John Leslie's (1817) view of arithmetic and its relevance for modern pedagogy

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Abstract

John Leslie in his book The Philosophy of Arithmetic *(1817) presented a hypothesis that arithmetic didn't start with the process of counting, but with partitioning a collection of objects into two equal parts with or without a remainder. He described how our ancestors could have developed arithmetic not only without written language, but also without number words and other linguistic skills. They only needed manual skills to manipulate small objects on primitive counting boards. This paper shows how the method described by Leslie can be used to introduce numbers in early grades, and to teach basic arithmetic skills.*

Keywords: counting boards, arithmetic tasks

Brief history of concepts presented

Sources

This paper develops the ideas of three mathematicians, John Napier (1550-1617), John Colson (1680-1760), and John Leslie (I766-1832).

John Napier, mostly known for his invention of logarithms, also designed an interesting counting board utilizing a geometric progression to carry out multiplication, computation of roots, and other more complex arithmetic operations efficiently. This work was described in one chapter of his book *Rabdology* (1617/1990), and remained mostly unknown until an article about it was published by Martin Gardner (1986).

John Colson introduced the concept of 'negative digits', which simplified arithmetic computations. They were not considered to be negative numbers, but positive numbers marked to be subtracted instead of being added. This work also was mainly forgotten until Donald Knuth quoted it in his *The Art of Computer Programming* (1997). The same concept was used in ancient China (Hart, 2011) where computations were carried out with bamboo sticks, before the invention of the suanpan (the Chinese abacus).

Bjarnadóttir, K., Furinghetti, F., Krüger, J., Prytz, J., Schubring, G. & Smid, H. J.(Eds.) (2019). "*Dig where you stand" 5. Proceedings of the fifth International Conference on the History of Mathematics Education*. Utrecht: Freudenthal Institute.

John Leslie in his book *The Philosophy of Arithmetic* (Leslie, 1817) provided two parallel versions of elementary arithmetic: 'palpable', where computations are carried out on a counting board, and 'figurate', where computations are carried out in writing. His book contains two original ideas. He takes 'halving with a remainder', which is partitioning a set into two almost equal subsets, instead of 'counting', for a basic arithmetic operation. This allows him to develop the arithmetic processes and his notation for numbers without any reference to a person's linguistic skills. According to Leslie, number words are optional and not a prerequisite for learning arithmetic. Also, no writing skills are required during palpable computations with counters. He also uses 'negative digits' in all computations, in a manner that was proposed by Colson (Colson, 1726). Leslie introduced his own terminology for negative counters used on boards, and for written digits, calling them 'empty' or 'deficient counters' and 'deficient figures'.

In his book Leslie displayed a very impressive knowledge of the history of mathematics, including non-European traditions, but because he did not provide any specific references, we may only assume that he took negative digits from Colson and the name "palpable" from Nicholas Saunderson (1740).

Leslie's approach, combined with concepts from Napier and Colson, adapted to the modern school setting, can be used in early grades. It may be useful in multilingual classrooms and when some pupils have inadequate writing skills[1].

Counting boards

Counting boards are often considered to be precursors of more modern mechanical computing devices, and of modern calculators and computers. This is a rather misleading analogy. Counting boards played the same role as modern tablets, white- and blackboards, or even paper, when one writes on it with a pencil and not with a pen. A counting board provides an external 'memory' that allows one to record, modify, and erase partial results of computations and other information that is necessary during the process of computing, but may be forgotten when the final result is obtained.

Historical sources indicate that all societies that developed advanced arithmetic used some counting boards, even after full written systems of arithmetic were introduced. Unfortunately, information regarding how the boards were used and what algorithms were carried out exist only for post-medieval Western Europe about computation 'on lines' (on a 'Roman abacus'), and in Eastern Asia about computation on a 'Chinese abacus'.

1 Biographies of J. Napier, J. Colson, N. Saunderson, and J. Leslie are available on the internet; see References.

There are also data indicating that arithmetic was developed very early and spread among prehistoric societies. But it seems that at present the internet is the best source for finding the current state of knowledge and a variety of different hypotheses concerning counting boards[2].

The counting boards described by Leslie are inadequate for modern school use, but they can be replaced with more 'modern' counting boards (Baggett & Ehrenfeucht, 2016) designed on the principle described by John Napier.

