HPM AND IN-SERVICE MATHEMATICS TEACHERS' PROFESSIONAL DEVELOPMENT IN CHINA

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ABSTRACT

Conceptualization of professional development have moved from “deficit” and “workshop or training” models to models of “professional growth” where teachers engage actively in collaborative inquiry into their own practice to enhance their knowledge of pedagogy, and students (Widjaja et al. 2017). However, through researching papers from Proceedings of HPM Satellite Meeting, ESU, CERME, ICME, in books and journals in the 21st century, we find that the studies on HPM and mathematics teachers' change or professional development mainly happen in universities, are mainly for pre-service teachers and the direct approach in this context is to give graduate courses (Barbin and Tzanakis, 2014).

In Mainland China, the situation is somewhat different. As a teaching research system has been practiced nationally since the 1950s (Wang, 2009), we have combined the teaching research system – mainly a Lesson Study (LS) – with HPM, which we call HPM Lesson Development (HPMLD). In the course of a LS, the teacher who conducts the LS would follow a procedure as shown in Fig.1 (Wang, Qi and Wang, 2017) with support from professional learning community (PLC), which is made up of a school-based group, an HPM research group, and a teaching expert group. Each group in the community has its own expertise, which is the reason why they get together, and without collaboration, it may be impossible to develop a sufficiently complete HPM Lesson. Through HPMLD, the teachers' knowledge, beliefs, attitude, and even instructional competencies would improve (Wang, 2013; Yue and Wang, 2016).

However, the influence of HPM has to be extended. So we publish the developed lessons, and make them open to all teachers. Some teachers interested in HPM adopt the instructional designs of the developed lessons in their own teaching, which is called HPM Lesson Sharing (HPMLS). On the other hand, we use the developed HPM Lessons to teach Pre- and In-service mathematics teachers, which is called HPM Lessons-based Teaching (HPMLbT), as one way of spreading the conception of HPM. What is the procedure of HPMLD, HPMLS and HPMLbT? What are the effects of HPMLD?

In this workshop, we organized a LS according to Fig. 1. Firstly we provide text from a textbook and curriculum’s requirements in China, the link with learned knowledge and knowledge learned later, and a historical resource about the studied topic. Participants in small groups design a lesson based on the above resource, the knowledge of the students they would teach, and the format of the design in China (Fig.2), under the guidance of the workshop organizers. Secondly each group presents its design. Then we play a video of an exemplary lesson, and participants watch and assess the lesson based on our assessment worksheet. Thirdly, we conduct a post-lesson debriefing, and participants revise their

design. Lastly, we use a case study to introduce teachers' professional development during the HPMLD and the procedure of HPMLS and HPMLBT, and construct a preliminary framework of HPM and mathematics teachers' professional development (shown as Fig.3).

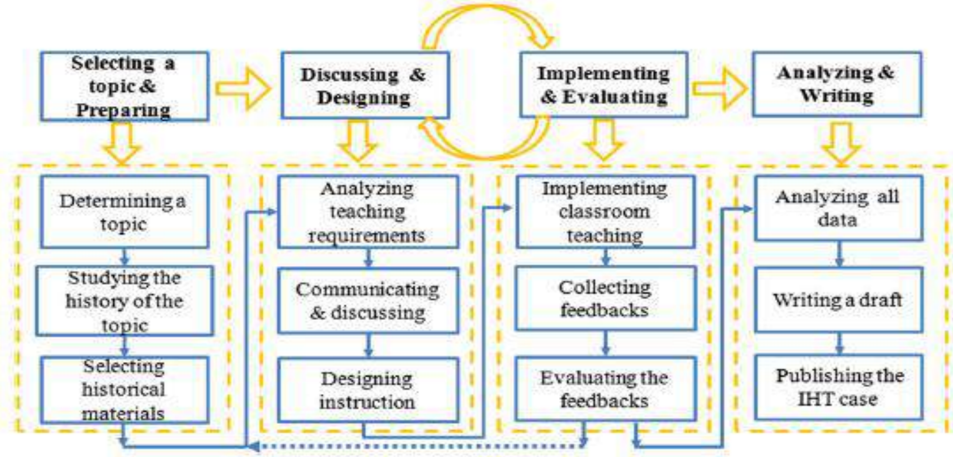

Figure1: The procedure of the HPMLD (Wang, Qi &Wang,2017)

