. as the mathematization of the equation story into a symbolic expression

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above. A *non-congruent* conversion is, however, much more complicated. For example, this could be doing the opposite translation in our example, i.e. going from the symbolic expression, $2x+3 = 23$, to an equation story, since there are infinitely many stories to be told based on this equation.

Duval also distinguishes between *discursive representations* resulting from one of the three kinds of discursive operations:

1. Denotation of objects (names, marks...)
2. Statement of relations of properties
3. Inference (deduction, computation...)

and *non-discursive representation* which consist of

1. Shape configurations (1D/2D, 2D/2D, 3D/2D)

Discourse here refers to something like ‘articulation’. It is d*ifficult to articulate geometrical representations and transformations through words, hence these ar*e non-discursive. Together with the distinction between monofunctional and multifunctional, this results in four types of semiotic system representation registers.

	Discursive registers	**Non-discursive registers**
Multifunctional registers (non-algorithms)	Basically the use of natural language, spoken or written	Typically depicting drawings, sketches, figures, patterns
Monofunctional registers (algorithms)	Refers to symbol containing and symbol using systems	Diagrams, graphs, etc. subject to rules of construction

Figure 2.1: Duval’s classification of registers mobilized through mathematical processes.

In relation to the students’ difficulties with mathematical proof and proving, Duval offers the following insights:

> Now we can only mention the important case of language in geometry. We can observe a big gap between a valid deductive reasoning using theorems and the common use of arguments. The two are quite opposite treatments, even though at a surface level the linguistic formulations seem very similar. A valid deductive reasoning runs like a verbal computation of propositions while the use of arguments in order to convince other people runs like the progressive description of a set of beliefs, facts and contradictions. Students can only understand what is a proof when they begin to differentiate these two kinds of reasoning in natural language. In order to make them get to this level, the use of transitional representation activity, such as construction of propositional graphs, is needed. (Duval, 2006, p. 120)

It appears obvious that digital technologies, in the sense of DGS, has a role to play in this respect as well.

3 Instrumental genesis, instrumental orchestration and instrumental distance

Instrumental genesis involves the process of transforming artefacts, such as digital tools, into mathematical instruments (Guin and Trouche, 1999). These instruments then become part of a student's cognitive scheme (Vergnaud, 2009) and can be used epistemically to support their learning of mathematical concepts (Guin & Trouche, 1999; Artigue, 2002). In more detail, a student can internalise an artefact's constraints, resources and procedures in the process of instrumental genesis (Guin & Trouche 1999). There are two processes involved. The process of *instrumentation*, which is how a digital tool shapes and affects the user's thinking, and the process of *instrumentalisation*. During the instrumentalisation process, students may acquire knowledge that may lead to a different use of the artefact, and once achieved, then the student is able to critically reflect upon the activity they are engaged in, potentially reinterpret it but also creatively use these artefacts. We do recognise of course that this process of transforming digital tools into mathematical instruments is lengthy and that instrumental genesis develops over time (Artigue, 2002). Moreover, the interrelation between technical knowledge about the artefact, i.e. knowing how to use and using the artefact, and knowledge of mathematical concepts can prove crucial in succeeding instrumental genesis (Drijvers et al., 2010). Drijvers and Gravemeijer (2005), for example, argued that the apparent technical difficulties students had were maybe due to cognitive difficulties with mathematical concepts. It is quite crucial therefore to consider how best and why to support students during their interactions with a digital tool and achieve the initial step of instrumental genesis (instrumentation). Once this step is achieved, we could say that the digital tool has served its epistemic purpose, i.e. being used to support students' understanding and learning within their cognitive system (Artigue, 2002; Lagrange, 2005; Trouche, 2005). The instrumentalisation process, involving how a student discovers the various functionalities of artefacts and transforms them in their own personal way, can then follow and allow a digital tool to serve its pragmatic purpose too, i.e. being used to create a difference in the world external to the student (ibid.).

Considering though the instrumental approach and how digital tools can be integrated in the mathematics classroom, we should discuss Trouche's (2004) notion of instrumental orchestration. It provides a framework for expressing teachers' work before and during their lessons and interactions with their students. As Drijvers and colleagues (2010) described it with reference to Trouche: "An *instrumental orchestration* is defined as the teacher's intentional and systematic organisation and use of the various artefacts available in a – in this case computerised – learning environment in a given mathematical task situation, in order to guide students' instrumental genesis" (pp. 214-215). Trouche (2004) distinguishes three elements of the instrumental orchestration framework, namely a didactical configuration, an exploitation mode and a didactical performance. A *didactical configuration* describes the arrangement of artefacts in a certain environment and the configuration of the teaching setting. For example, in a mathematics classroom, such an arrangement would involve a certain orchestration of mathematical discourse. An *exploitation mode* regards the strategies the teacher uses to exploit a didactical configuration in order to achieve their teaching objectives. For example, a mathematics teacher would need to make decisions on how to introduce and model a mathematical task using an artefact, on the possible roles an artefact they use for their own teaching, but also for students to interact with, can have, and on the schemes and techniques students should develop and establish. Finally, a *didactical performance* involves the decisions a teacher should take during a lesson and how best to perform in their chosen didactic configuration and exploitation mode. For example, a mathematics teacher would consider what the best probing questions to use to develop students' mathematical thinking or understanding of a concept are, how to respond to certain students' comments, shared solutions and answers and their justifications, how to improvise and identify the best approach when an unexpected aspect of the mathematical task or the technological tool surfaces, or any other emerging goals appear in a lesson.

For digital tools to be integrated in the mathematics classroom, besides looking at teachers' instrumental orchestrations, we need to consider a number of factors. Some of these were described by Haspekian's (2005; 2014) research work and the introduction of the notion of *instrumental distance*. Haspekian explains:

> For a given tool, if the distance to the 'current school habits' is too great, this acts as a constraint on its integration [...]. On the other hand, the didactical potential of technology relies on the distance it introduces with regards to paper-pencil mathematics as, for instance, by providing new representations, new problems, increasing calculation possibilities, etc. This is the case for the dynamic figures in geometry softwares, with respect to the static figures in paper-pencil geometry. The didactic potentialities of these dynamic objects and their benefits for students' learning have been evidenced by many research studies [...] (Haspekian, 2014, p.246).

The notion of distance in Haspekian's work refers to the distance between the praxeologies involved in two different environments. There is a distance between praxes involved when interacting with a digital environment and the praxes involved when interacting with a paper and pencil activity for example. There is also a distance between the scope of a digital environment and how it was designed to be used and the culture of a mathematics classroom and the school's policies.

