Mathématique moderne: A pioneering Belgian textbook series shaping the New Math reform of the 1960s

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Abstract

In 1963 the Belgian mathematician and mathematics educator Georges Papy published the first volume of his groundbreaking textbook series entitled Mathématique moderne (MM) (in collaboration with Frédérique Papy), intended for students from 12 to 18 and based on several years of classroom experimentation. It marked a revolution in the teaching of mathematics and in the art of textbook design. Papy reshaped the content of secondary school mathematics by basing it upon the unifying themes of sets, relations and algebraic structures. Meanwhile, he proposed an innovative pedagogy using multi-colored arrow graphs, playful drawings and 'visual proofs' by means of drawings of film strips. During the 1960s and early 1970s, translations of the volumes of Mathématique moderne *appeared in European and non-European languages and were reviewed in mathematics education journals of that time. Papy's MMs influenced the national and international debates and became major guides for shaping the New Math reform in several countries.*

Keywords: Frédérique Lenger; Georges Papy; Mathématique moderne; New Math; textbook analysis

Introduction

In Belgium, the 'New Math' reform movement or 'modern mathematics', as it was commonly called in Europe, was inextricably linked with the personality of Georges Papy. Papy was born in Anderlecht, a municipality in Brussels, on November 4, 1920 and died in Brussels on November 11, 2011. He grew up in the pre-World War II period and studied mathematics at the Université libre de Bruxelles (ULB). During the war, he was an active member of the armed underground resistance forces in Belgium, serving in particular in the areas of intelligence and action, and teaching clandestine courses at the ULB and in a clandestine Jewish school (Gotovitch, 1991). After the end of the war, Papy obtained a PhD in mathematics (in 1945) and was granted an advanced teaching diploma by the Faculté des Sciences of the ULB in 1951. He was appointed lecturer at the ULB in 1956 and promoted to full professor in 1962, in charge of the chair of algebra which he occupied until his retirement in 1985 (Van Praag, n.d.). After a relatively short career in pure mathematical research, with several studies in the fields of algebra, topology, analysis and differential geometry, work being awarded at that time and still being recognized as important (Dieudonné, 1992), Papy reoriented his professional career. It was probably in 1959

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that he became involved in the Belgian – and very soon also in the international – New Math educational reform movement. From then on, the modernization of mathematics teaching 'from kindergarten to university' became his life mission.

We briefly sketch Papy's career in mathematics education. The impulse for Papy's involvement in the New Math movement was a question by Frédérique Lenger and Willy Servais, two influential Belgian mathematics teachers at the time and both, from the early 1950s on, members of the Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques [International Commission for the Study and Improvement of Mathematics Teaching] (CIEAEM). In 1958, in the margin of 12th CIEAEM meeting in Saint Andrews (Scotland, UK), Lenger and Servais had developed an experimental program for the teaching of modern mathematics to future kindergarten teachers (who were at that time in Belgium trained from the age of 15 in a special kind of secondary schools) (Félix, 1985). The program was run during the school year 1958-59 in two such schools, but Lenger and Servais realized that they needed the assistance of a research mathematician for the further development of their experimental actions and consulted Papy to advise them. Papy not only accepted their invitation but immediately took charge of the project and in the school years 1959-60 and 1960-61 he started teaching experimental classes in Berkendael, a school for kindergarten teachers in Brussels (G. Papy, 1960). There he formed his ideas about the construction of the content and laid the basis of his innovative didactic approach. In this period Lenger and Papy married each other. Also professionally Frédérique and Georges became a team, permeated with the same ideas and with a firm determination to actually implement the reform (Noël, 1993).

In May 1961, Papy published his *Suggestions pour un nouveau programme de mathématique dans la classe de sixième* [Suggestions for a new mathematics curriculum in the first year of secondary schools] (G. Papy, 1961). This curriculum proposal was based on Papy's experiences in Berkendael and already in September 1961, a new and large-scale experiment was set up in the first year of secondary school (and from then on gradually in the subsequent years). To coordinate and to develop meticulously these and other initiatives related to the upcoming reform, Papy founded on May 24, 1961 the Centre Belge de Pédagogie de la Mathématique (CBPM) [Belgian Center for Mathematics Pedagogy] of which he also became the chairman. The Centre, which brought together all main actors in the reform movement, organised, beside other meetings, the Journées d'Arlon [Days of Arlon], a series of large-scale teacher (re) education courses and, from 1968 on, also published its own journal NICO. The Centre's actions finally led to a generalized introduction of modern mathematics in the first year of all Belgian secondary schools from September 1968 on (and from then on gradually in the subsequent years). At that time however, Lenger's and Papy's interests were already re-directed towards the primary level for which they started in September 1967 an extensive experiment in two classes of six-year-old children (F. Papy, 1970). In September 1976 modern mathematics was introduced in the Belgian primary schools of the catholic network and two years later in the publicly run schools.