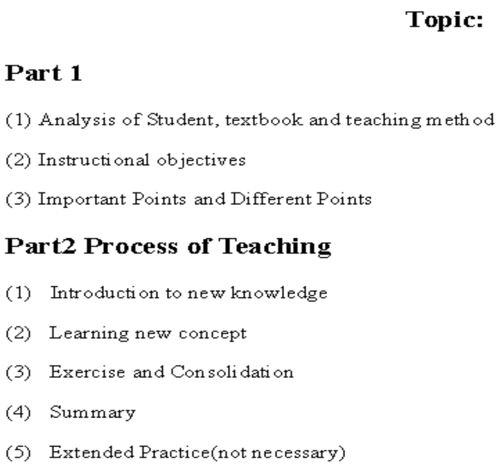

Figure 2

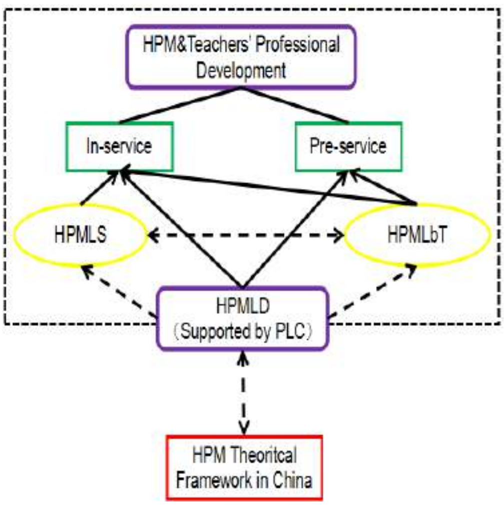

Figure 3

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USING ANCIENT INSTRUMENTS IN THE TEACHING OF GEOMETRY WITH BACHELARD'S PHENOMENO-TECHNOLOGY

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ABSTRACT

In this paper we explore Gaston Bachelard's notion of phenomeno-technique to explain how a geometrical instrument can be conceived as a "connaissance-en-action" (knowledge-in-action) in the construction of geometry. In particular, we analyze relations between a field of notions to teach and a field of problems to solve. So, firstly, we examine a hierarchy of instruments (Gerbert d'Aurillac, Jean Errard, Oronce Fine) that can be used to construct notions in an elementary geometrical teaching. They illustrate two kind of genesis of instruments defined by Pierre Rabardel as instrumentalization and instrumentation. Secondly, we analyze how the hierarchy of problems in *Dioptra* of Hero of Alexandria leads to a global notion of similitude in a geometrical space. As conclusion, we present what we call "an instrumental approach" in teaching of geometry.

1 Introduction: Gaston Bachelard's phenomeno-technology

Since around 25 years, the role of instruments of mathematics for teaching had been the subject of meetings and works in French IREMs (Hébert 1994, Johan 1996). More recently, several experiments using ancient instruments had been proposed in geometrical teaching, for students aged 11 until 14 years, in the same time of a revival of this teaching (Barbin 2014, Barbin et al. 2018). Often, these works propose to use an instrument only and the main didactical purpose is to give to students interesting applications of knowledge learned in classroom before. While in our purpose, linked to Bachelard's phenomeno-technology, instruments are sawn as knowledge-in-action and they correspond to notions or theorems that can be introduced and explored in teaching in the same time than the instruments are used.

Gaston Bachelard is a famous French philosopher of sciences, who wrote on physical sciences principally. His historical epistemology had been a basis of works in French IREMs (Institute for Research in Mathematical Education). He introduced the notion of phenomeno-technology in *Le nouvel esprit scientifique* (1934). He began to explain that scientific observation is neither a naked situation: "scientific observation is always polemical; it either confirms or denies a prior analysis, a pre-existing model, an observational protocol" (Bachelard 1984, p.12). Then he continued with experimentation (Bachelard 1984, p. 13):

> "And once the step is taken from observation to experimentation, the polemical character of knowledge stands out even more sharply. Now phenomena must be selected filtered, purified, shaped by instruments; indeed, it may well be the instruments that produce the phenomenon in the first place. And instruments are nothing but theories materialized. The phenomena they produce bear the stamp of theory throughout."

And he concluded:

> "A truly scientific phenomenology is therefore a phenomeno-technology. Its purpose is to amplify what is revealed beyond appearance. It takes its instruction from construction."

We will specially focus on this Bachelard' conception: "instruments are nothing but theories materialized". Precisely, we would like to show that instruments are "knowledge-in-action" (Barbin 2004, Barbin 2016).

For this purpose, we can quote two other authors: Gilbert Simondon is an important philosopher of techniques and Pierre Rabardel is a researcher on psychology and ergonomics. The first one wrote that "object that comes out of technical invention takes with it something from the human being who produced it […]; we could say that there is human nature into technical being" (Simondon 1969, p. 248). For the second one, "instrument is a means of capitalizing on accumulated experience (some authors say cristallized experience). In this way, any instrument is knowledge." (Rabardel 1975, p. 73).

