Let them speak; hear them speak — old Chinese wisdom on mathematics education

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Abstract

Since the mid-1990s there has been an upsurge of interest in the process of learning and teaching in a classroom environment dominated by the so-called Confucian heritage culture (CHC), which is brought into focus in the form of two paradoxes, namely, the CHC Learner Paradox and the CHC Teacher Paradox. This paper takes a look at how in the past Chinese scholars and mathematicians viewed the subject of mathematics and what kind of underlying philosophy in learning and teaching of the subject they believed in, by studying some selected passages from mathematical texts from very early times to the nineteenth century.

Keywords: Confucian-heritage-culture classroom; Chinese mathematical classics

Introduction

Since the mid-1990s there has been an upsurge of interest in the process of learning and teaching in an environment dominated by the so-called Confucian heritage culture (CHC), which is brought into focus in the form of two paradoxes (Leung, 2001; Watkins & Biggs, 1996, 2001; Wong, 1998), namely, (i) The CHC Learner Paradox: CHC students are perceived as using low-level, rote-based strategies in a classroom environment which should not be conducive to high achievement, yet CHC students report a preference for high-level, meaning-based learning strategies and they achieve significantly better in international assessments! (ii) The CHC Teacher Paradox: Teachers in CHC classrooms produce a positive learning outcome under substandard conditions that Western educators would regard as most unpromising! A more detailed discussion on these two points can be found in Chapter 1 of (Watkins & Biggs, 2001).

It would therefore be of interest, both for mathematics educators and historians of mathematics, to take a look at how in the past Chinese scholars and mathematicians viewed the subject of mathematics and what kind of underlying philosophy in learning and teaching of the subject they believed in. To this end selected mathematical texts, particularly prefaces or accompanying commentaries or essays, from very early times since the second century B.C.E. to the late-nineteenth century before China more or less fully adopted the Western way and content of mathematics in education, will be quoted. If a Chinese text is quoted, the source of the English translation will be noted, but to keep to the length of the paper the original Chinese

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text will not be shown. If no source of translation is noted, the translation is that of the author's, may not be the best but hopefully adequate to convey the meaning. Many of the original Chinese mathematical books are collected in (Guo, 1993).

An Early Discussion on Learning Mathematics

Let us begin by looking at a passage taken from *Zhou Bi Suan Jing* [周髀算經 The Arithmetical Classic of the Gnomon and the Circular Paths], which is believed to be compiled between the fifth and second centuries B.C.E., and is perhaps the earliest extant Chinese treatise on astronomy and mathematics. In the book there is an interesting dialogue between Rong-fang [榮方] and Chen-zi [陳子]:

> *Rong-fang* : […] Can a fellow as stupid like me learn the way of mathematics?
>
> *Chen-zi*: Of course you can. What you have learnt in elementary arithmetic is sufficient to let you go on to learn it. But you must be willing to think continually in earnest.
>
> [A few days later, *Rong-fang* came back to *Chen-zi*.]
>
> *Ron-fang*: I cannot figure it out. May I enquire again?
>
> *Chen-zi*: This is because you have thought about it but not yet to the point of maturity […]. You cannot yet generalize what you have learnt. […] The mathematics is simple to explain but has wide applications. After understanding one category of problems one can infer the reasoning for a variety of other categories. […] What makes it difficult to be well versed in the way of mathematics is that when one has learnt it one worries about a lack of breadth; when one has attained breadth, one worries about a lack of practice; when one has attained practice, one worries about a lack of ability to understand. To be able to compare and contrast different categories of problems, that is the mark of an intelligent person.