4 An illustrative case based on the 'guided reading' approach

In this section, we offer a potential teaching scenario for introducing Euclid's Proposition 22 with GeoGebra based on the approach of guided reading. The GeoGebra digital tool can be used to present this proposition, but also allow students to explore the idea Euclid shared with the ultimate goal of proving the proposition. We rely on the approach of *guided readings* of original sources developed by Barnett, Lodder and Pengelley (2014), and also used by Jankvist (2013). This approach offers a sensible way of dealing with the occasional inaccessibility of primary original sources. The main idea is to supply or interrupt the reading of an original source by explanatory comments and illustrative tasks along the way as orchestrated by the teacher. One feature of guided reading is, as claimed by Barnett, Lodder and Pengelley (2014) that "the primary source is now being used not just to introduce the mathematics in an authentically motivated context, but also as a text which the student is explicitly challenged to actively "interpret" as part of their personal process of making modern mathematics their own. In alignment with this shift, the tasks we now write for students increasingly adopt a more active "read, reflect, respond" approach to these sources" (p.10).

In our teaching scenario below, we will showcase how we also made use of Barnett and colleagues' so-called *read-reflect-respond* type of tasks. Barnett and colleagues (2014) aimed at students achieving "a deep understanding of both the similarities and differences between past and present mathematics, not merely the past as a convenient or most natural avenue to the present" (p. 23). In our proposed teaching scenario, we focus on two resources (or media for acquiring mathematical knowledge), paper and digital technologies. "Radical engagement with the disparate discourses of original sources selected from various mathematical communities appears to also support student learning by providing the scaffolding necessary to become a participant in a new (e.g., modern) mathematical discourse" (ibid., p. 24). We focus on how to connect and "unpack" past resources using a modern medium, such as the GeoGebra digital tool designed for mathematical learning, a tool that is familiar to the student, as already mentioned in the introduction. Another goal of Barnett and colleagues was to promote students' mathematical reasoning skills and further develop their ability to create valid mathematical arguments, a goal that we did adopt too.

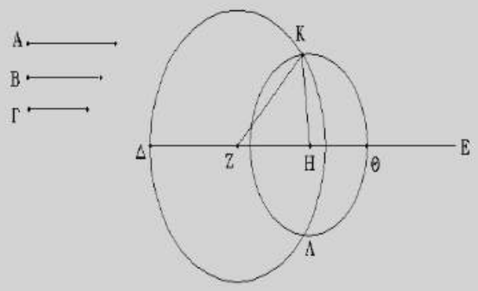
Euclid's Proposition 22 states that "To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one" (source: https://mathcs.clarku.edu/~djoyce/elements/bookI/propI22.html). This is the English translation from the original source in Ancient Greek, which is presented in Figure 4.1. In Figure 4.2, the same proposition is presented in Latin from the 1482 version of Euclid's Book of Elements. Both sources offer a diagrammatic representation for describing Proposition 22. Can you make a comment/argument here about Duval's registers here?

Πρότασις κβ΄. [22]

Ἐκ τριῶν εὐθειῶν, αἵ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις [εὐθείαις], τρίγωνον συστήσασθαι· δεῖ δὲ τὰς δύο τῆς λοιπῆς μείζονας εἶναι πάντῃ μεταλαμβανομένας [διὰ τὸ καὶ παντὸς τριγώνου τὰς δύο πλευρὰς τῆς λοιπῆς μείζονας εἶναι πάντῃ μεταλαμβανομένας].

Ἔστωσαν αἱ δοθεῖσαι τρεῖς εὐθεῖαι αἱ Α, Β, Γ, ὧν αἱ δύο τῆς λοιπῆς μείζονες ἔστωσαν πάντῃ μεταλαμβανόμεναι, αἱ μὲν Α, Β τῆς Γ, αἱ δὲ Α, Γ τῆς Β, καὶ ἔτι αἱ Β, Γ τῆς Α· δεῖ δὴ ἐκ τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συστήσασθαι.

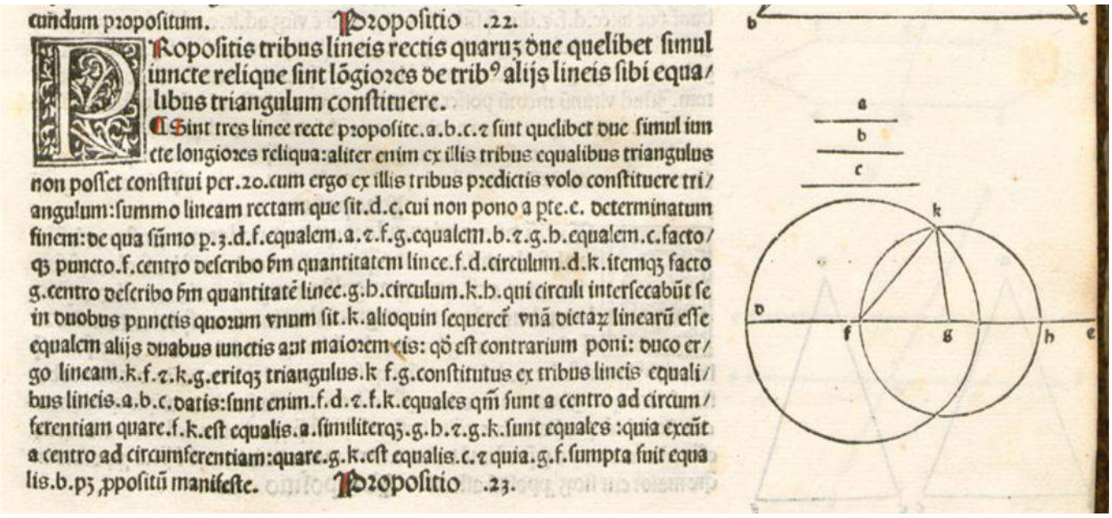
Ἐκκείσθω τις εὐθεῖα ἡ ΔΕ πεπερασμένη μὲν κατὰ τὸ Δ ἄπειρος δὲ κατὰ τὸ Ε, καὶ κείσθω τῇ μὲν Α ἴση ἡ ΔΖ, τῇ δὲ Β ἴση ἡ ΖΗ, τῇ δὲ Γ ἴση ἡ ΗΘ· καὶ κέντρῳ μὲν τῷ Ζ, διαστήματι δὲ τῷ ΖΔ κύκλος γεγράφθω ὁ ΔΚΛ· πάλιν κέντρῳ μὲν τῷ Η, διαστήματι δὲ τῷ ΗΘ κύκλος γεγράφθω ὁ ΚΛΘ, καὶ ἐπεζεύχθωσαν αἱ ΚΖ, ΚΗ· λέγω, ὅτι ἐκ τριῶν εὐθειῶν τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συνέσταται τὸ ΚΖΗ.