During the 1960s, Papy was also a prominent actor on the international mathematics education scene. As an expert he was invited to present his reform proposals at the major forums of that period, including the UNESCO symposium in Budapest (1962), the OECD conference in Athens (1963), the ICMI symposia in Frascati (1964) and Echternach (1965), the 15th International Congress of Mathematicians in Moscow (1966) and the UNESCO colloquium in Bucharest (1968). Papy also played a major role within the CIEAEM (as vice-president since 1960 and as president since 1963), but in 1970, due to strong disagreements about the Commission's future, Papy left the Commission (Bernet & Jaquet, 1998) and founded, with some loyal followers, the Groupe International de Recherche en Pédagogie de la Mathématique (GIRP) [International Group for the Study of the Pedagogy of Mathematics] (1971). However the meetings of the GIRP, as well as the other international activities of Papy during the 1970s, had limited impact on further developments in mathematics education. The tide had turned; modern mathematics was already on the way back in most countries.

The *MM*s

In 1963, Papy started with the publication of *Mathématique moderne* (in collaboration with Frédérique Papy) (G. Papy, 1963, 1965, 1966, 1967a, 1967b), a revolutionary textbook series, both in terms of content and layout. The series was intended for teaching modern mathematics to students aged 12 to 18 and was partly based on Papy's previous classroom experimentation in Berkendael (G. Papy, 1960). However, the series did not follow any specific program approved by the Ministry. In the next subparagraphs we briefly review the different volumes of this series to gain insight in Papy's viewpoints as revealed through this work.

MM1 – *The language of modern mathematics*

MM1 (G. Papy, 1963, see also Fig. 1), "particularly suited for students aged 12" (p. vi)[1], has 24 chapters with the following mathematical contents: (1-5) Algebra of sets (pp. 1–52), (6) First elements of geometry (pp. 53–87), (7-13) Relations, properties, composition (pp. 88–215), (14-15) Transformations of the plane (pp. 216–235), (16-18) Natural numbers (cardinal numbers), operations (pp. 236–295), (19) The binary system of numeration (pp. 296–315), (20) Integers (pp. 316–341), (21 23) Equipollence, translations, vectors, central symmetries (pp. 342–440), and

1 Unless otherwise stated, all translations were made by the authors.

(24) Groups (pp. 441–459). On the basis of this overview, it might be clear that this textbook proposed a completely new structure and foundation of initial secondary school mathematics of that time. In a note for students Papy explains the need for a renewed approach to mathematics education:

> If you want to play an effective part in the world of tomorrow […] you must master the mathematics of today. You will have to get to grips with het basic ideas of modern mathematics – which are used by all scientists – without wasting any time. And that is why we cannot teach you mathematics in the way your parents or your grandparents were taught – though you will be discovering (by a different route) all the basic things they were taught. (p. 45)

Papy's mathematical universe is built on (naïve) set theory. Sets are also the unifying element in this universe. The set-theoretical concepts are visualized by Venn diagrams. Venn diagrams are used for concept development, but also for reasoning and proof, e.g. of the properties of set operations. The algebra of sets is studied extensively because of its intrinsic value and its interesting new applications (which, however, are not discussed in the book). Moreover, because the algebra of sets resembles, but in some respects differs from the usual 'algebra of numbers', this study can also contribute to a better understanding of the latter. Implicitly the symmetric difference provides a first example of a group structure.