Another prominent Chinese mathematical classics is *Jiu Zhang Suan Shu* [九章算術 Nine Chapters on the Mathematical Art], which is believed to be compiled between 100 B.C.E. and 100 C.E. with part of the content already quite well-known much earlier, as evidenced by another book (written on bamboo strips) *Suan Shu Shu* [算數書 Book of Numbers and Computation] dated to 200 B.C.E, excavated in Zhangjiashan in the Hubei Province in 1983 (Peng, 2001). In the mid-third century the noted Chinese mathematician LIU Hui [劉徽 *ca*. 225-295] wrote a detailed commentary on *Jiu Zhang Suan Shu*. In his preface he said, "I studied *Jiu Zhang* [*Suan Shu*] at an early age and perused it when I got older. I see the separation of the *Yin* and the *Yang* and arrive at the root of the mathematical art. In this process of probing I comprehend its meaning. Despite ignorance and incompetence on my part I dare expose what I understand in these commentaries. Things are related to each other through logical reasoning so that like branches of a tree, diversified as they are, they nevertheless come out of a single trunk. If we elucidated by prose and illustrated by pictures, then we may be able to attain conciseness as well as comprehensiveness, clarity as well as rigour. Looking at a part we will understand the rest."

What LIU Hui meant by "elucidated by prose and illustrated by pictures" points out a special feature in traditional Chinese mathematics, namely, that geometry and algebra/arithmetic, or that shapes and numbers, are intimately integrated. How these two aspects come together will be best seen through examples taken from the commentaries of LIU Hui, so let us take a more detailed look at Chapter 9 of *Jiu Zhang Suan Shu*.

The design of this final chapter (titled *Gou Gu* [勾股 Right Triangles]) well exemplifies what is known as the variation theory in teaching/learning (à la Ference Marton and his team) in the Western community of educators, and as the *bian shi* method [變式] (à la GU Ling-yuan [顧泠沅] and his team) in the Eastern community of educators. For instance, the first three problems in the chapter, which simply ask for one side of a right triangle given the other two sides, are set at "level zero" of variation! By moving to some word problems in different contexts learners are urged to move to a higher level in Problem 4 (about cutting a rectangular plank out of a circular log) and Problem 5 (about a winding vine around a tree). Learners are further urged to move to yet higher and higher levels of variation in subsequent problems that place a higher demand on making use of knowledge about a right triangle, at the same time unfolding a systematic theory according to a well-designed framework. (A more detailed discussion on this framework is offered in (Li, 1990).)

Let us see how shapes and numbers go hand in hand in traditional Chinese mathematics by studying the solutions of these problems. For fun (and as a token of thanks to the host of this 5th ICHME) allow me to make use of a recruitment advertisement that appeared in a Dutch magazine (April 27, 1999) with the title "Ben jij ook altijd zo benieuwd naar de ontknoping?" The advertisement shows six amusing mathematical problems, one of which is actually a problem with a rich heritage. It is Problem 13 in Chapter 9 of *Jiu Zhang Suan Shu*, which also appeared (with different data) in *Lilavati* written by the Indian mathematician Bhāskara II (1114-1185) in the twelfth century, and in a European text written by the Italian mathematician Filippo Calandri in the fifteenth century. The problem asks: "Given a bamboo 1 *zhang* high, which is broken with its tip touching the ground 3 *chi* away from the base. What is the height of the break?" In modern day mathematical language it means that we want to calculate the side b of a right triangle given a and $c + b$, where c is the hypotenuse.

The answer given in *Jiu Zhang Suan Shu*, expressed in modern day mathematical language, is given by the formula

$$b = \frac{1}{2}\left[(c+b) - \frac{a^2}{(c+b)}\right]$$

A school pupil of today would be able to obtain this formula by invoking Pythagoras' Theorem and solving a certain equation. However, how was it done more than two thousand years ago, when the Chinese did not yet have the facility afforded by symbolic manipulation at their disposal? Both LIU Hui, and later another noted Chinese mathematician YANG Hui [楊輝 1238-1298], explained how they obtained the answer in their commentaries. Their method is ingenious. Not to spoil the fun for readers who like to seek their own methods I will leave this ingenious explanation to a *Geogebra* applet which may be accessed at the link https://ggbm.at/2772025.

Let us now move to Problem 15 in Chapter 9 of *Jiu Zhang Suan Shu*: "Given a right triangle whose *gou* is 5 *bu* and whose *gu* is 12 *bu*. What is the side of an inscribed square? The answer is 3 and 9/17 *bu*." (See Fig. 1.)