Modern counting boards

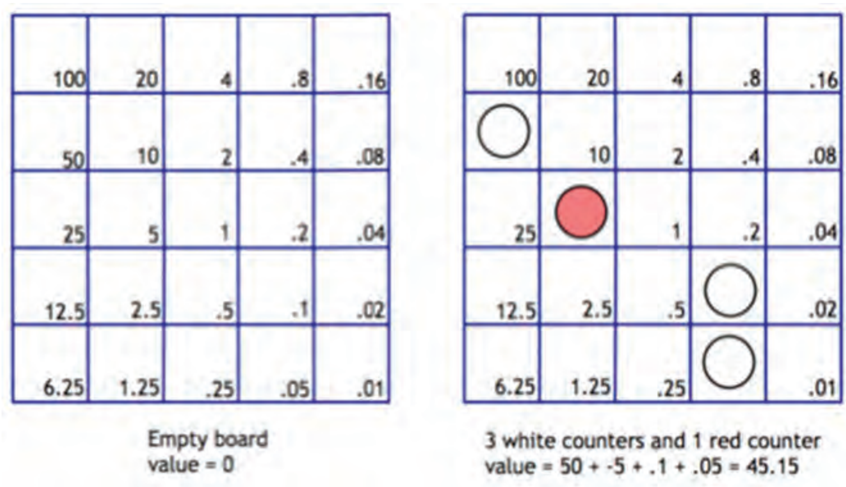

Fig. 1. A typical modern counting board for computation with decimals

Numbers in each column (a column is also called a rod) form a geometric progression with ratio 2. (On this board, rows also form a geometric progression with ratio 5, but this will not be the case on some other boards.)

Counters are two-colored; one side is white and the other is red. A white counter on a board has the value of its location (which is positive) and a red counter has the opposite value (which is negative).

The value of the whole board is the sum of the values of all counters.

One can stack several counters on one square.

2 See History in the References.

There are three rules of regrouping counters on this board that are sufficient to transform a configuration of counters into any other configuration of the same value.

One can extend this board in all four directions, left, right, up, and down, as long as numbers in rows and columns form the geometric progressions, because the rules of regrouping remain unchanged.

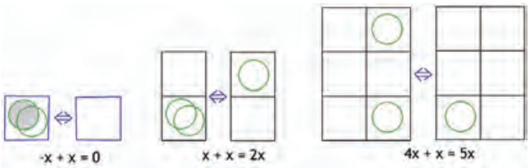

Fig. 2. Three rules of regrouping corresponding to three algebraic equalities, x + -x = 0, x + x = 2x, and x + 4x = 5x.

In Figure 2, a white circle represents either a white or red counter. And a gray circle represents its opposite.

Arithmetic operations performed on a board

Addition and subtraction is straightforward. One puts both numbers on the same board and regroups the counters to a desired final format.

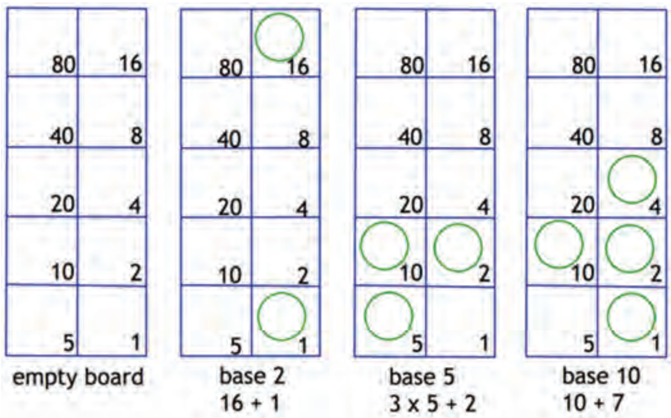

Fig. 3. The number 17, represented in bases 2, 5 and 10, on a decimal board.

Multiplication of decimals is also easy. Multiplication and division by 5 correspond to shifting all counters one square to the left or one square to the right. Multiplication and division by 2 correspond to shifting counters up and down. Thus, the product of any two numbers can be computed by shifting and adding patterns of counters.

Introducing the concept of numbers in grades kindergarten, 1st, and 2nd

In US schools learning arithmetic starts in kindergarten and elementary grades with verbal counting, and children need to master this skill to at least 20. This is followed by addition and subtraction of whole numbers, multiplication and division of whole numbers, common fractions and operations on them, decimals and finally negative numbers. These topics strongly overlap, so for example, addition often starts as counting up, multiplication starts as repeated addition, and common fractions start with fractions having small denominators.

Here is a different sequence of material for kindergarten and the first two grades. It starts with addition and subtraction of whole numbers on the board with two-color tokens (leaving open the question whether they represent negative numbers or positive numbers marked to be subtracted). It is followed by counting, and by multiplication, and the set of whole numbers can be extended by binary fractions.