In this paper, we will examine phenomeno-technology about geometrical instruments only and its consequence for implanting an instrumental approach in teaching of geometry. We will begin with the role of instruments for construction of a geometrical world, then we will analyze links between genesis of instruments and construction of knowledge, and then we will examine the invention of a geometrical space with the dioptre of Hero of Alexandria. We will conclude to precise what we name "an instrumental approach in teaching of geometry".

2 Instrument and invention of a geometrical world: the quadrant of Ionians

Historians of Greek mathematics explained that, in VIth century BC, Ionians used a quadrant to measure the distance of a boat in sea, which is an inaccessible distance by land surveying. A quadrant is made with a quarter of circle and a rod that turns and with which we can make sights (fig. 2.1).

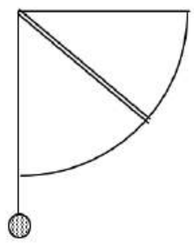

Figure 2.1: Quadrant of the Ionians

I let you imagine the real situation with sea, boat and us on the beach. Now, suppose that somebody asks me the question: how do you use the instrument? It will be more comprehensive to draw a layout. With this layout, I shall explain that I can climb on the top of a tower, make a first sight in direction of the boat, then return myself in direction of the ground and make a second sight (fig. 2.2).

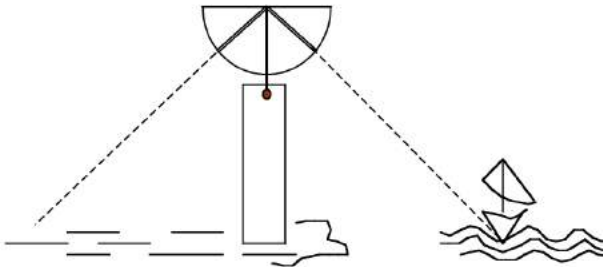

Figure 2.2: Distance of a boat: layout

Then suppose that somebody asks me: how do you find the distance of the boat? It will be better that we go together on the beach and that I draw a diagram on the sand to explain that “an equality of sights implies an equality of distances”, and it is a first rational discourse (fig. 2.3).

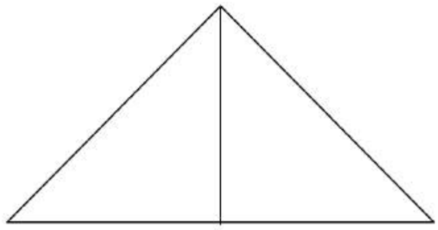

Figure 2.3: Distance of a boat: diagram

More generally, we can precise the different roles of layout and diagram. The layout permits a description of doings to answer to the question: what do you do with the instrument? While diagram is the basis of a first rational discourse to answer to the question: how do you find the result? Moreover, a diagram coordinates elements of a particular configuration, which can be activated, transformed or generalized by its re-cognition in various situations.

Now, suppose that somebody asks me: how do you know that your discourse is true? I shall add letters to the diagram to specify some of its parts called angles and lengths and I shall obtain a figure that permits a theoretical discourse on magnitudes, which is “if angles *BAD* and *DAC* are equal then lengths *BD* and *DC* are equal” (fig. 2.4). From a problem of inaccessible distance, Ionians invented a geometrical world made of figures and of relation between figures. The quadrant can be considered as “knowledge-in-action”, because it contains in itself a knowledge associating angles and distances.

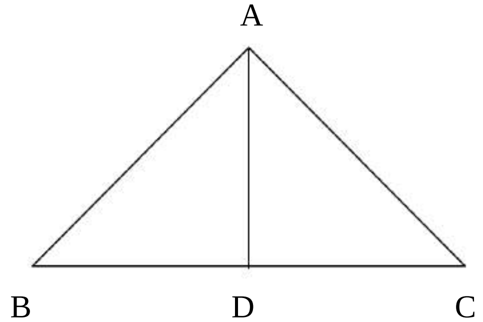

Figure 2.4: Distance of a boat: figure

3 Genesis of instruments and construction of knowledge: instrumentalization and instrumentation

In this second part, we will examine genesis of instruments; that is, birth or invention or creation of instruments, if you prefer. To understand relations between knowing subjects and instruments, we will go on by using history because, as Simondon wrote, “we have to grasp the historicity on how instruments become through how human being become” (Simondon 1969, pp. 107-109).

Rabardel distinguished two types of process in genesis of instruments (Rabardel 1995, p. 109). In the first type, there is an “enrichment” of an instrument (as artefact) by the subject without modification of the underlying diagram. He called it an “instrumentalization”. In the second type, there is a change of diagram by the human being with a modification of the instrument. He called it “instrumentation” (fig. 3.1). We are going to illustrate these two processes with ancient instruments taken in history.