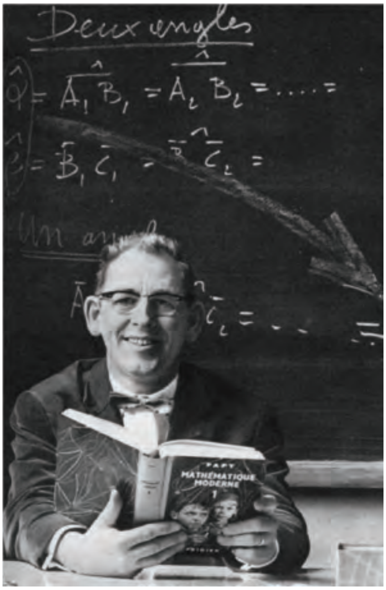

Fig. 1. Papy with a copy of the first volume of *Mathématique moderne*

The plane (an infinite set of points) is denoted by Π, and straight lines are subsets of Π. Two straight lines are equal sets or their intersection is the empty set – in both cases they are called parallel – or their intersection is a singleton (a set with exactly one element). These possible mutual positions are illustrated with Venn diagrams ("The set diagrams provide intuitive support for the logical structure of the theory", p. vi). In his first introduction to geometry Papy already brings in some basic topological notions, e.g. he differentiates between an open and a closed disk, and a circle (which only includes the perimeter). To visualize these notions the red-green convention (for parts that are ex/included) is introduced. Papy devotes ample attention to proving and logical-deductive reasoning. Therefore some initial propositions are selected as axioms. These axioms are not given all at once, but are released subsequently. In the chapters on geometry Papy proves some properties of parallelism and perpendicularity. These properties are simple and also intuitively clear, which makes them particularly suited for learning to reason correctly and for understanding the essence of proof. Noteworthy is Papy's axiom Π4, a reformulation of Euclid's parallel postulate as "Every direction is a partition of the plane" (p. 74). It typifies his concise and abstract style.

Relations (sets of ordered pairs), their properties and composition, are discussed in great detail. Special attention is paid to relations of equivalence and order, functions and permutations. Papy further develops his pedagogical method based on arrow diagrams or papygrams (Holvoet, 1992). Papygrams made their entrance in Papy's Berkendael course (G. Papy, 1960), but reappear here in multiple colours (up to six!) and in different geometrical constellations. New concepts are typically introduced with some simple and familiar situations with which "the student is encouraged to take an active part in building the mathematical edifice" (p. vi). Then this situation is abstracted to prepare a precise concept definition. Although the situations are common, they are especially designed for the above purpose and hence often a bit artificial. Papy never uses newly learned concepts to analyse (really) realistic situations although he argues that sets and relations are versatile and widely applicable instruments of thought.

> The scope of the material studied in the first 13 chapters goes far beyond the boundary of mathematics. The student is initiated into types of reasoning constantly used in all spheres of thought, science and technology. (p. vii)

Functions and their graphical representations by arrow graphs reappear in geometry as 'transformations of the plane'. Papy first discusses the constant and the identical transformation, the simplest – but not necessarily the most relevant – cases. For the third case, the parallel projection onto a line, again the way 'from simple to more complex' is followed: first points, then line segments and finally some other 'sets of points' are projected. In the last chapters on geometry, the concept of equipollence of ordered pairs of points is defined. It is proved that equipollence is an equivalence relation, of which the equivalence classes are called translations or vectors, and the set of translations forms a group under composition. In this section Papy introduces the didactical tool of proof by film fragments: a sequence of suggestive images, from which a line of thought may be read easily, is presented and students are asked to add justifications.

> The assimilation of a proof involves several stages which we should try to keep separate. The first step is for the pupil to understand the film so that he can explain it in informal language. Next he must be able to reconstruct the argument himself. After this comes the stage where more formal justifications are required. Only after all this do we turn our attention to the proper setting out of the proof. (p. viii)

Regarding algebra Papy first founds students' pre-knowledge about numbers and their operations in a set-theoretical framework. Natural numbers are defined as cardinal numbers of finite sets and the addition and multiplication of such numbers is related to, respectively, the union and Cartesian product of sets. The positional notation of numbers is revisited with the study of the binary system for which some kind of abacus is developed. This didactical tool, already prefiguring Frédérique's minicomputer for introducing numerical calculation at the primary level (F. Papy, 1969), is also used for the introduction of integers and their addition. For that Papy introduces a combat game with red and blue counters, representing oppositely signed numbers which 'kill' each other when coming in the same compartment. Properties of the operations with integers are strongly emphasized and lead to the discovery of a group and ring structure. MM1 concludes with a chapter on (abstract) groups, bringing together and systematizing several 'concrete' examples from the previous chapters.