During these three years, children would use only boards built from four rods of length at most 6, consisting of at most 4 rods containing the numbers 1, 3, 5, and 15 (and possibly binary fractions). The order in which the rods are arranged to form a board may vary.

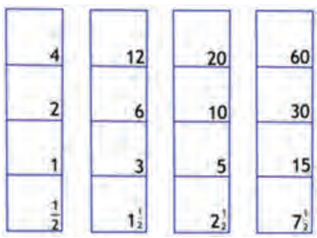

Fig. 4. Examples of rods used in grades K, 1st, 2nd.

The only rules needed for regrouping on these boards are rules for the decimal board and a rule corresponding to the algebraic equality, 2 + 1 = 3.

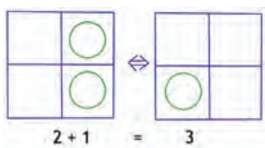

Fig. 5. The new regrouping rule; it corresponds to 2 + 1 = 3.

Reading location numbers

Numbers labeling locations on a board are written as positive decimals or as mixed numbers that combine a counting number with a proper binary fraction.

Reading number-words, even without understanding, is a difficult task that children acquire rather slowly in their native language, and it is even more difficult for children in multi-lingual classrooms. So a different reading method, that is simple and easy to learn, is needed in each classroom.

On each board there is one location that holds the number 1. Any other location can be reached by moving some number of steps left or right, followed by some number of steps up or down.

So for example on the board in Figure 1, instead of saying "white counter on square five hundredths", one can say, "white, one right, two down".

This reading has one big advantage. If you already know some fractions and that the label .2 means one fifth, you would know that .05 stands for one twentieth (because numbers in every column form a progression with ratio 2).

Three small boards and their ranges

Fig. 6. Three small boards for grades kindergarten, 1st, and 2nd

The range of a board is the set of all numbers represented on it with at most one counter for each location. (Because we use two-color counters, the range of each board always contains negative numbers, which we don't list here.)

Only integers are represented on these boards, and their ranges are 0 to 12, 0 to 28, and 0 to 42.

The first board can be used to introduce all arithmetic information about numbers up to 12 that children may need later, providing that they already know the numbers 1, 2 and 3.

Examples

Question:	What is the number 5?
Answer:	Make a stack of five white counters on the square labeled 1; regroup two counters on 1 to one counter on 2; regroup a counter on 1 and a counter on 2 to a token on 3; and finally, regroup two counters on 1 to one counter on 2. So we found that 5 equals 2 plus 3.

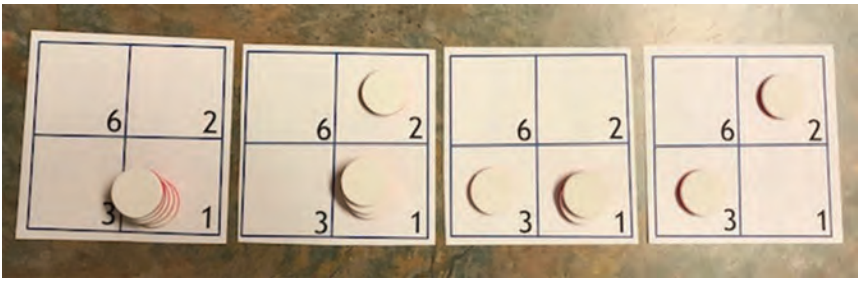

Fig. 7. Representing 5 on the board

Question: How to represent 0 with exactly 3 counters? (Find all the answers.)

The two other boards are used for actual calculations. Children should be shown how to represent, on each board, numbers from 0 to 12, with at most 2 tokens. (Notice that it requires the use of both white and red counters.)

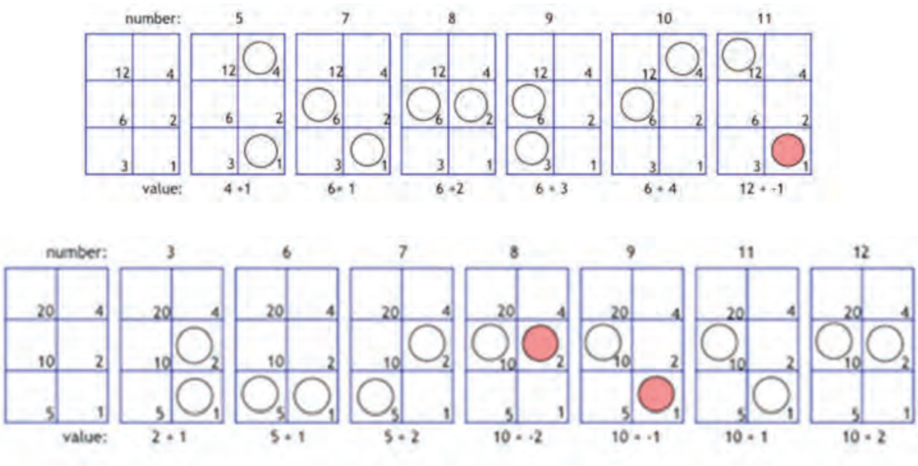

Fig. 8. Numbers up to twelve on two small boards

The board with the smaller range is easier to use, but the other board is necessary for introducing the concept of base 10.