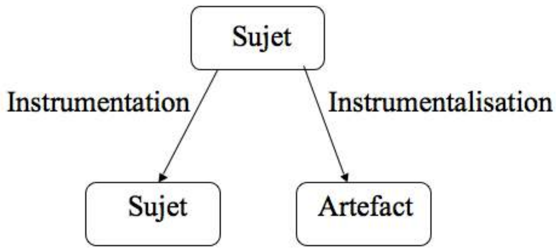

Figure 3.1: Instrumentation and instrumentalization according to Rabardel

3.1 Instrumentalization: from Gerbert’s stick to Errard’s instrument

The stick of Gerbert d’Aurillac is more a tool that an instrument, in the sense that it does not contains knowledge. Indeed, for our purpose, that is to analyze an instrument as a “connaissance-en-action” we have to distinguish an “instrument“ from a “tool” (Barbin 1994). The word “instrument” (in French linked with the verb “instruire”), is taking by us as an object whose conception integrates a knowledge. It is not the case for a simple stick. This distinction is useful in this paper later to compare the knowledge integrated in different instruments.

Let examine how Gerbert of Aurillac solved a problem of inaccessible distance, which is the width of a river, in his *Isagoge Geometriæ* (around 1000). Gerbert d’Aurillac wrote that a geometer has to have a stick with him always. How to do with the stick? Let see the layout and the figure (fig. 3.2). Similarity of triangles permits to write a proportion, in modern writing, the ratio *BD* : *CD* is equal to the ratio *BP* : *OP*. As *BP* is equal to the sum of the lengths *BD* and *DP*, we can calculate *BD* from accessible measures.

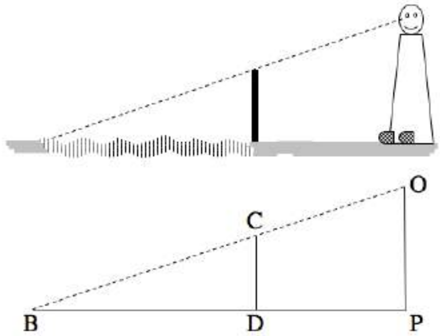

Figure 3.2: Width of a river with Gerbert’s stick

Now, to give an example of instrumentalization, we will go from the stick to Jean Errard's instrument, presented in his *La géométrie et pratique générale d'icelle* (2d ed., 1602). The instrument is more sophisticated, since it is composed of three rulers: *AB* is horizontal, *AC* can turn around *A* and *EF* can glide along *AB* (fig. 3.3).

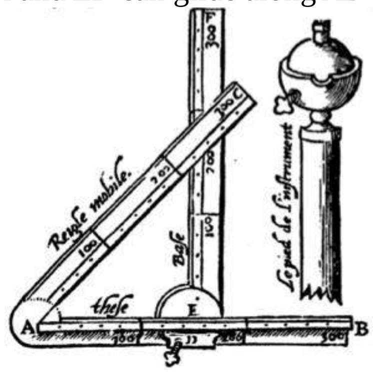

Figure 3.3: Errard's instrument (Errard 1602, p. 18)

The book contains two layouts corresponding to two problems of inaccessible distances: width of a river and height of a tower (fig. 3.4).

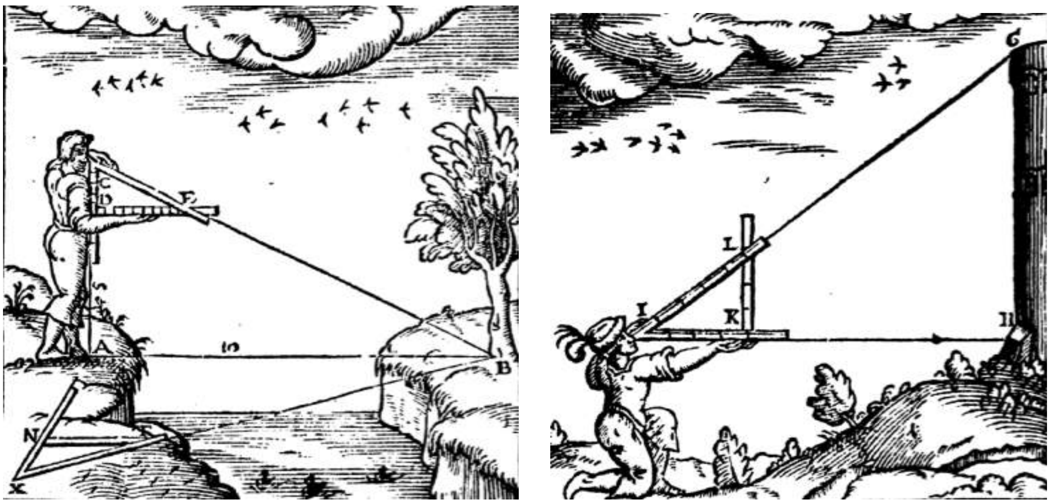

Figure 3.4: Two inaccessible distances with Errard's instrument (Errard 1602, pp. 21-22)