MM2 *and* MM3 – *Real numbers and the Euclidean vector plane*

MM2 (G. Papy, 1965), subtitled *Real numbers and the vector plane*, provides a rigorous, but didactically elaborated construction of the field of real numbers and the vector plane. The textbook, "intended for 13-year olds" (p. vi) has 18 chapters and deals with: (1-2) The group $\Pi_O,+$ (Π_O is the plane in which a point has been fixed so that each point represents a vector) (pp. 1–40), (3-5) Graduation of the line and the axiom of Archimedes (pp. 41–92), (6-7) Real numbers (pp. 93–154), (8) The theorem of Thales (pp. 155–177), (9) Homotheties (pp. 178–207), (10-13) The multiplication of real numbers (pp. 208–317), (14) Rational and irrational numbers (pp. 318–355), (15) Vector spaces (pp. 356–384), and (16-18) Equation of a straight line in the plane (pp. 385–434).

The major part of *MM2* is devoted to a mathematically sound construction of the real numbers and to the equipment of the set of real numbers with order and with an additive and multiplicative structure. Papy uses a process of binary graduation of a straight line to build up the real numbers. By inserting the axioms of Archimedes and continuity he is able to establish a one-to-one correspondence between the points on a straight line and the set of numbers, represented by terminating or non-terminating binaries, at a certain moment called 'real numbers'. Then the order and additive structure of the points (vectors) on that line is transferred to the set of real numbers. For the multiplicative structure, Papy first defines multiplication of real numbers by means of homotheties (= homothetic maps): if h1 and h2 are homotheties with factors a and b, then a · b is the factor of the homothety h2 ∘ h1 (the composition of h1 and h2). Finally, the basic properties of multiplication are deduced from the corresponding properties of composition of homotheties. The ordered field of the real numbers appears as the ultimate reward (on page 275). Papy believes that introducing the real numbers in this manner "enriches both the geometrical notions and the real number concept" (G. Papy, 1962, p. 6). The rational numbers are defined after the real numbers (!) – their structure appears to be an ordered subfield of that of the real numbers – and finally, some attention is paid to general vector spaces (exemplified with the principles of the vector plane) and to elements of vector-based affine analytic plane geometry.

In *MM3 – Euclid now* – (G. Papy, 1967a) the axiomatic-deductive building up of plane geometry is continued and will finally result in a contemporary vector-based exposition of Euclidean (metric) geometry for 14–15-year old students.

> Euclid's Elements exposed the basic mathematics of his time, about 300 years before J-C. The monumental work of Nicolas Bourbaki presents, at the highest level, the basic mathematics of today. The MMs want to expose the Elements of today's basic mathematics for adolescents ... and people of any age and schooling who wish to initiate themselves in the mathematics of our time. (p. vii)

The 19 chapters cover the following topics: (1-3) Point reflections, (oblique and perpendicular) line reflections (pp. 1–46), (4-8) Isometries, classification (pp. 47–141), (9-13) Distance, circle, scalar product of vectors (pp. 142–281), and (14-19) Angles (pp. 282–441).

Transformations and groups which are generated by these transformations, play a key role in Papy's construction of (Euclidean) geometry. Isometries are defined via the composition of a finite number of (perpendicular) line reflections. The different types – translations, rotations, reflections and glide reflections – and their possible compositions receive considerable attention. Colorful classification schemes based on Venn diagrams are presented and group structures are highlighted. Each time a group is discovered, it can provoke an Aha-Erlebnis: when a student recognizes a known abstract structure in a new setting, he might be able to apply all previously-learned knowledge and skills about this structure to that setting, an example of Ernst Mach's 'economy of thought' principle (see, e.g., Banks, 2004).

> Over the past half century, mathematics has switched from the artisanal stage to the industrial stage. The machine tools of our factories made it possible to save human muscular effort. The great structures of contemporary mathematics allow to save the human mind. (p. vii)

Transformations are also promoted as an alternative for traditional methods in school geometry in comparison with which they are much more intuitive and universal.