Ἐπεὶ γὰρ τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ΔΚΛ κύκλου, ἴση ἐστὶν ἡ ΖΔ τῇ ΖΚ· ἀλλὰ ἡ ΖΔ τῇ Α ἐστιν ἴση. καὶ ἡ ΚΖ ἄρα τῇ Α ἐστιν ἴση. πάλιν, ἐπεὶ τὸ Η σημεῖον κέντρον ἐστὶ τοῦ ΛΚΘ κύκλου, ἴση ἐστὶν ἡ ΗΘ τῇ ΗΚ· ἀλλὰ ἡ ΗΘ τῇ Γ ἐστιν ἴση· καὶ ἡ ΚΗ ἄρα τῇ Γ ἐστιν ἴση. ἐστὶ δὲ καὶ ἡ ΖΗ τῇ Β ἴση· αἱ τρεῖς ἄρα εὐθεῖαι αἱ ΚΖ, ΖΗ, ΗΚ τρισὶ ταῖς Α, Β, Γ ἴσαι εἰσίν.

Ἐκ τριῶν ἄρα εὐθειῶν τῶν ΚΖ, ΖΗ, ΗΚ, αἵ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις εὐθείαις ταῖς Α, Β, Γ, τρίγωνον συνέσταται τὸ ΚΖΗ· ὅπερ ἔδει ποιῆσαι.

Figure 4.1: The Euclid proposition 22 in its original presentation in ancient Greek (source: http://www.physics.ntua.gr/mourmouras/euclid/book1/postulate22.html)


cundum propositum. Propositio .22.

Propositis tribus lineis rectis quarum due quelibet simul iuncte relique sint lõgiores de trib⁹ alijs lineis sibi equalibus triangulum constituere.

Sint tres linee recte proposite.a.b.c.et sint quelibet due simul iuncte longiores reliqua:aliter enim ex illis tribus equalibus triangulus non posset constitui per.20.cum ergo ex illis tribus predictis volo constituere triangulum:summo lineam rectam que sit.d.e.cui non pono a pte.e. determinatum finem:de qua sũmo p.3.d.f.equalem.a.et.f.g.equalem.b.et.g.h.equalem.c.facto que puncto.f.centro describo sm quantitatem linee.f.d.circulum.d.k.itemque facto g.centro describo sm quantitaté linee.g.h.circulum.k.h.qui circuli intersecabũt se in duobus punctis quorum vnum sit.k.alioquin sequeret vnã dictarum linearũ esse equalem alijs duabus iunctis aut maiorem eis: qd est contrarium poni: duco ergo lineam.k.f.et.k.g.eritque triangulus.k f.g.constitutus ex tribus lineis equalibus lineis.a.b.c.datis:sunt enim.f.d.et.f.k.equales qm̃ sunt a centro ad circumferentiam quare.f.k.est equalis.a.similiterque.g.h.et.g.k.sunt equales :quia exeũt a centro ad circumferentiam:quare.g.k.est equalis.c.et quia.g.f.sumpta fuit equalis.b.pz ppositũ manifeste. Propositio .23.

Figure 4.2: The Euclid proposition 22 in its original presentation in Latin from 1482.

There are attempts at recreating this proposition in GeoGebra. For example, such a resource is: https://www.geogebra.org/m/bp2mpZVz. In this construction, the lengths of the three sides of the constructed triangle are variable and can be dragged, i.e. dynamically moved, while the construction reflects that movement and change of the lengths of the three sides. In Figures 4.3 and 4.4, we present how dragging one line segment, CD, impacts the construction.

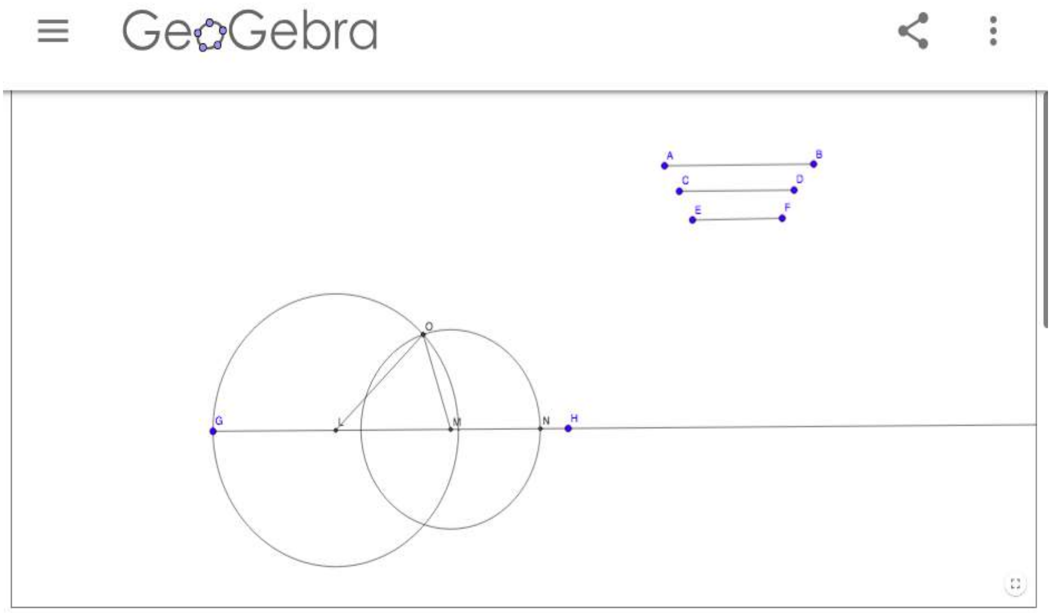


Figure 4.3: GeoGebra construction of Proposition 22
(source: https://www.geogebra.org/m/bp2mpZVz).

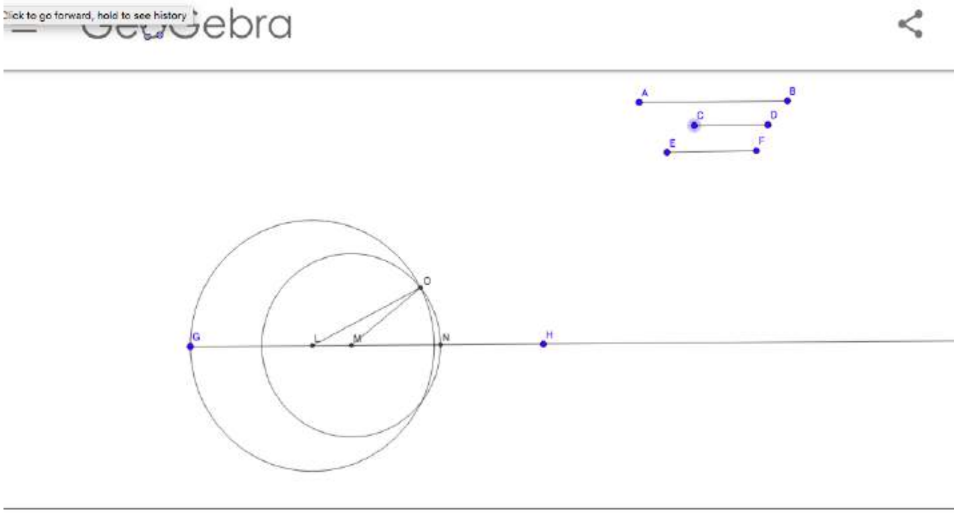


Figure 4.4: GeoGebra construction of Proposition 22, after dragging line segment CD from its original position presented in Figure 4.3 and making its length shorter than the original (source: https://www.geogebra.org/m/bp2mpZVz).