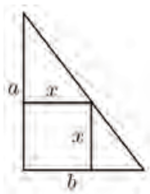

Fig. 1. Inscribed square in a right triangle.

The text follows with a method: "Let the sum of the *gou* and the *gu* be the divisor; let the product of the *gou* and the *gu* be the dividend. Divide to obtain the side of the square." In modern day mathematical language it means the formula $x = ab / (a + b)$.

Again, a school pupil of today can handle this problem easily using similar triangles. However, unlike the ancient Greeks the Chinese did not develop a geometric theory involving the notion of parallel lines and similar figures. How did LIU Hui solve the problem at the time? His ingenious method using dissection-and-reassembling provides a nice "proof without words"! (See Fig. 2.)

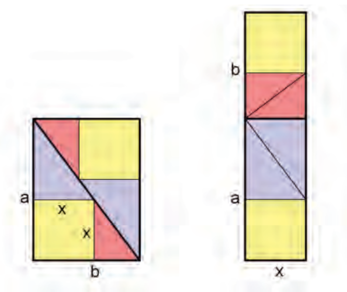

Fig. 2. Calculation of side of the inscribed square.

An animated solution is provided in a *GeoGebra* applet that can be accessed at the link https://ggbm.at/2812253.

Readers may like to try their hands in applying a similar method to solve the next problem (Problem 16) in Chapter 9 of *Jiu Zhang Suan Shu*: "Given a right triangle whose *gou* is 8 *bu* and whose *gu* is 15 *bu*. What is the diameter of its inscribed circle? The answer is 6 *bu*." For a more detailed discussion see (Siu, 1993). An animated solution is provided in a *GeoGebra* applet that can be accessed at the link https://ggbm.at/697695.

A Chinese Essay on Mathematics Education in the Song Dynasty

In the latter part of the thirteenth century the mathematician YANG Hui of the Southern Song Dynasty wrote several books that were later compiled as a treatise known as *Yang Hui Suan Fa* [楊輝算法 Yang Hui's Methods of Computation], which includes the three books with titles *Cheng Chu Tong Bian Suan Bao* [乘除通變算寶 Precious Reckoner for Variations of Multiplication and Division] of 1274, *Tian Mu Bi Lei Cheng Chu Jie Fa* [田畝比類乘除捷法 Practical Rules of Arithmetic for Surveying] of 1275, and *Xu Gu Zhai Qi Suan Fa* [續古摘奇算法 Continuation of Ancient Mathematical Methods for Elucidating the Strange Properties of Numbers] of 1275.

The preface to the first chapter of *Cheng Chu Tong Bian Suan Bao* is perhaps the first paper on mathematics education in China. With its title *Xi Suan Gang Mu* [習算綱目 A General Outline of Mathematical Studies] this preface offers a re-organized syllabus of the traditional curriculum in a comprehensive programme of 260 days, which is comparable to a modern school programme of about 1500 hours (Zhou, 1990).

Let us take a look at some passages in *Xi Suan Gang Mu* to see how the author sequenced the learning of the basics. Some other passages in the other chapters of

Yang Hui Suan Fa would indicate how the author valued comprehension to mere rote learning and drilling[1].