> The outdated artisanal technique based on congruence of triangles must be abandoned in favor of translations, rotations and reflections, which are much more intuitive and whose scope goes far beyond the framework of elementary geometry alone. (p. ix)

Once the group of isometries is established, the fundamental concepts of Euclidean (metric) geometry can be introduced. The distance of a pair of points and the length of a line segment are defined by means of isometries (and from then on, isometries gain their etymological meaning of 'length preserving transformations'). Definitions of the norm of a vector and the scalar product of two vectors follow. The natural structure for Euclidean geometry – a vector space equipped with an inner product – is thus created. Classical results, such as the Pythagorean Theorem – the cosine rule formulated in terms of vectors – can be proved easily within this structure.

> Certain statements, once fundamental, are reduced to the rank of simple corollaries. That they now stop cluttering up the memory of our students. If necessary, they would be able to retrieve these results by routine use of one of the machine tools of modern mathematics. (p. ix)

Angles are equivalence classes of rotations (in Papy's words "rotations that have lost their center", p. 289). The sum of angles is defined by means of the composition of rotations and so the group of angles, isomorphic to the group of rotations, is created. MM3 ends with some (very) basic elements of trigonometry.

MM4 *to* MM6 – *The series' closing in a minor key*

We can be short about *MM4*: the book remained unpublished. Late 1975, early 1976 Gilberte Capiaux, assistant at the CBPM, has worked on a volume about real functions referred to as *MM4*, supervised and guided by Papy, but the initiative has not yielded more than a hand-written draft (G. Papy, n.d.).

MM5 – Arithmetic – (G. Papy, 1966), intended for 14–18-year old students, presents a contemporary introduction to discrete mathematics. It only relies on contents that were exposed in *MM1* and *MM2*. The book has five major sections: (1) Combinatorics (pp. 1–50), (2) The arithmetic of integers (pp. 51–136), (3) The arithmetic of rational numbers (pp. 137–156), (4) An introduction to commutative rings and fields (pp. 157–230), and (5) Arithmetical properties of groups and finite fields (pp. 231–280).

In combinatorics Papy deliberately avoids the "disused terminology" (p. vii) of variations, combinations, groupings with or without repetition, … and consequently bases his exposition on the theory and language of sets and relations (in particular mappings), both for formulating counting problems and for developing and defining the necessary instruments. This is the only domain in the *MM*s in which some real problem situations are presented and discussed. The arithmetic of integers, dealing with divisibility, prime factorization and related issues, is embedded in the ring of integers (*MM1*). This structure and its substructures are studied in depth, and further abstracted and generalized to commutative rings and fields. In the last section, "particularly intended for students […] preparing for mathematics studies" (p. x), one can find theorems as, e.g., 'every transformation of a finite field is a polynomial function' and 'the multiplicative group of every finite field is cyclic', results that normally go beyond secondary school mathematics.

In *MM6 – Plane geometry* – (G. Papy, 1967b), published in a somewhat more sober style than the other volumes, Papy takes up the geometric thread for 15–16-year olds. The 11 chapters cover the following contents: (1-3) Repetition/summary of *MM1*, *MM2* and *MM3* (pp. 9–82), (4-7) The (Euclidean) vector plane, linear transformations, matrices (pp. 83–183), (8-9) Orthogonal transformations, similarity (pp. 184–234), (10) The complex plane (pp. 235–258), and (11) Trigonometry (pp. 259–267).

Papy first retraces in brief the laborious path, from the original 'intuitive' (synthetic) axioms of geometry to the establishment of a Euclidean vector plane structure, the path that the students had followed from the age of 12 to 15. This summary, mainly clarifying the math-educational methodology of the first three *MM*s, must prepare these students for the second step which is described as a 'psychological reversal': the structure of a Euclidean vector plane is taken as a new and unique starting axiom for the further development of plane geometry (from p. 84 on). This approach also opens perspectives for the future study of higher-dimensional Euclidean spaces, in particular for building up solid geometry. Although it is possible, in principle, to rebuild all mathematics from previous years on the basis of the new axiomatic, this is not suggested.

> This wonderful machine tool should not be used to rediscover what we already know. It should allow new conquests. (p. v)

Linear transformations of the vector plane play a key role in the continuation of the book. Special types, such as orthogonal transformations (= linear transformations that preserve the scalar product) and similarities, are studied in depth. As in *MM3*, the transformational aspect as well as the identification and classification of (sub) structures receive ample attention and are illustrated with multicolored papygrams, playful drawings and Venn diagrams. In *MM6*, transformations are also algebraically typified by means of matrices. At the end of the book complex numbers are introduced as direct similarities. By relying on the structure of the latter and isomorphism, it is proved that the complex numbers form a field extending the field of the real numbers.