We have to remark that the underlying diagram is the same for these two problems, and also the same than for Gerbert's situation (fig. 3.5). But now, we can say that the diagram is incorporated into the instrument and that this instrument is a "knowledge-in-action" (Barbin 2004, Barbin 2016). The knowledge corresponds to proposition 4 of Euclid's Book VI (Euclid, pp. 200-202). We have an instrumentalization: this instrument is an enrichment of the stick that does not engage new diagram.

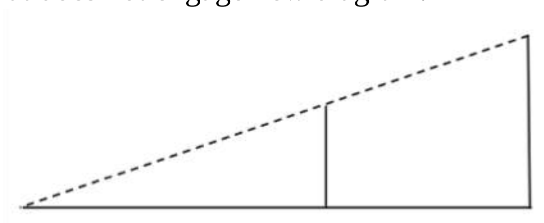

Figure 3.5: Diagram for inaccessible distances

3.2 Instrumentation: Gerbert's instrument and Oronce Fine's articulated set square

We go on with a process of instrumentation that engages new diagram. Gerbert d'Aurillac proposed an instrument more elaborated than a simple stick. A layout with letters permits to show how to use this instrument for measuring an inaccessible height of a tower (fig. 3.6).

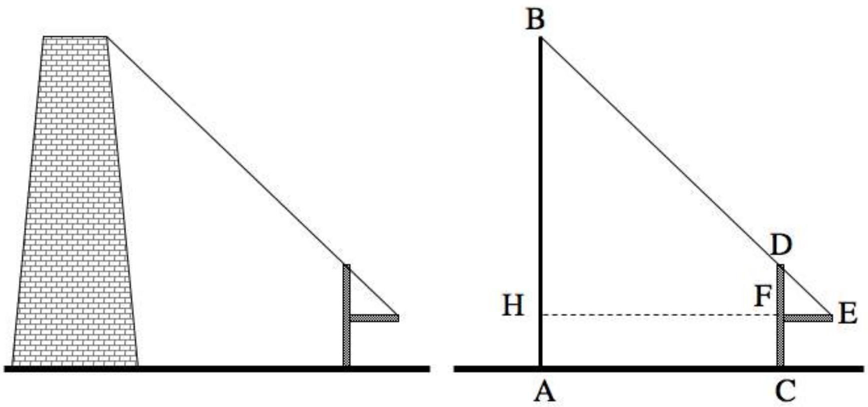

Figure 3.6: Height of a tower with Gerbert's instrument: layout and figure

The instrument is composed of two perpendicular sticks *FE* and *DC*, such that *DF*, *FE* and *FC* are equals. Here, we have a new diagram and we can explain on the figure how to determine the height. By equalities of angles, the triangle *BHE* is isosceles. So *AB* is equal to the sum of the lengths *HE* and *FC*. Now, the user doesn't need to calculate ratios to obtain the result. But the instrument incorporates a new knowledge, which concerns isosceles triangles.

Another example of instrumentation is given with the instrument of Oronce Fine presented in his *Protomathesis* (1532). It is composed of a stick and of a square set turning around the bottom of the stick. The layout shows how to use the instrument to measure the width of a river (fig. 3.7). The instrument is down on a bank of the river, and an alidade is aligned with the other bank.

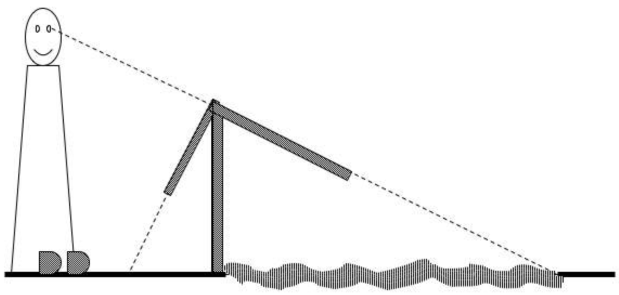

Figure 3.7: Width of a river with Oronce Fine's instrument: layout

How to obtain the width of the river? We have a new diagram and a figure on which we can state the theorem on the height of a rectangular triangle, that is theorem of the height of a rectangular triangle, in modern writing: $AH^2 = BH \times HC$ (fig. 3.8). If we take *AH* equal to 1 then *HC* is equal to $1/BH$. The distance is very easy to calculate, but the instrument incorporates a strong theorem, that is proven by Euclid two times, as a consequence of Pythagoras theorem in Book II, and as a consequence of the theorem on similar triangles in Book VI of his *Elements*.