> First learn the multiplication tables. < Start with 1 times 1 equals 1 up to 9 times 9 equals 81, from the smaller numbers to the larger ones. Their applications are not shown here. > Learn the rules of multiplication and how to fix the place-values. < Lesson for one day. > […] Learn the rules of division and how to fix the place-values. < Lesson for one day. > […] In the chia (addition 加) method the number is increased, while in the chien (subtraction 減) method a certain number is taken away. Whenever there is addition there is subtraction. One who learns the chien method should test the result by applying the chia method to the answer of the problem. This will enable one to understand the method to its origin. Five days are sufficient for revision. […] In learning the chiu kuei (tables of division 九歸) one will need at least five to seven days to become familiar with the recitation of the forty-four sentences. However if one examines carefully the explanatory notes of the art on chiu kwei in the Hsiang Chieh Suan Fa (A Detailed Analysis of the Methods of Computation 詳解算法 — a lost treatise by YANG Hui), one can then understand the inner meaning of the process and a single day will suffice for committing the tables and their applications to memory. Revise the subject on chiu kwei. < one day. > […] In mathematics, root-extraction constitutes a major item. Under the headings kou ku, p'ang yao, yen tuan and so chi can be found numerous common examples of root-extraction. […] Learn a method a day and work on the subject for two months. It is essential to enquire into the origins of the applications of the methods so that they will not be forgotten for a long time. […] When the methods of multiplication, division, fractions and root-extraction in the 246 problems of the Chiu Chang have been revised, then start with chapters fang t'ien and su mi. These will require one day to know thoroughly. The next chapter ch'ui fên deals with proportional parts. The whole of shao kuang is devoted to the addition of fractions and all of shang kung is on substitution of equivalent volumes. The chapter chün shu employs the cross-multiplication of ordered numbers. Take three days to peruse each chapter. The applications of the methods in the three remaining chapters ying pu tsu, fang ch'êng and kou ku are more complicated, so take four days to study each chapter. Make a detailed study of Chiu Chang [Suan Fa] Tsuan Lei, so that the rules of applications are thoroughly known. Then only will the art of the Chiu Chang be fully understood.

It would be of interest to compare such a curriculum with what students in the state-run School of Mathematics in the Tang Dynasty [唐 681–907] went through. The period of the Tang Dynasty is chosen for its most established form of the curriculum and the state examination system in mathematics, which later dynasties either modelled after it or it was even no longer in place. Beginning with the Sui Dynasty [隋 581–618], further consolidated in subsequent dynasties, a comprehensive official

1 The translated text of the quoted passages are taken from (Lam, 1977).

system of education was established with a well-planned curriculum, including the syllabus and the adopted textbooks, for each of several chosen disciplines. State examinations for these chosen disciplines were held regularly and successful candidates were appointed to official posts according to merit in their performance at examinations. (Siu, 1995; Siu & Volkov, 1999)

At an Imperial Order the Tang mathematician LI Chunfeng [李淳風 602– 670] collated *Suan Jing Shi Shu* [算經十書 Ten Mathematical Manuals], which was adopted as the official textbook in the School of Mathematics in 656. This compendium comprised ten classics that were compiled by different authors at different times, listed below roughly in chronological order: (1) *Zhou Bi Suan Jing* [周髀算經 The Arithmetical Classic of the Gnomon and the Circular Paths, ca. 100 B.C.E.], (2) *Jiu Zhang Suan Shu* [九章算術 Nine Chapters on the Mathematical Art, between 100 B.C.E. and 100 C.E.], (3) *Hai Dao Suan Jing* [海島算經 Sea Island Mathematical Manual, third century], (4) *Wu Cao Suan Jing* [五曹算經 Mathematical Manual of the Five Government Departments, sixth century], (5) *Sun Zi Suan Jing* [孫子算經 Master Sun's Mathematical Manual, between fourth and fifth centuries], (6) *Xia Hou Yang Suan Jing* [夏侯陽算經 Xia Hou Yang's Mathematical Manual, fifth century], (7) *Zhang Qiu Jian Suan Jing* [張丘建算經 Zhang Qiu Jian's Mathematical Manual, fifth century], (8) *Wu Jing Suan Shu* [五經算術 Arithmetic in the Five Classics, sixth century], (9) *Qi Gu Suan Jing* [緝古算經 Continuation of Ancient Mathematics, seventh century], (10) *Zhui Shu* [綴術 Art of Mending, fifth century]. The original text of *Zhui Shu* was lost in about the tenth century, with its role in the compendium being subsequently replaced in the Song Dynasty by *Shu Shu Ji Yi* [數術記遺 Memoir on Some Traditions of the Mathematical Art, a book of doubtful sixth century authorship].