Considering these resources and with the aim of using GeoGebra as a tool for 'unpacking' Proposition 22 and making it more accessible to students, we have designed the following teaching scenario using the guided reading approach, as mentioned earlier. Besides taking into account aspects of Duval's semiotic registers, we considered both the instrumentation and instrumentalisation processes a student is expected to go through in achieving instrumental genesis, but focused mostly on the teacher's potential perspective and in fact a teacher's instrumental orchestration processes in achieving their students' instrumental genesis. We also added the analytical lens of instrumental distance. This considers the distance between the praxeologies of Euclid's Proposition 22 original resource and potentially how it was intended to be used for the teaching and learning of Geometry and the praxes involved when students interact with GeoGebra to interpret Proposition 22 and paper and pencil to record their reflective comments and arguments. We neither intend to discuss the culture of a mathematics classroom nor any school's policies or how this teaching scenario could be carried out considering such factors, as these differ from school to school.

Students are presented with the following **Learning Objectives (LOs)**.

In this learning sequence, you are going to be introduced to some of Euclid's work, and in particular his Proposition 22, as presented in his book of Elements. Euclid was an ancient Greek mathematician, who is often referred to as the "founder of Geometry" (http://www-groups.dcs.st-and.ac.uk/history/Biographies/Euclid.html) and his book, Elements, is one of the most influential work in the history of mathematics which has been used as a core textbook for the teaching and learning of Geometry. "In the *Elements*, Euclid deduced the theorems of what is now called Euclidean geometry from a small set of axioms.". Your Learning Objectives (LOs) are:

LO1. to interpret Euclid's Proposition 22 using paper and pencil and GeoGebra
LO2. to argue about the 'correctness' of this proposition and improve your mathematical reasoning skills.

Using a paper and pencil can support you in the first steps towards interpreting Proposition 22 and potentially creating a diagram. GeoGebra can support you in recreating and analysing Proposition 22 with an interactive and dynamic diagram, as opposed to the static diagram on paper.

The above LOs should set the scene for the activity sequence that supports in bridging the instrumental distance between the paper and pencil medium and GeoGebra. The teacher presents Proposition 22 on the board and printed for students, but without any diagrams. This strategy is used to exploit the didactical configuration of reading the proposition and using a familiar and easy to use medium, that of paper and pencil, as preparation for using the digital medium later on (*exploitation mode*).

TASK 1:

Read the following Proposition 22, as it was presented by Euclid and think for a few minutes on your own about its meaning. What does this proposition state ?

Proposition 22

To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one. I.20

Students are expected to read carefully this sentence and interpret its meaning using any means available to them (e.g., either mentally or by drawing a diagram on paper), before moving on to a paired task, Task 2, (*exploitation mode*). At this stage, GeoGebra is not used as we envisage students need to consider what the proposition states in its paper presentation before interacting with the GeoGebra tool to explore its meaning by carrying out constructions of triangles. This process could be viewed as the first stage of students' *instrumentalisation*, as it is planned with the aim of students' acquiring knowledge that may lead to a different use of the GeoGebra tool, e.g. not just as a tool to construct and explore, but as an analytical tool for reflecting upon their construction through a dynamic exploration (*exploitation mode*).

TASK 2:

In your pairs, share your interpretations of Proposition 22 with each other. Say to your partner what you believe this proposition says. Please use paper and pencil to record your agreed statement of what this Proposition says.

Students are expected to come up with any diagramatic or symbolic representations on paper in their efforts to make sense of the proposition. They are also expected to work as a pair and agree upon a statement for what Proposition 22 states using their own words. Such a preparatory work is aimed at giving meaning to students' interactions with GeoGebra that follow in Task 3. They should have reached a certain level of understanding of what Proposition 22 states and may use GeoGebra as a tool for validating their conjectures by constructing a triangle and dynamically manipulating it (instrumentalisation process), as opposed to carrying out meaningless actions.

Students are then presented with the GeoGebra tool. In this scenario, students are expected to be familiar with GeoGebra and its main functionalities.

TASK 3:

Using GeoGebra, construct a triangle of lengths 3cm, 4cm and 5cm.

Students could potentially use GeoGebra as a drawing tool and create triangles by constructing three line segments of 3cm, 4cm and 5cm respectively, and place them in such a way so that each two segments are connected at a corner. Such an approach cannot be criticised when considering the instrumental distance between the paper medium, which they are used to and are familiar with and their tendency to follow a similar strategy in GeoGebra. It could though be considered as students' initial steps of instrumentation since students 'drawing' with GeoGebra differs from that of drawing on paper and students are 'forced' to use GeoGebra's tools for constructing line segments of given lengths as opposed to a ruler and measuring a line segment in order to draw it. The rationale for this task 3 is to give students the opportunity to consider how to construct a triangle with sides of three different fixed lengths. After a 10-15 minutes exploration of how to construct a triangle in GeoGebra when being given the lengths for its three sides, the teacher runs a class discussion focusing on the strategies students followed to construct their triangles. Such strategies may involve: (a) using the GeoGebra 'polygon' feature, with which students may form a polygon of 3 sides, or (b) using the GeoGebra 'segments with given length' feature, with which students could construct three line segments of 3cm, 4cm and 5cm respectively and join them in such a way so that a triangle is formed. These two didactical configurations of course are not general enough and the students restrict themselves from using a dynamic tool such as GeoGebra to its full potential; (c) Students could construct a 'segment with given length', e.g. AB = 5cm, and then construct two circles. One circle of centre 'A' and radius 3cm and one circle of centre 'B' and radius 4cm. Either of the two points where the circles intersect can be chosen as the third vertex, 'C', of the triangle. This second construction path will be referred to as the 'triangle construction' for the rest of the paper. It is also worth considering that students may have been taught already how to construct a triangle of given lengths (or not) by using a ruler and compass. Such prior knowledge would certainly influence their instrumentalisation process as they would potentially use similar strategies of constructing line segments and circles to recreate such a construction of a triangle on GeoGebra. On the other hand though, such prior knowledge may act as a bridge for the instrumental distance between the paper and pencil medium and GeoGebra. Students would look for ways to construct line segments and circles in GeoGebra mapping their prior experiences on paper. Hovering over the various GeoGebra features, but also the iconic representation of these features can support, shape, as well as affect their thinking processes (*instrumentation*).

The teacher then refers to the Proposition 22 and asks students to compare their actions in Task 3 to what the Proposition states.