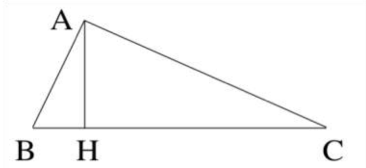

Figure 3.8: Weight of a river with Oronce Fine: figure and theorem

We can conclude with three comments on process of instrumentation concerning teaching. Firstly, in the two chosen examples of instrumentation, the inaccessible distance can be easily obtained from the accessible. Secondly, in each process of instrumentation, a new knowledge is incorporated into the instrument and the underlying diagram changes: isosceles triangle for Errard’s instrument, theorem on height in a rectangular triangle for Oronce’Fine’s instrument. Thirdly, we can write that more the instrument is “instructed” by theory, then less the user has to be instructed.

3.3 Instrumentalization and instrumentation: the place of the subject

The two processes has to intervene in teaching because they engage activities of the learner. Rabardel emphasized the two places of the subject in the two processes by writing:

> “These two types of processes are the fact of the subject […]. What distinguishes them is the orientation of this activity. In the process of instrumentation, the activity is turned towards the subject himself, while in the correlative process of instrumentalization, the activity is oriented towards the component artefact of the instrument” (Rabardel, 1995, pp. 111-112).

We can complete the scheme above by indicating the three instruments that can illustrate the two processes (fig. 3.9).

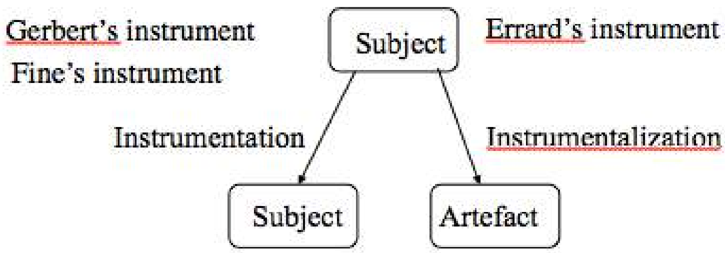

Figure 3.9: Instrumentation and instrumentalization accordingly with Rabardel

In a previous paper (Barbin, 2016), I remarked the difference between this scheme and the one given by Luc Trouche (fig. 3.10). This last scheme corresponds to what Trouche wrote: “Rabardel distinguishes, in the genesis of an instrument, two crossed processes, instrumentation and instrumentalization: the instrumentalization concerns the personalization of the artefact by the subject, the instrumentation concerns the apparition of schemes into the subject (that is to say the manner with which the artefact contributes to pre-structure the action of subject for carrying out the task in question)” (Trouche, 2015, p. 267).

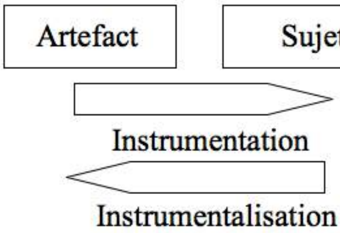

Figure 3.10: Instrumentation and instrumentalization with Trouche

In our epistemological and didactical purpose, we do not see the instrument as a way to pre-structure the action of the subject. But, like in Bachelard's phenomeno-technology, we rather consider that the instrument is structured by the subject's knowledge.

4 Instrument and invention of a geometrical space: on Hero of Alexandria's dioptre

In his *Dioptra* (IIIrd century), Hero of Alexandria considers one instrument only: the dioptre. It is not a complicated instrument. There is an upper part which can only turn around or to be inclined on a leg (fig. 4.1). Moreover, we can make sights through two holes of the upper part, like it is shown on the layout by the discontinued line. Moreover, technical means permit to obtain the horizontal position when it needs for the upper part and the vertical position for the leg of the instrument.