Students in the state-run School of Mathematics were divided into two programmes. In Programme A students studied *Sun Zi Suan Jing* and *Wu Cao Suan Jing* for 1 year, *Jiu Zhang Suan Shu* and *Hai Dao Suan Jing* for 3 years, *Zhang Qiu Jian Suan Jing* for 1 year, *Xia Hou Yang Suan Jing* for 1 year, *Zhou Bi Suan Jing* and *Wu Jing Suan Shu* for 1 year. In Programme B students studied *Zhui Shu* for 4 years, and *Qi Gu Suan Jing* for 3 years. In addition to these books, students in each of the two programmes must also study two more manuals, *Shu Shu Ji Yi* and *San Deng Shu* [三等數 Three Hierarchies of Numbers, written not later than the mid-sixth century but was lost by the Song Dynasty].

It is recorded in the chronicles Tang Liu Dian [唐六典 The Six Codes of the Tang Dynasty, 739] and Xin Tang Shu [新唐書 The New History of the Tang Dynasty, 1060] that regular examinations were held throughout the seven years of study, and at the end of each year an annual examination was held. Any student who failed thrice or who had spent nine years at the School of Mathematics would be discontinued. Judging from the age of admission at fourteen to nineteen years of age, we know that a mathematics student would sit for the state examination at around twenty-two-year-old, meaning that they spent some seven to ten years studying mathematics (Siu & Volkov, 1999). We can now compare the comprehensive programme of 260 days suggested by YANG Hui with the seven-year programme in the official system!

In a later chapter of the book *Cheng Chu Tong Bian Suan Bao* the author said:

> The working of a problem is selected from various methods, and the method should suit the problem. In order that a method is to be clearly understood, it should be illustrated by an example. If one meets a problem, its method must be carefully chosen.

In another book *Tian Mu Bi Lei Cheng Chu Jie Fa* the author said:

> It is difficult to see the logic and method behind complicated problems. Simple problems are hereby given and elucidated. Once these are understood, problems, however difficult, will become clear.

In the third book *Xu Gu Zhai Qi Suan Fa* the author said:

> The author has placed the small diagram on the island in the sea before him, and came to understand a little of the method employed by his predecessors. If the complete method is handed down, is not the secret purport being slighted? And if it is not handed down, then there is nothing to further the good work of his predecessors. […] Therefore after this one single problem, the learned reader should be able to examine and solve by analogy other remaining problems. What is the need of passing it on so easily to the uninitiated?

Back to the CHC Learner Paradox

What kind of teaching environment would conduce to good learning? In the Western literature it is normally agreed that such an environment should include factors like (i) varied teaching methods, (ii) student-centered activities, (iii) content presented in a meaningful context, (iv) warm classroom climate, (v) high cognitive level outcomes expected and assessed, (vi) classroom-based assessment in a non-threatening atmosphere (Biggs & Moore, 1993). However, in a Confucian-heritage culture (CHC) classroom, Western observers see just the opposite! It would therefore be natural to conclude that (1) CHC classrooms should give rise to low quality outcomes: rote learning and low achievement, and (2) CHC students are perceived as using low-level, rote-based learning strategies. But paradoxically what instead happens is to the contrary! CHC students have significantly higher levels of achievement than those of Western students (e.g. in IEA, PISA studies)[2], and CHC students report a preference for high-level, meaning-based learning strategies (Biggs, 1994). A more detailed discussion about this "Western misperceptions of the Confucian Heritage learning culture" can be found in Chapter 3 of (Watkins & Biggs, 1996).

Is it true that rote learning was the norm in education in ancient China? Perhaps a reading of some texts in ancient Chinese classics can shed some light on this question. (These texts are about moral philosophy, but the principle applies to mathematics education as well.) In the well-known Chinese classics Lun Yu [論語 Confucian Analects] it is written: "The Master said: Learning without thought is labour lost; thought without learning is perilous." In another classics Zhong Yong [中庸 Doctrine of the Mean] it is written:

> He who attains sincerity chooses the good and holds fast to it. This involves the extensive study of it, close inquiry into it, careful deliberation of it, clear distinction of it, and earnest practice of it[3].