General comments

- If one wants to extend the range of whole numbers that are used, each board can be extended up.
- Using four-column boards and boards with fractions is optional.
- Many tasks require using more than one board. They are easy to make, and there is no restriction on how many of them are used at one time.

- Writing numbers should be done only for keeping records and communication; all computations can be done mentally or on counting boards.

Examples of tasks for use in grades K, 1st and 2nd

These tasks are not lesson plans because they do not include stories that usually accompany and justify the tasks, or address pedagogical issues such as whether children should work in groups or individually, or how to teach them procedures and how to assess their work.

First task

Students learn how to compute all products of whole numbers up to 12*15 (multiplication 'facts'). Each product is computed by adding (at most) two numbers on one counting board. But three boards, called multiplier, multiplicand, and product are used. How the boards are laid out on a table is a matter of convenience.

Each number from 1 to 12 is represented by at most two tokens on the multiplier board. But 4, 8 and 9 require the use of red counters, 4 = 5 + -1,

8 = 10 + -2, and 9 = 10 + -1.

When a number has multiple representations with the same number of tokens, students may use any one of them, depending on which one is most useful in a given context.

The numbers 1 to 15 are represented on the multiplicand board with white counters only. The number 15 requires 4 counters, 15 = 8 + 4 + 2 + 1.

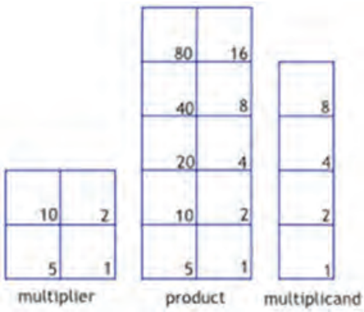

Fig. 9. Boards for multiplication

Algorithm for multiplication

Call the multiplicand m, and start with an empty product board.

To add 1*m to the multiplier, just copy m onto the product;

to add 2*m, copy m, and shift it one square up;

to add 5*m, copy m, and shift it one square to the left;

to add 10*m, copy m and shift it one square left and up.

When multiplying by a negative number, flip over all counters of the multiplicand.

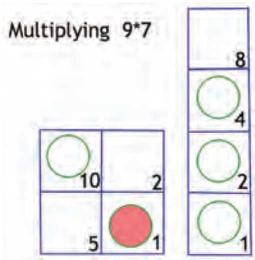

Fig. 10. Multiplying nine times seven

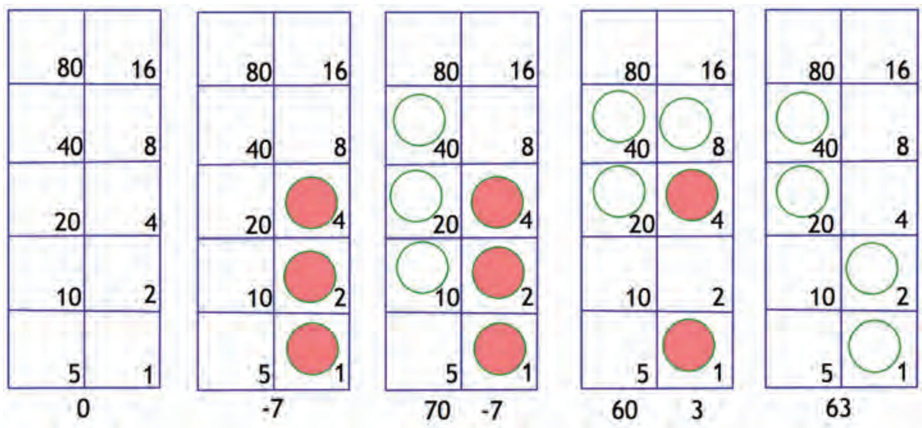

Fig. 11. Multiplying nine times seven step-by-step

Second task

A student is given a cup of beans or other small objects such as river pebbles, pennies, or non-rolling marbles. The number of objects, unknown to the child (but known to the teacher), can vary from 2 to 63.