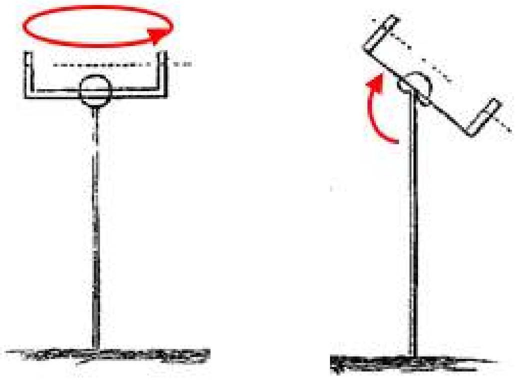

Figure 4.1: Hero of Alexandria's dioptre

Hero wrote: "in general, dioptre is used for measuring distances of any kind, when this operation only can be made by far" (Hero 1858, p. 177). Indeed, the book is composed by a series of problems of inaccessible distances ranged in a "deductive" order, in the sense that one problem is solved by using solving of previous problems. Reading Hero is very interesting because he explained steps by steps how to solve problems. Let read the second problem: "Problem 2. Two points *A* and *B* are given such that it is not possible to see one of them from the other: to join them by a straight line". Hero explained how to bypass the obstacle, which prevents to see one point from the other point, And, as for each problem, we have to make a diagram. Hero wrote: "while making these operations, we write them on a paper, that is we represent the layout, with indicating the summits of the broken line and the lengths of its several parts" (fig. 4.2). We have to put the dioptre in *A* and to mark *AG* on the ground, a straight line with an arbitrary length. Then we have to put the dioptre in *G* and to mark *GD* perpendicular to *AG*, with an arbitrary length. Then we have to put the dioptre in *D* and to mark *DE*, etc The points *G*, *D*, *E*, etc. are accessible points and they have been supposed to be in the same plane.

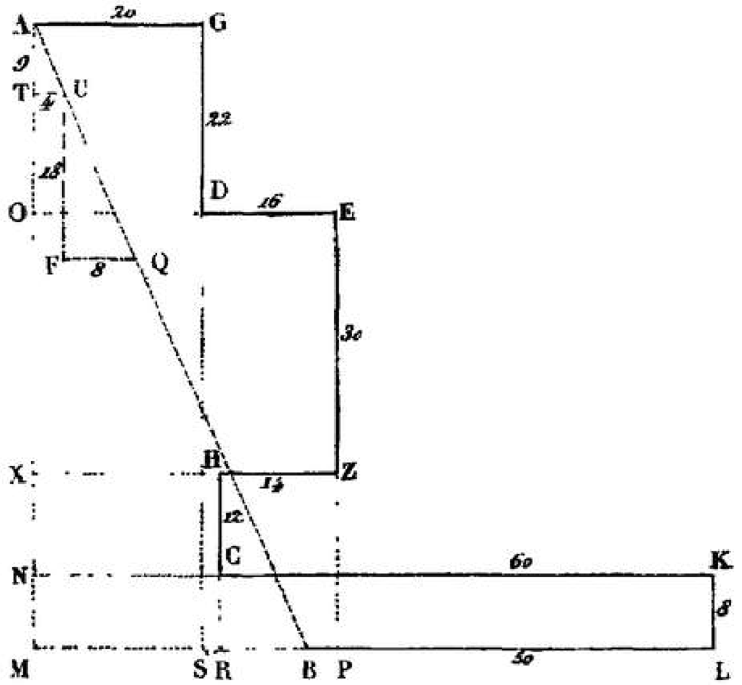

Figure 4.2: Problem of inaccessible distance in Hero: diagram and figure

After which, to obtain different points belonging to the straight line AB, we have to draw a figure with two steps. Firstly, we have to draw "real" lines that are on the ground with their measures. Secondly, like Hero wrote, we have to imagine and to draw other lines, the "imaginary" lines *AM*, *MB*, *AT*, etc., on which the reasoning will be explained. Their drawings are discontinued lines (fig. 4.2). We can calculate *AM* and *MB* from the "real lines" and their ratio. In Hero's example, we obtain 72 and 32, and the ratio *AM*: *MB* is equal to 72 : 32. Now, let take (for example) *AT* equal to 9 on *AM* and *TU* perpendicular to *AT*. Hero explained that the ratio 72: 32 is equal to the ratio 9: *TU* and so *TU* equal 4 . In the same manner, we can obtain the point *U* of *AB*, etc. Hero wrote: "by observing the same ratio always". That means there is the conception of a "global similarity" between several rectangular triangles, so, this similarity operates in a geometrical space. So, we can consider that dioptre is a simple instrument that needs an instructed user.

This "global similarity" is different of the notion of similarity in Euclid's *Elements* (Barbin PUR). Indeed, Euclid's geometry is a study of figures without introduction of space. In Book VI, Euclid only proved similarity between two geometrical forms of same type (two triangles, two rectangles, two polygons, etc). Hero did not state Euclid's fundamental theorems on similar forms. But he represented figures accordingly to an implicit "scale", where an explicit ratio operates on forms of a geometrical space by a "global similarity".

Let consider two other figures corresponding to Hero's problems: the problem of the tunnel of Samos (fig.4.3) and the problem of the depth of a well (fig. 4.4). To solve these problems, Hero introduced rectangular triangles that can be deduced thanks the calculation of one ratio only. These figures are linked by a "global similarity".