Shaping the New Math reform

Although conceived as textbooks the *MM*s have never been used for that purpose, except in experimental classes. When from 1968-69 on New Math was made compulsory in Belgian secondary schools, the official programs were different from and less ambitious than those developed by Papy. In response Papy and collaborators started the *Minimath* series, a 'light version' of the *MM*s, but only the first two volumes of that series ever appeared (G. Papy, 1970, 1974). Papy's *MM*s have thus served primarily as a major source of inspiration, both in terms of content and style, for mathematics educators and textbook developers during the 1960s, the period in which the New Math reform was prepared and implemented in several countries. More specifically with respect to the reform in Belgium, Warrinnier (1984) noted that "all textbooks essentially go back to the remarkable series *Mathématique moderne* by G. Papy" (p. 120).

The *MM*s were also translated into several different languages, including Danish (Vols. 1, 2, 3), Dutch (Vols. 1, 2, 3, 5), English (Vols. 1, 2), German (revised version of the geometric chapters of Vols. 1 and 2), Italian (Vol. 6), Japanese (Vol. 1), Romanian (Vols. 1, 2), and Spanish (Vols. 1, 2, 3, 5).

The impact of Papy and his *MM*s on the international mathematics education debates during the 1960s can hardly be overestimated. As mentioned before, Papy acted as an uncompromising New Math ambassador at major international conferences of that period, reported about his successful experiments and defended, with verve and authority, his views on the modernization of mathematics teaching. Already at the 1963 OECD conference in Athens, Papy presented an extended sneak preview of the mathematical content and methodological approach of his first two *MM*s (G. Papy, 1964). Papy's design of teaching modern mathematics was well received by the other OECD experts:

> The example given by Mr. Papy […] was stimulating as to what can be accomplished by a proper blend of modern mathematical ideas with very conscious psychological methods of presentation. When students are directed toward the discovery of mathematical patterns and the self-construction of mathematical entities (such as the real numbers), motivation and permanency of learning are greatly enhanced. (OECD, 1964, p. 296)

International debates on the teaching of geometry

Of particular interest is the influence of Papy's experimental approach, as documented in the *MM*s, on the international debate on the teaching of geometry that took place in Europe during the 1960s. This debate was launched by Jean Dieudonné who rejected, in combative terms, the traditional teaching of Euclidean geometry at the 1959 Royaumont Seminar. A group of experts that met at Dubrovnik in 1960 in order to work out a detailed synopsis for a modern treatment of the entire mathematical curriculum, only partially succeeded for the geometry part. Also at related international conferences specifically devoted to 'the case of geometry' (Aarhus, 1960 and Bologna, 1961), only compromises could be reached and the debate increasingly narrowed to the search for the most adequate axiom system for the teaching of geometry at the secondary level.

The debate heated up in 1964. That year Gustave Choquet published his *L'enseignement de la géométrie* [The teaching of geometry] in the Introduction of which he stated that the perfect "royal road to geometry is based on the notions of vector space and scalar product" (Choquet, 1964, p. 11). However, Choquet acknowledged that children benefit from an approach to geometry based on concepts drawn from the real world such as parallelism, perpendicularity and distance. To reconcile this pedagogical concern with the mathematically most valid method, Choquet set out intuitively clear synthetic axioms to demonstrate the algebraic structure of the plane. Then using the tools of linear algebra he developed his course of geometry. Also Jean Dieudonné published a book on geometry teaching that year, entitled *Algèbre linéaire et géométrie élémentaire* [Linear algebra and elementary geometry] (Dieudonné, 1964). Dieudonné uncompromisingly based his geometry course on linear algebra and made absolutely no concessions to synthetic methods. Moreover, in the Introduction he launched a violent attack on Choquet's more realistic and evolutionary axiom system which demonstrated "a remarkable ingenuity which shows the great talent of its author, but that I consider as completely useless and even harmful" (p. 17).