> **TASK 4:**
>
> Using GeoGebra, construct a triangle of lengths 3cm, 4cm and 5cm Consider your constructed triangle of sides 3cm, 4cm and 5cm and read again Proposition 22. Using your constructed triangle as an example, describe what Proposition 22 says. Write down any arguments to support your claims and be ready to share with your peers in a class discussion.

The rationale for this discussion is for students to recognise the condition for being able "To construct a triangle out of three straight lines which equal three given straight lines", or in other words to construct a triangle when being given three line segments of certain lengths. Interacting with and exploring their constructed triangle of fixed side lengths in GeoGebra, while bearing in mind the two sentences of Proposition 22 and stating these in their own words, should prompt students' critical reflection on the validity of Proposition 22. Recognising that their triangle is 'fixed' in GeoGebra and none of its corners can be dragged in such a way that their triangle changes size and a different triangle is formed, should reinforce the idea that any constructed triangle of given lengths for its three sides is unique. They should start thinking then about the generalisability of the Proposition. Should this proposition be true for any triangle of given lengths for each side? Are there any conditions for the proposition to hold true? Such probing questions may be used by the teacher to support students' reflections and their efforts in writing down arguments regarding the Proposition. In their written arguments, students are expected to reveal their instrumentalisations, as they may exploit GeoGebra's features and their interactions to form arguments and support their claims. Moreover, students are expected to start thinking about the value of the second sentence in the Proposition, "thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one". Why is this a "necessary" condition to construct a triangle? Students' written arguments will be valuable information as they will reveal the current state of their instrumentation and instrumentalisation processes as well as their mathematical reasoning skills.

Considering Duval's theory, in Task 1, the representation of the translated to English Proposition 22 on paper serves multiple purposes (*multifunctional semiotic register*). It communicates to students Euclid's original presentation of the proposition and makes students aware of the condition for a triangle of given lengths for its three sides to exist.

Students are expected to process the shared information and using for example, their imagination, creativity and/or visualisation skills to ‘translate’ or convert this proposition using different registers, such as a diagram and/or mathematical notation and symbols. Such a process can lead to the creation and use of a non-discursive representation and it is expected to happen in Task 2. In pairs, students have to argue about their interpretations and could potentially ‘move’ between different registers. After articulating their thinking using natural language (*multifunctional discursive register*), they could decide to draw a triangle on paper (*multifunctional non-discursive register*) combined with the use of mathematical notation to describe the condition for their triangle to exist (*monofunctional discursive register*). In Task 3, students make shifts between various registers, but these occur between the paper medium and GeoGebra and within GeoGebra. They create a geometrical figure, i.e. a triangle with sides 3cm, 4cm and 5cm, and interpret the icons representing GeoGebra features, which also involve natural language (hovering over them shows what they do) and symbolic language (algebraic notation to represent angles, points, lengths, etc.). Then, in Task 4, students reflect on what the Proposition states once again and are asked to articulate arguments for supporting their claims and convince their peers. In this case, we could argue that students should be ready to share their beliefs, facts and contradictions, but they are not maybe ready to produce a valid deductive reasoning, as per Duval’s (2000) distinction.

TASK 5:

Have a look at the triangle the teacher made in GeoGebra. What happens if you drag corner **A** of the triangle? What happens to the triangle?

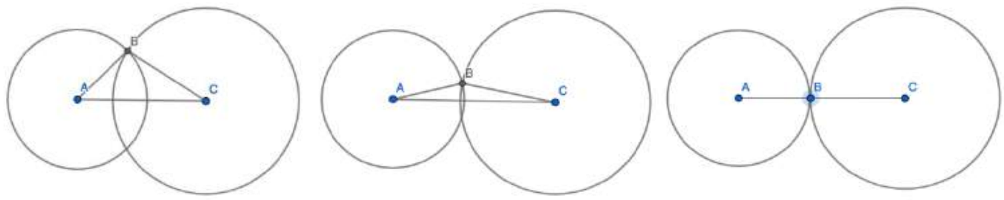


Figure 4.5(a), (b), (c): Students are expected to drag corner A of the triangle in GeoGebra and explore what happens. In (a) the triangle is formed with the longer side as the base. In (b) the corner A is dragged towards the left and the base gets longer, while corner B gets closer to the base. In (c) the corner A is dragged further to the left and corner B ‘lands’ on side AB of the triangle and as a result the triangle ABC ceases to exist and instead a longer line segment AC is created.

The rationale for Task 5 is for students to explore a triangle constructed in GeoGebra with non-fixed lengths for its three sides. This didactical configuration should trigger students’ instrumentalisation of using GeoGebra as a tool for critically reflecting upon the second statement in Proposition 22. Students are expected to recognise that if the corner *A* gets dragged further to the left in such a way that the length *AC* gets longer than the sum of *AB* and *BC*, then the triangle ceases to exist. This realisation is key in understanding the importance of the condition stated in Proposition 22. We could argue for the benefits of using GeoGebra and in fact a triangle of varied lengths for its sides to ‘unpack’ and validate Proposition 22.

Considering a triangle of no given lengths for their three sides in GeoGebra as a didactical configuration can certainly trigger students' critical reflection of Proposition 22, but at the same time some students may still have difficulties thinking in such an abstract way and require working with triangles of specific lengths. The teacher could present students with a number of sets of 3 lengths for constructing a number of triangles. Such a strategy will prompt students to reflect upon sets of lengths that can produce a triangle and others that won't and with some further 'guided reading' and the teacher's intervention, students should recognise the condition of having the sum of two lengths being greater than the remaining length for a triangle to be created, as stated in Proposition 22, and gain a deeper understanding.

> **TASK 6:**
>
> Work in pairs and construct in GeoGebra the following triangles for each set of 3 lengths for the sides:
>
> 2, 3, 3
> 2, 3, 4
> 2, 3, 5
> 2, 2, 2
> 2, 2, 5
>
> What do you notice?

In this process, students are expected to recognise that only for (a), (b) and (d), a triangle can be constructed. The teacher asks for any instances the students couldn't make a triangle, encouraging them to share their reflections and arguments. What does Proposition 22 state in relation to constructing a triangle? Were you all able to construct all triangles? Why? Why not? Anything 'special' about these lengths?

The teacher runs a class discussion aiming at supporting students in reaching a conclusion about the condition for being able to construct a triangle from three given straight lines. Why can't we have a triangle of lengths [2, 2, 5] and [2, 3, 5]? Why can we have a triangle of lengths [2, 2, 2], [2, 3, 4] and [2, 4, 5]? What's the same and what's different between these sets of lengths?

All the above probing questions are aimed at supporting students' development of critical thinking regarding Proposition 22. GeoGebra can support them in constructing quickly and accurately the different triangles and comparing their constructions to reach their conclusions and potentially articulating valid arguments relying upon the specific examples.