The child is asked to count the objects and write down the result in Hindu-Arabic notation. All computations have to be done mentally and on a counting board, having only one rod as shown in Figure 12.

The child has to use the algorithm 'Counting by halving', which does not require knowing number words in any language.

Algorithm for counting by halving

Initial step:

Put one white counter on square 1 on the board. Empty the cup, making one pile of beans on the table.

Next step:

Use both hands to partition the pile into two equal parts. Notice if one bean is leftover, or no bean is leftover. (This is important!)

Case 1: If no bean is left, move the top counter on the board one square up.

Case 2: If there is one leftover bean, put one new white counter on the square above the top counter on the board.

Put one of the two piles of beans from the table and the leftover bean (if any) back into the cup.

Look at the pile that is left on the table.

If it contains 2 or more beans, repeat this step.

If it contains only one bean, put it into the cup, and you are done.

Writing the result

Mentally compute the total value on the board and write it down.

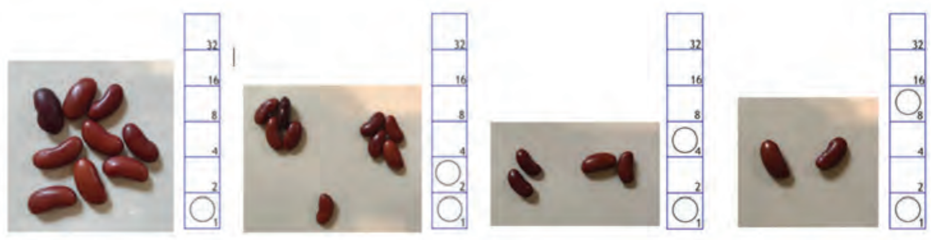

Figure 12. Four steps in counting nine beans

Third task

In this task each student works alone. But a student who has already finished the task should help anyone who needs help.

Each student is given an irregular flat piece of modeling clay and a plastic knife.

The sharing-task instructions: Cut the ball of clay into pieces so that each student in the class can get one piece. In each step you are going to cut one piece of clay into two or three parts. Try to make them approximately of the same volume. Their shape is unimportant.

Algorithm the students need to learn:

Count the number of students in the class and represent it with counters on a one–column board of length 6. Now you will be working with "piles" of pieces.

Start: Put the piece of clay on a rather large flat surface. It is the first 'pile', consisting of only one piece.

There are two kinds of moves, even and odd.

Even move: Cut each piece from the pile into two, approximately equal, parts..

Odd move: Cut one piece (preferably the biggest one), into three, approximately equal parts; and all other pieces into two parts, as in an even move

You need to make the right number of moves in the right order to make the required number of pieces; and you get this order from reading the number from the counting board from the top down. When there is a counter on the location, you make an *odd* move. When the location is empty, you make an *even* move.

Example: 37 pieces are needed.

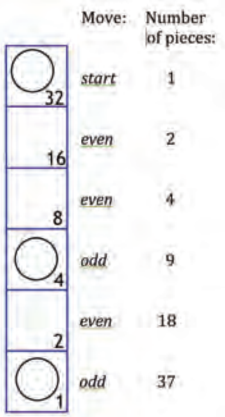

Figure 13. Seven stages of a sharing task

Remark.

This algorithm is easy to use, but rather difficult to learn. So before individual students would try it, they should be quite familiar both with representing the numbers on this board and with cutting clay. Also the teacher may demonstrate to the whole class how to do it.

Final comments

1. More classroom materials using counting boards are available at *Breaking Away from the Mathbook* under *Number Boards.* Most of the available materials have been tested with current and practicing teachers taking college math courses. But so far we have no data showing how a proposed curriculum for early grades would work as a whole. So we present it only as a possible alternative approach and not as one that is recommended.

2. We have avoided discussing pedagogical issues involving learning arithmetic in early grades. (For example: In a bilingual classroom, should all children learn spoken number words in both languages?) In material for teachers we try to present only the mathematical content, the task, and the organization of student work that is appropriate for it, leaving all other decisions to the classroom teacher.

3. None of the ideas presented in this paper concerning mathematical concepts and methods of computation is original or new. All of them, as shown at the beginning of the paper, were either used before or at least considered, but somehow they were never fully developed and they never broadly spread.

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John Leslie http://www-groups.dcs.st-and.ac.uk/history/Biographies/Leslie.html

History of counting boards:

The earliest surviving counting board http://www.historyofinformation.com/expanded.php?id=1664

The abacus: A brief history https://www.ee.ryerson.ca/~elf/abacus/history.html