The noted neo-Confucian Zhu Xi [朱熹 1130-1200] of the Southern Song Dynasty compiled *The Four Books* (the original copies believed to be compiled between the 6th and 5th century B.C.E) and one passage reads:

> In reading, if you have no doubts, encourage them. And if you do have doubts, get rid of them. Only when you've reached this point have you made progress.

Another passage reads:

> Generally speaking, in reading, we must first become intimately familiar with the text so that its words seem to come from our own mouths. We should then continue to reflect on it so that its ideas seem to come from our own minds. Only then can there be real understanding. Still, once our intimate reading of it and careful reflection on it have led to a clear understanding of it, we must continue to question. Then there might be additional progress. If we cease questioning, in the end there'll be no additional progress.

Still another passage reads:

> Learning is reciting. If we recite it then think it over, think it over then recite it, naturally it'll become meaningful to us. If we recite it but don't think it over, we still won't appreciate its meaning. If we think it over but don't recite it, even though we might understand it, our understanding will be precarious. […] Should we recite it to the point of intimate familiarity, and moreover

2 In recent years there have emerged views which question the effectiveness and validity of such kind of international assessment studies, but that is not what we wish to enter into discussion here.

3 Translated texts are taken from (Legge 1893/1960).

> think about it in detail, naturally our mind and principle will become one and never shall we forget what we've read[4].

All these passages indicate clearly that repetitive learning is not to be equated with rote learning.

Let us further illustrate this point with what was supposed to be tested in the State Examination in Mathematics in the Tang Dynasty. In the state examination in mathematics for either Programme A or Programme B candidates were examined on two types of question. The first type was described in *Xin Tang Shu* as:

> [The candidates should] write [a composition on] the general meaning, taking as the basic/original task a problem and answer. [They should] elucidate the numbers/ computations, [and] construct an algorithm. [They should] elucidate the structure/principle of the algorithm in detail.

For Programme B there was added the remark,

> If there is no commentary, [the candidates should] make the numbers/computations correspond [to the right ones?] in constructing the algorithm.

The second type of question was known as *tie du* [帖讀 strip reading] in which candidates were shown a line taken from either *Shu Shu Ji Yi* or *San Deng Shu*, with three characters covered up and were asked to answer what those three missing characters were, that is, what nowadays we call "fill in the blank". It is interesting to note that *Shu Shu Ji Yi* is a short text with only 934 characters, which could be committed to memory with reasonable ease (not to mention that a candidate had seven years to do it!). *San Deng Shu* was probably a book of similar nature (Siu & Volkov, 1999).

What is meant by that added remark? Since no trace of any examination question is extant, we can only attempt to "re-construct" an examination question, with an eye to supporting the thesis that the curriculum in mathematics in ancient China was not so elementary nor was mathematics learnt by rote. It is hard to believe that a group of selected young men spent seven of their golden years in simply memorizing the mathematical classics one by one without understanding them, just to regurgitate the answers in the state examination at the end! (Siu, 2004) Let us illustrate with one "fictitious" examination question that might appear in a state examination, that of calculating the volume of a truncated pyramid with a rectangular base. For convenience we will describe it using modern day mathematical language: Compute the volume of an "(oblong) *ting* [亭 Pavilion]" of height h with bottom and top being rectangles of sides a_1, a_2 and b_1, b_2 respectively ($a_1 \neq a_2$, $b_1 \neq b_2$) (Siu, 2004).