> **TASK 7:**
>
> Read Proposition 22 again and focus on the second condition: "thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one". What does this mean? How many sums do you need to find and check if the condition holds? Record your answers and arguments.

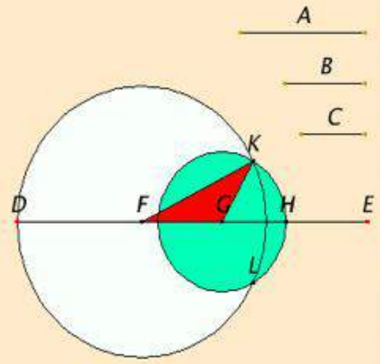
In Task 5, students were presented with the teacher's constructed triangle of varied lengths for its sides, whereas in this Task 7 students are expected to use GeoGebra to test the second condition in Proposition 22 by choosing different lengths for a triangle they construct. GeoGebra then becomes a validation tool for their own constructions, but also a tool that helps them critically reflect upon an original source (*instrumentalisation*).

Considering Duval's (2000) work on registers, working with specific examples could also help them in articulating their arguments regarding Proposition 22 and why and when a triangle can be created when being given a set of lengths for its three sides. The coherency of those arguments though would reveal the type of reasoning students would have achieved at this stage in the learning sequence. Students are prompted to use multifunctional semiotic registers and discursive representations, such as their verbal communications, their strategies for constructing the requested triangles in GeoGebra, and non-discursive representations, such as their constructed triangles in GeoGebra. By the end of Task 7, students should feel confident about what the Proposition 22 states and have a better understanding of the condition. Their arguments may still be based on specific examples and certain cases of lengths for the three sides of a triangle and may not be general enough. Depending on their confidence with mathematical notation, some students may bring algebraic notation into their written statements. In their collaborations, some students may go through a valid deductive reasoning process at this stage, where they argue about the correctness, truthfulness, or in other words, the proof of Proposition 22, using their constructions as specific examples. Students' explorations and arguments most likely based on specific examples, though, on paper and in GeoGebra, cannot be considered as formal proofs. The next step is for students to be exposed to the proof as presented in one of the original sources, but translated into English.

TASK 8:

Read the proof for Proposition 22 presented in the Figure below and follow the steps to construct a triangle of 3 lengths of your choice in GeoGebra.

Let the three given straight lines be *A*, *B*, and *C*, and let the sum of any two of these be greater than the remaining one, namely, *A* plus *B* greater than *C*, *A* plus *C* greater than *B*, and *B* plus *C* greater than *B*.



It is required to construct a triangle out of straight lines equal to *A*, *B*, and *C*.

Set out a straight line *DE*, terminated at *D* but of infinite length in the direction of *E*. Make *DF* equal to A, FG equal to *B*, and *GH* equal to *C*. I.Post.2 I.3

Describe the circle *DKL* with center *F* and radius *FD*. Again, describe the circle *KLH* with center *G* and radius *GH*. Join *KF* and *KG*. I.Post.3 I.Post.1

I say that the triangle *KFG* has been constructed out of three straight lines equal to *A*, *B*, and *C*.

Since the point *F* is the center of the circle *DKL*, therefore *FD* equals *FK*. But *FD* equals *A*, therefore *KF* also equals *A*. I.Def.16 C.N.1

Again, since the point *G* is the center of the circle *LKH*, therefore *GH* equals *GK*. But *GH* equals *C*, therefore *KG* also equals *C*. I.Def.16 C.N.1

And *FG* also equals *B*, therefore the three straight lines *KF*, *FG*, and GK equal the three straight lines *A*, *B*, and C.

Therefore out of the three straight lines *KF*, *FG*, and *GK*, which equal the three given straight lines *A*, *B*, and C, the triangle *KFG* has been constructed.

Q.E.F.

Students are presented with this image taken from https://mathcs.clarku.edu/~djoyce/elements/bookI/propI22.html and are asked to use

GeoGebra to create the presented construction as described in those steps. Students are expected to choose three lengths for their triangle and then recreate the above construction in GeoGebra following the given steps. Depending on students' attainment and other factors, such as time or students' engagement and their characters, these steps could be given as individual separate tasks (presented below) as opposed to being presented in one figure as a lengthy sequence of steps where students are trusted to follow the steps accurately. Such didactical configurations should be decided by the teacher.

TASK 8a:

Construct a triangle in GeoGebra following the following instructions:

> It is required to construct a triangle out of straight lines equal to *A*, *B*, and *C*.
> Set out a straight line *DE*, terminated at *D* but of infinite length in the direction of *E*. Make *DF* equal to A, FG equal to *B*, and *GH* equal to *C*.
> Describe the circle *DKL* with center *F* and radius *FD*. Again, describe the circle *KLH* with center *G* and radius *GH*. Join *KF* and *KG*.
> I say that the triangle *KFG* has been constructed out of three straight lines equal to *A*, *B*, and *C*.

By the end of this task, all students should have their triangles constructed provided that they chose lengths for the three sides that comply with the triangle inequality condition (the sum of the lengths of any two sides is bigger than the length of the remaining side). Their construction should be similar to the one they were presented with on paper (see Figure within Task 8a above). Students could potentially have difficulties with understanding what "Set out a straight-line *DE*, terminated at *D*, but of infinite length in the direction of *E*" means. The given diagram can support them with this statement, but also the "Ray" feature within the 'Lines' GeoGebra tools can help them in 'unpacking' this statement and constructing such a line (*instrumentation*).

TASK 8b:

Write down what the next step in the proof states in your own words:

> Since the point *F* is the center of the circle *DKL*, therefore *FD* equals *FK*. But *FD* equals *A*, therefore *KF* also equals *A*.

Do you agree with this statement? Why? Why not?
Use GeoGebra to help you recognise whether and why KF = A.

In this step, we should comment on how a length is represented by a capital letter, *A*, whereas usually students are presented with line segments of given lengths, e.g. *AB* = 3cm. Students are expected to interpret and potentially use mathematical notation, which is a requirement a teacher can share with their students:

$FD = FK$

$FD = A$, so $FK = A$

Their arguments should involve the fact that *FD* and *FK* represent the radius of the circle *DKL* and since the radius of this circle was constructed to be equal to the given length of side *A* of the triangle, then all three lengths are equal, $FD = FK = A$.

Using GeoGebra for this step in the proof, students should validate the equalities as described in the written statement above by potentially dragging certain points in their construction and comparing the lengths of line segments in question. Students could argue about the validity of the statements based on their constructions and in essence argue about the validity of the statements by using 'proof by construction'.

TASK 8c:

Write down what the next step in the proof states in your own words:

Again, since the point *G* is the center of the circle *LKH*, therefore *GH* equals *GK*. But *GH* equals *C*, therefore *KG* also equals *C*.

Do you agree with this statement? Why? Why not?
Use GeoGebra to help you recognise whether and why KG = C.