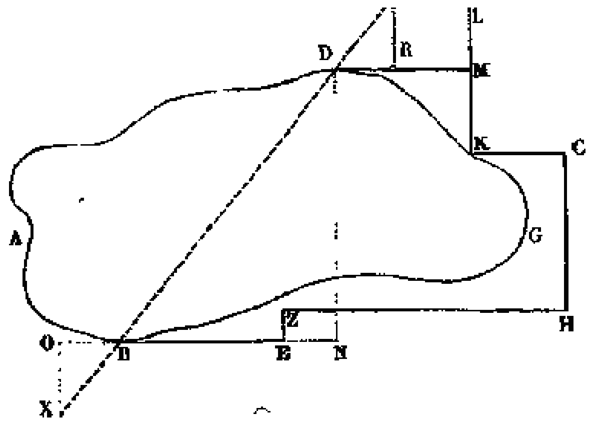

Figure 4.3: Problem of the Tunnel of Samos: diagram and figure

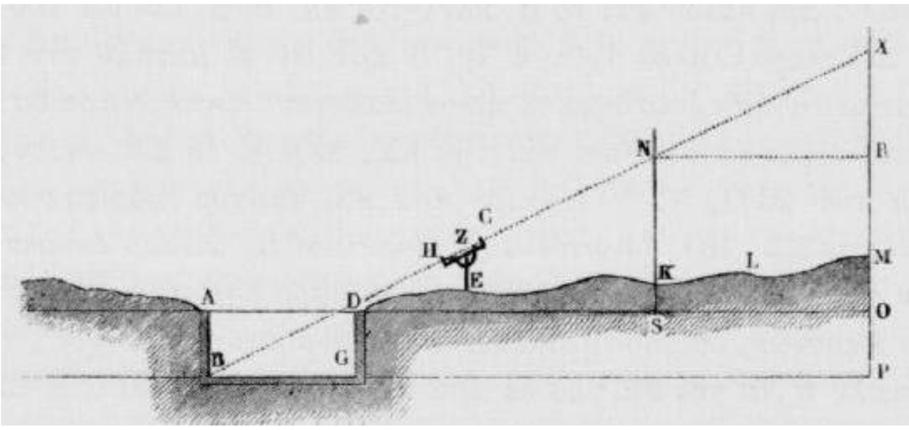

Figure 4.4: Problem of the depth of a well: layout, diagram and figure

5 An instrumental approach in teaching of geometry with phenomeno-technology

As we remarked above, several works on using instruments in geometrical teaching propose to use an instrument only to solve one problem only, and the main didactical purpose is to give to students interesting applications of knowledge learned in classroom before. Also, it is chosen instruments, different of those presented in this chapter, because these instruments permit to read measures of angles by numbers. Thus, these instruments lead to trigonometry rapidly (on instruments and angles see Chatelon & Troudet 2018, Mercier 2018, Guichard 2018).

While in our purpose, linked to phenomeno-technology, instruments are sawn as knowledge-in-action and they correspond to notions or theorems that can be introduced and explored in teaching in the same time than the instruments are used. They are simple, in the sense that they did not include a way to read angles. Here, an "instrumental approach" means a teaching where students use instruments, not in a disparate manner, but use a set of instruments following a cognitive and mathematical order that can permit to construct a geometrical knowledge. Also, following Simondon, it is a teaching where historical elements are integrated thanks to the use of ancient instruments (Barbin et al., 2018).

We can characterize an instrumental approach of teaching by three features. Firstly it is a teaching by problem-solving, where problems of inaccessible distances play a major role to introduce objects and theorems as tools to solve problems. Secondly, it is teaching beginning with spatial situations and going on with two explicit types of activities for learners. There are at first activities of drawings with scales representing situations on the ground (layouts and diagrams), and then activities of enouncing rational discourses on figures with letters, like in Hero's *Dioptra*. For this purpose, an instrument can be used to solve a hierarchy of problems, organized in a deductive manner. Thirdly, it is a teaching where a hierarchical field of instruments and an ordered field of knowledge are constructed in a mutual "enrichment" (Barbin, 2016).

We propose that learners themselves distinguish different steps in activities of problem-solving, corresponding to steps in learning (on activities on the ground and drawings, see Chatelon & Troudet 2014). It is interesting to differentiate the roles of drawing layouts, diagrams, figures with real and imaginary lines, because, in this instrumental approach, an instrument is considered as a knowledge-in-action corresponding to an underlying geometrical diagram. For this purpose, we stress on the mutual "enrichment" between constructions of new geometrical knowledge and of new instruments introduced in teaching. It is important to well conceive and to use the two possibilities in teaching: we can introduce the same knowledge to conceive several instruments and we also can conceive an instrument corresponding to a new geometrical knowledge.

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