Problem 10 in Chapter 5 (titled S*hang Gong* [商功 Consultation of Engineering Works]) of *Jiu Zhang Suan Shu* is about the volume of a "*fang ting* [方亭 Pavilion with

4 Translated texts are taken from (Gardner, 1990).

square base]", which is a truncated pyramid with square base. If *a* and *b* are respectively the side of the bottom and top square and *h* is the height, then the volume is given in the text as

$$V = \frac{1}{3}(a^2 + b^2 + ab)h.$$

In his commentary LIU Hui explained how to arrive at this formula by an ingenious method of assembling blocks of standard shapes, called by him "*qi*" [棊 chess piece], including three standard types, namely, (i) *li fang* [立方], which is a cube of side *a*, with volume a^3 , (ii) *yang ma* [陽馬], which is a pyramid of square base of side *a* and one vertical side of length *a* perpendicular to the base, with volume (iii) *qian du* [塹堵], which is a triangular prism with isosceles right triangle of side *a* as base and height *a*, with volume . (See Fig. 3.)

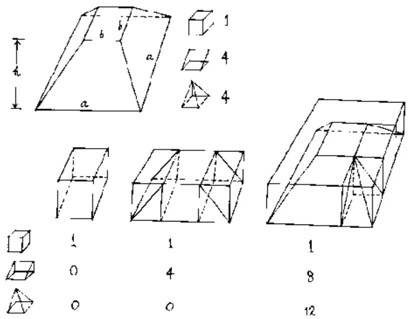

Fig. 3. Assembling a truncated pyramid.

He also explained how to arrive at an alternative formula by using another way of dissection. To keep to the length of this paper we omit all the detailed calculation. Interested readers may like to consult (Siu, 2004), or go to an animated solution provided in a *GeoGebra* applet at the link https://ggbm.at/6829110 .

A student who understands the argument by LIU Hui can easily modify either method to arrive at the correct answer, which is

$$V = \frac{1}{3}\left[a_1a_2 + b_1b_2 + \frac{1}{2}(a_1b_2 + a_2b_1)\right]h.$$

But if a student merely memorizes the formula given in the textbook by heart, it is not easy to hit upon this correct formula. In view of practical need a candidate in the School of Mathematics might face in his subsequent career as an official engineer, such a demand imposed on the candidate in the state examination is quite reasonable.

Interested readers may go to an animated solution provided in a *GeoGebra* applet at the link https://ggbm.at/6834973 .

Some Thoughts on Mathematics Education in the IT Age

My university (University of Hong Kong) adopts as its motto SAPIENTIA ET VIRTUS in Latin, rendered in Chinese as "明德格物", which is derived from the text of the very ancient Chinese text *Da Xue* [大學 the Great Learning]. It is meaningful to quote part of the full text:

> The point where to rest being known, the object of pursuit is then determined; and, that being determined, a calm unperturbedness may be attained to. To that calmness there will succeed a tranquil repose. In that repose there may be careful deliberation, and that deliberation will be followed by the attainment of the desired end.

This ancient passage motivated me to raise a set of questions in a lecture given in the 3rd International Congress of Chinese Mathematicians held in December of 2004 in Hong Kong, particularly because after reading a cover story of *Newsweek* (Aug.25-Sept.1, 2003) that featured "Bionic Kids: How technology is altering the next generation of humans", in which one of the featured articles with the title "Log on and learn" mentions two points:

> Children's brains are growing adept at handling a variety of visual information. […] Kids are getting better at paying attention to several things at once. But there is a cost, in that you don't go into any one thing in much depth.

We need to recognize, like it or not, that the IT age breeds a generation with a different working habit and a different learning habit, even a different mentality, from that of their parents and teachers. The set of questions I like to raise is, in my opinion, what mathematics educators would do well to deal with in this IT age. How should IT be employed to enable students to learn better but not to limit their ability to think critically and in depth? How can we ensure that a discovery approach is not to be equated with a hit-and-miss tactic? How can we ensure that imaginative thinking is not to be equated with a cavalier attitude, that multi-tasking needs not be identified as sloppy and hasty work, and that the use of IT is not to be identified as following instructions step by step without thinking? (Siu, 2008)

A Chinese Essay on Learning Mathematics in the Late Nineteenth Century

I like to conclude with an account on the personal learning experience described by the nineteenth century Chinese mathematician Hua Heng-fang [華蘅芳 1833-1902],

a well-known mathematician of the latter part of the Qing Dynasty who also translated many Western books of science and mathematics in collaboration with the English missionary John Fryer (1839-1928). In two books he wrote, *Xue Suan Bi Tan* [學算筆談 Essays on Studying Mathematics] of 1882 and *Suan Zhai Suo Yu* [算齋瑣語 Trivial Talks in the Mathematics Study Room] of 1896, he offered a perceptive and detailed discussion on how he studied mathematics and acquired new learning when Western mathematics was introduced into China.