Similarly to Task 8b, students could follow a similar process to derive that:

$GH = GK$

$GH = C$, so $KG = C$

Their arguments should involve the lengths of GH, GK and C, all being equal to the radius of the circle LKH and therefore $GH = GK = C$. Some students may notice in this statement the change in the order of the letters for representing the line segment GK.

As mentioned above regarding Task 8b, GeoGebra can support students' thinking processes and argumentation in interpreting a mathematical statement presented in an original source. We believe though that using a combination of representations (i.e. the construction in GeoGebra and mathematical notation on paper) is crucial in understanding the statement, but also recognising the presentation of a formal proof. Writing down the actual equalities ($GH = GK$, $GH + C$ and $GH = GK = C$) could be re-enforced by the teacher as a valid strategy in developing students' mathematical thinking. And even though GeoGebra is designed to bring together different forms of representations, students may not necessarily recognise this feature during their interactions as they are focusing on constructing triangles following the steps in the given proof.

TASK 8d:

Write down what the next step in the proof states in your own words:

And *FG* also equals *B*, therefore the three straight lines *KF*, *FG*, and GK equal the three straight lines *A*, *B*, and C.

Do you agree with this statement? Why? Why not?
Use GeoGebra to help you recognise whether and why KF = A, FG = B and GK = C.

Students are expected to justify why:

$KF = A$

$FG = B$

$GK = C$.

GeoGebra can certainly help them visualise and test why these 3 sides are equal to the given length. Students could argue that the above mathematical statements are true by construction. But would that be enough? GeoGebra's power lies in enabling students to vary the lengths of the three sides, and if these sides are linked to the constructed triangle, then students can see the impact of their dragging and changing the given lengths of the triangle. Students could be asked to choose a different set of lengths and go through the same process of constructing a triangle in GeoGebra to reflect upon the condition for a triangle to be formed.

TASK 8e:

Write down what the next step in the proof states in your own words:

Therefore out of the three straight lines *KF*, *FG*, and *GK*, which equal the three given straight lines *A*, *B*, and C, the triangle *KFG* has been constructed.

Q.E.F.

Do you agree with this statement? Why? Why not?

Depending on their constructions and provided that their chosen lengths meet the required condition, students should argue about their beliefs on the validity of this proof. They are prompted to use GeoGebra to argue about the constructions and instantly see the outcome of any action they take, e.g. choosing different lengths for the sides and investigating when a triangle ceases to exist. If they were to choose the three lengths and place them in a straight line (see Figure 4.6 below), then they would receive instant and accurate feedback from GeoGebra showing that the circle with centre *G* and radius *GH* is not big enough to intersect the circle with centre *F* and radius *DF*.

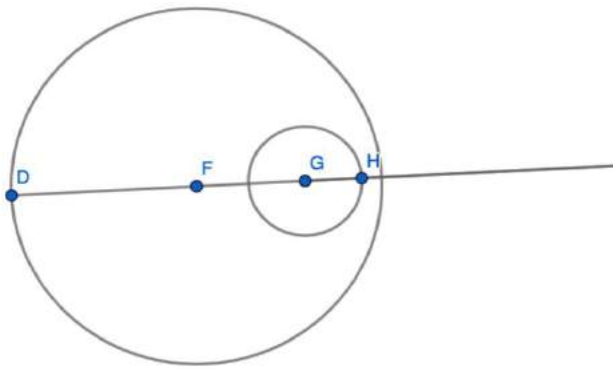


Figure 4.6: An attempt to construct a triangle using GeoGebra, when the sum of the 2 lengths FG and GH is less than the length of DF, i.e. |FG| + |GH| < |DF|.

Considering students' possible GeoGebra construction described earlier, students may find it easier to 'see' why the triangle ceases to exist when the sum of the lengths of the two sides is less than the length of the third side. For example, if they were to construct a triangle *ABC* (see Figure 4.7), where |*AB*| + |*B*'*C*| is less than |*AC*| by constructing the line segment *AC* and then two circles *C*(*A*, *AB*) and *C*(*C*, *B*'*C*), they could see that the two circles never intersect and the third vertex of the triangle cannot be constructed. Comparing these two constructions (the proof construction and the triangle construction) and how they could be presented and explored in GeoGebra by students, we could argue about the different level of difficulty with each one of them and how GeoGebra can make them more accessible to students and support students' valid deductive reasoning, as per Duval's (2000) theory. Considering the triangle construction in GeoGebra, students, for example, may keep the length of the sides *AC* and *B*'*C* constant and increase or decrease the radius *AB*. This exploratory process can help students recognise the importance of the triangle inequality as a condition for forming a triangle. Even though it cannot be considered as a formal proof, GeoGebra certainly supports students in getting a better understanding of Proposition 22 by allowing the creation of interactive diagrams (*multifunctional registers and discursive and non-discursive representations*) and promoting students to analyse, evaluate and confirm (or falsify) the derived results by carrying out various conversions (*discursive representations and multifunctional and monofunctional registers*) (Duval, 2000). Considering the proof construction in GeoGebra, students can compare the lengths while following carefully each step of the proof and convincing themselves of the equalities suggested by the formal proof.

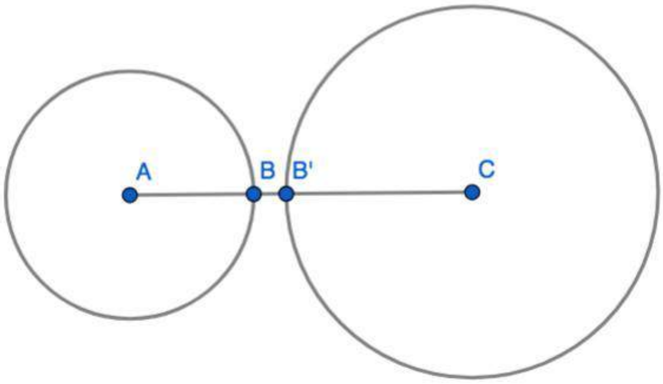


Figure 4.7: When |AB| + |BC| < |AC|, then no triangle could be formed.

Going back to the proof presented in the translation of Proposition 22 (see the presentations of Task 8 and Task 8e), it is worth referring to the three letters on the bottom right, “Q.E.F.”. These represent the Latin phrase “quod erat faciendum”, which means “that which was to be done”. In Euclid’s original source, as presented in Figure 4.1, this is written as “ὅπερ ἔδει ποιεῖσαι”, and which is commonly presented in other original sources as “ὅπερ ἔδει δεῖξαι” and which means “that which was to be shown” and which in essence means “that which was to be proved”. This was used at the end of a proof to indicate that the proof has been completed.