André Revuz tried to reconcile Choquet's and Dieudonné's points of view. Therefore he enlisted the help of Papy whose approach was very similar to that of Choquet[2]. Moreover, Dieudonné himself paid tribute "to the remarkable and promising trials of our Belgian neighbours" (Dieudonné, 1964, p. 17), referring to Papy's experiments with 12–13-year olds as documented in *MM1*. So Revuz could authoritatively defend the systems of Choquet (and Papy) as 'intermediate steps' between students' intuition and the 'ideal' (purely linear algebra based) system proposed by Dieudonné.

2 However, there are areas of difference. For example, whereas Choquet deliberately assumed the distance on a line and the structure of the real numbers, Papy gradually developed these concepts.

> However, if one believes that geometry is not only a mathematical theory, but also a physical theory, if one thinks that the role of education is not only to know mathematics, but also to learn to mathematize reality, one can think about Choquet's system as an intermediate step, which will not only allow teachers to change their mentality, but perhaps also will enable any student to move easily from the intuitive space to the mathematical theory. (Revuz, 1965, p. 273)

On the initiative of Revuz, the disagreement between Choquet and Dieudonné was officially settled in April 1965 at the 19th CIEAEM meeting in Ravenna (Italy), with a statement about the role of geometry in the education of 10–18-year old students, agreed by all CIEAEM members present (but in the absence of the two disputants). In this statement the special place of geometry was recognized. More concretely, an approach in two stages, inspired by the ongoing experimentation of Papy and his CBPM, was recommended. The first stage (for 10–11- to 14–15-year olds) aimed at mathematizing the concrete space of the student by means of axioms and deductive reasoning, leading to the construction of a Euclidean real vector space structure. In the second stage (for 14–15- to 17–18-year olds) this structure was taken as a new axiomatic for the further development of geometry. The Ravenna manifesto was approved by Choquet and Dieudonné at the ICMI meeting in Echternach in June 1965 (Félix, 1985).

Contribution of the MM*s to national reform debates*

The contribution of Papy and his *MM*s to the New Math debates in the different countries that were involved in this reform movement, has not been investigated systematically. With respect to France and the Netherlands, we are, to a certain extent, informed about the reception of Papy's ideas by reviews of the *MM*s in journals of professional organizations of mathematics teachers.

Papy's *MM*s were enthusiastically welcomed in France. In the *Bulletin de l'Association des Professeurs de Mathématiques de l'Enseignement Public* (APMEP) [Bulletin of the Association of Mathematics Teachers of Public Education], Gilbert Walusinski, influential member and former president of the APMEP, reviewed the first two *MM*s in glowing terms (1963, 1966). For him, Papy's experiments, as documented in the *MM*s, demonstrated that it is really possible to teach modern mathematics in an active and non-dogmatic way, in real classes of 12–13-year olds, including students who are not necessarily gifted for mathematics. Walusinski characterised Papy's actions as encouragements and inspiring examples for future developments in France:

> Papy's *MM1*, the printed testimony of his experience, it is the reform in act. [...] The time will come when a reconstruction of our public schools will be possible; it will not be a time of rest, but one of an action that should be

> fast and efficient. We must prepare for it from now. Papy is helping us. Very friendly, I want to thank him. (Walusinski, 1963, p. 126)

Piet Vredenduin, a prominent mathematics educator from the Netherlands, reviewed Papy's experimental approach in global terms (1967a), as well as the different MM volumes upon publication, in *Euclides*, the journal of the Nederlandse Vereniging van Wiskundeleraren [Dutch Association of Mathematics Teachers] (1964, 1966, 1967b, 1967c, 1968). As a former Royaumont delegate, Vredenduin was favourably disposed to New Math. He acclaimed Papy's radical and uncompromising modernization efforts, and recommended all mathematics teachers to read and to enjoy the *MM*s. Nevertheless, he raised questions about six main differences with what was common practice in his country at that time: (1) Papy pushes mathematical rigour to the extreme, (2) from the outset, the emphasis is on the structure of the mathematical systems, (3) the training of mathematical techniques is neglected, (4) the interest in the triangle, criteria for congruence, the special types of quadrilaterals, … has decreased, (5) the use of symbols is strongly enforced, and (6) the acquisition of a specific mathematical language is promoted to a central goal. He concludes that also in the Netherlands a modernization is desirable, but he is concerned that too many valuable things from the past will be dumped (Vredenduin, 1964, 1967a).