In Xue Suan Bi Tan he wrote:

> If computation and reasoning are two different activities, then those who compute need not reason. But can those who reason compute? And is what they compute the same as what those who do not reason compute? Perhaps those who reason do not prefer to compute, not that they cannot compute. If they are forced to compute, what they perform is exactly the same as what those compute will perform. However, since those who compute have carried out the activity with such familiarity, they compute with celerity and accuracy, usually not to be surpassed by those who prefer to reason."

In *Suan Zhai Suo Yu* he wrote:

> I hold the view that in writing a book in mathematics one must offer discussion in words besides writing in symbols and formulas and drawing figures. Symbols, formulas and figures can only explain what is in a problem but not necessarily what is in the periphery of the problem. Confined to the problem one can only explain what one understands rather than what one has not yet understood. Furthermore, while reading the book and finding every page covered with *1*, *2*, *3*, *4* or *a*, *b*, *c*, *d*, one may inevitably feel bored and tiresome. Insertion of discussion would facilitate the flow of ideas and offer a new and refreshing appearance, thus open up one's thought.

The more senior Chinese mathematician LI Shan-lan [李善蘭 1811-1882] was a mentor of Hua. LI translated in collaboration with the English missionary Alexander Wylie (1815-1887) a textbook on calculus *Elements of Analytical Geometry and of Differential and Integral Calculus* (1850) written by the American mathematician Elias Loomis in 1859, and later presented a copy of the translation to Hua, who worked hard on studying this branch of mathematics new to him. Hua recounted his learning experience in perusing the translation which bears the Chinese title *Dai Wei Ji Shi Ji* [代微積拾級 Geometry and Differential and Integral Calculus Step by Step] in Chapter 5 of *Xue Suan Bi Tan* that ends with a rather poetic metaphor:

> […] After browsing through a few pages I was at a loss, not comprehending what the book said. […] I asked Master Li about it, and he told me it was not easy to express in words the subtlety contained therein, but everything had been written down in the book with nothing hidden from the reader. One has to read it many times to acquire the meaning. How can one expect to

> accomplish full understanding overnight? I believe in what he said and study the content of the book repeatedly. I began to get somewhere, just like seeing stars appearing at dusk. At first I only saw one, then a few more, then several tens more, then several hundreds more, and finally the starlit sky was full of them!"

An Epilogue

It is mentioned in the introductory section that China more or less fully adopted the Western way and content of mathematics in education since the late nineteenth and early twentieth centuries. However, it seems that the Chinese maintain a traditional attitude to learning despite such an adaptation, which makes the so-called CHC classroom (in all subjects besides just mathematics) a topic which attracts so much attention of some scholars that its study leads to the rich discussion of the two CHC paradoxes. In Chapter 1 of (Watkins & Biggs, 2001) several positive features of this tradition of the CHC classroom are highlighted and explained, grouped into six categories:

- (1) memorizing and understanding,
- (2) effort versus ability attributes,
- (3) intrinsic versus extrinsic motivations,
- (4) general patterns of socialization,
- (5) achievement motivation: ego versus social,
- (6) collective versus individual orientation.

Readers can consult (Watkins & Biggs, 1996, 2001) for a more complete discussion on all these aspects, while in this paper we focus on (1) and (2), in that repetitive learning is different from rote learning, and that understanding is regarded as a long process which requires considerable mental effort. All in all, the issue is deeply entrenched in a social and cultural setting so that merely replicating on the surface the way of teaching, either the Western world from the Eastern world or vice versa, without considering the social and cultural setting in which the learners are brought up since childhood may not give a satisfactory solution.

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