5 Concluding discussion

Through our outlined teaching scenario above, we believe to have shown how a use of digital technologies can assist in ‘unpacking’ an original source, and hence make this more accessible to potential students. Surely, our example of Euclid’s proposition 22 is not a long and comprehensive source, as those described by Barnett and colleagues (2014). Yet, the manageability of this limited excerpt from the Elements seems to serve well as an illustrative case for our line of argument in this particular paper. While our example seconds the claims of the previous studies considering the use of digital technologies in relation to history, as first laid out by Isoda (2000a), e.g. the benefits of multiple representations, support of students’ reflective thinking and mathematical inquiry, or those closer related to students’ concept formation (e.g. Chorlay, 2015), it does so by attempting to ground these arguments in the mathematics education literature. More precisely, we have attempted to articulate the aspects concerning multiple representations, and underlying concept formation, by using Duval’s framework of semiotic representations.

The aspects concerning proofs and proving in relation to students' reflective thinking is also attempted and articulated through a use of Duval, while that of students' mathematical inquiry is addressed through the rich literature on digital technologies in mathematics education. More precisely, we used the theory of instrumental genesis to consider and discuss students' potential instrumentation processes, i.e. how GeoGebra shaped and affected their thinking, and instrumentalisation processes, i.e. how their acquired knowledge through their preparatory work on paper and then their reflective tasks in GeoGebra may lead to a potentially different use of the GeoGebra tool that may allow them to critically reflect upon their own interpretations and understanding of Proposition 22. We used the theory of instrumental orchestration to analyse the teacher's potential intentions and aims for using GeoGebra for 'unpacking' Proposition 22. We discussed mainly the chosen didactical configurations and the rationale for those decisions, but also a teacher's strategies for exploiting the chosen didactical configurations (exploitation mode). Since this work is preparatory in its nature, we did not discuss the 'didactical performance', as this concerns the various decisions a teacher takes throughout a lesson aiming at using the digital tool in question as best and as effectively as possible. Finally, we touched upon the notion of instrumental distance (Haspekian, 2005; 2014) that allowed us to consider how a teacher through carefully designing a task sequence can take into consideration strategies to bridge the gap between students' learning, classroom and culture norms, and the norms involved when interacting with a digital tool, such as GeoGebra.

In terms of the claimed 'unpacking' of the original sources through digital technologies – or making it more accessible – GeoGebra can enable students to (a) explore statements, such as the two sentences in Proposition 22, but also statements shared by their peers and/or their teacher; (b) translate or convert the written statements in geometrical figures using the various GeoGebra features; (c) validate such statements by creating accurate constructions, comparing those constructions and using them as objects to think and test conjectures; (d) critically reflect upon mathematical statements through dynamic interactions with accurate constructions; (e) consider and prove Proposition 22 by construction.

Duval's framework of semiotic representations allowed us to take into account in the design of the above teaching scenario how students reach a good understanding of mathematical ideas, concepts and statements, such as that presented in Proposition 22. Students can be presented with and interact with two media, 'paper and pencil' and GeoGebra, and may be prompted through a task sequence to shift within and between registers. For example, for Task 8, students would have to recreate the static diagram presented in the original source in GeoGebra and therefore create a 'dynamic' construction involving line segments, circles, intersection points and of course a triangle (provided that the condition was met). They need to consider the mathematical statements presented by the original source in combination with the use of some symbolic language (mainly the use of letters to refer to certain parts of the diagram) and the static diagram and convert these to a dynamic geometrical figure constructed in GeoGebra, also annotated by letters. They could also convert from the semi-natural and semi-symbolic language to a series of mathematical statements presented in symbolic language, e.g. $FD = FK$ and $FD = A$, so $FK = A$. Critically reflecting upon these registers and shifting between them could certainly enhance students' understanding of Proposition 22, and support our argument for the value of combining paper and pencil resource with a digital tool to access, explore and even prove mathematical statements presented in an original source.

Although the use of GeoGebra throughout the duration of the teaching scenario serves a number of minor pragmatic purposes, the overall purposes are epistemic ones. In particular, GeoGebra serves the role of letting the students grasp the nature of the construction by dragging the GeoGebra construction (Task 5, Figure 4.5) and by relying on GeoGebra in the proof of the proposition (Task 8, Figures 4.6 and 4.7). The digital technology is used to more than just solve a mathematical task. It is used to deepen the understanding of the mathematical content of the original source. The combination of the original source and the digital tool seem to draw the use of the digital tool in an epistemic direction.

So, while digital technologies assist in making the original source more accessible to the students, the original source seems to 'enforce' upon the students an epistemic use of the digital technologies. This indeed appears to be a promising and positive synergy.

6. Future perspectives and questions to answer

As for the use of digital technologies in relation to the work with original sources, there are still many stones that are left unturned. If we agree to the seemingly large potential in this relationship, then several new questions arise: questions which need addressing in order to exploit the fruitful interplay between original sources and digital technologies further. We shall end this paper with outlining such questions or issues that appear central to us.

Firstly, it should be considered which original sources may benefit from which digital technologies. The 'unpacking potential' may certainly differ from technology to technology in relation to a given source. In our illustrative case a DGS was of course the obvious choice, since the source concerns plane geometry, whereas, say, a CAS tool would not have provided us with much assistance.

Secondly, we should ask ourselves which mathematics education frameworks may be applicable in our pursuit to describe the interplay between a use of original sources and digital technologies. In our illustrative case, we made use of Duval's framework of semiotic registers to articulate potential benefits of register shifts, in particular conversions between mono- or multifunctional discursive registers and multifunctional non-discursive representations in the form of geometrical figures. But surely there are other mathematics education theoretical frameworks that would apply better to a different combination of sources and technologies. Say, for instance, we are working with a source where we only operate in the monofunctional discursive register, i.e. symbolic systems – this could be some symbolic proof in algebra, etc. Then Duval's framework appears not to be the most suitable, since no conversions would take place. Similarly, in relation to the first question above, a DGS tool might not be so suitable with such a source, whereas a CAS tool might be.

Thirdly, as argued above, and as claimed in some of the available studies (Balsløv, 2018; Olsen & Thomsen, 2017), it appears that the combination of original sources not only seems to assist the students in their reading of the source, but that the presence of the source appears to draw the use of the technology in a more epistemic direction.

While the two first questions above require *a priori* theoretical analyses, in line with what has been presented in this paper, the third question or hypothesis calls for empirical investigations. Yet, if the hypothesis holds, i.e. if the study of original sources ‘enforces’ upon the students an epistemic use of the digital technology, then both knowing that this is so and exactly how this is, is not only a result that is of interest to the HPM community. While it appears common knowledge that the use of digital technologies in schools most often serves pragmatic purposes, it is well known (e.g. Artigue, 2010) that any use which is only, or mainly, pragmatic is of little – or even negative – educational value. Hence, if indeed a use of original sources fosters a positive educational effect on the use of digital technologies, then this would not only be a significant contribution to the field of HPM, but to the mathematics education research field at large.

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