Vredenduin's compatriot Gerrit Krooshof, member of the editorial board of *Euclides* and leading author of a successful Dutch textbook series with the same title as Papy's *MM*s (*Moderne wiskunde* [Modern mathematics]), formulated his reservations in terms of a metaphor:

> We can confidently say that this means a new building of secondary school mathematics. As a Le Corbusier of mathematics education, Papy has created, from pre-stressed concrete and glass, a robust and (at least for us) transparent structure. But for our students the windows are too high. (Krooshof, 1967, p. 194)

Papy also left his footprints in the Nordic countries. The Danish translation of *MM*1 and *MM*2, dating from 1971, was reviewed in *Nordisk Matematisk Tidskrift* [Nordic Mathematical Journal] (Solvang, 1972). Ragnar Solvang praised Papy's audacity to present completely new materials for secondary school mathematics in an accessible, informal and 'entertaining' style, but raised the question whether the new topics can really defend their place in a mathematics syllabus for 12–13-year olds:

> It's clear that this can be discussed and the answer one gives will depend on the objectives that are sets up for the subject. If one believes that the goal should be to provide the students with an insight into the subject's structures, then it is clear that some of these topics motivate themselves directly. And when it concerns adjusting to the age level, Papy has realized a nice piece of work. In the *MM*s, the students are guided through the individual substructures and finally end up with the group concept. His use of filmstrips in the

> chapter on groups is methodology of the best class. (Solvang, 1972, p. 146, translated from Norwegian by Kristín Bjarnadóttir)

Moreover, Solvang wondered how the teaching, based on these books, can be organized practically, in particular how the practical arithmetic work and the applied side of mathematics can be integrated in the same spirit of what's already there. He nevertheless acknowledged the centrality of Papy's work in the debate about the new directions of lower secondary school mathematics.

Concluding remarks

The *MM*s were definitely a milestone in the history of the New Math reform movement of the 1960s, having inspired many mathematics educators worldwide. Papy did not present a cautious compromise by integrating new ideas in an existing tradition, but radically reshaped the content of secondary school mathematics by basing it upon the unifying themes of sets, relations, functions and algebraic structures. Meanwhile, he proposed a completely new teaching approach using multi-colored arrow graphs, playful drawings and 'visual proofs' by means of film fragments. However, these strengths were also weaknesses. Probably it was too ambitious to try to change at the same time both the content and the pedagogy of mathematics education. Although the approach proved to be successful in an experimental setting, directed by Papy or Frédérique, it went far beyond the range of competence of most common teachers.

Papy's voluminous books also did not offer a practical solution for actual classroom teaching. Although the *MM*s were conceived as textbooks, they didn't follow any official program. Moreover the series remained incomplete: the teaching of algebra and geometry for 12- to 16-year olds, is well elaborated in the first three *MM*s and in *MM*6, but the continuation to 16- to 18-year olds remained unclear. Only for the teaching of arithmetic, a convincing alternative in line with the other volumes was developed in *MM*5, but for several reasons, this topic never received a central position in secondary school mathematics. For the teaching of analysis, which constitutes the lion's share of the mathematical curriculum for 16- to 18-year olds, the *MM*s didn't provide the necessary material. This is strange as Papy (and Frédérique) definitively had some clear ideas about the teaching of analysis (see, e.g., G. Papy, 1968) and these ideas were already prepared, at least to a certain extent, in *MM*1. But unfortunately an integration of these ideas in the *MM* project was never realized. A possible reason is that by the end of the 1960s, Papy became discouraged about the actual implementation of the reform he had initiated. In 1972 he declared to the press:

> The message of the promoters of the reform of the teaching of mathematics, all over the world, has been lost ... the ship of the reform is stranded. (Debefve, 1972, p. 7)

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3 There may occur some confusion regarding references to publications by Frédérique Lenger and Georges Papy, wife and husband from October 1, 1960. From the 1960s, Georges typically signed his publications as 'Papy', while Frédérique, after marriage, used her husband's surname (and signed 'Frédérique Papy') or signed with only her first name 'Frédérique'. The MMs mention 'Papy' as an author, but it is stated that the series was written 'in collaboration with Frédérique Papy'. Lists of publications from both authors can be found at http://www.rkennes.be/.

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