Two textbooks on unconventional arithmetic: Reactions of influential persons

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Abstract

The paper describes two Icelandic arithmetic textbooks, published respectively in 1780 and 1911–14, both written at the beginning of an era when enhancing general education was placed on the agenda for restoring the Icelandic society after natural disasters. Both textbooks emphasized unconventional and mental arithmetic. Both books were criticized by influential persons. The focus is on the grounds of their arguments. Both books survived the criticism, considering the circumstances, while few positive comments on them have been found in sources from their times.

Keywords: Enlightenment, arithmetic textbooks, mental arithmetic, number tricks, Rule of Three

Introduction

The influence from the Enlightenment was strongly felt among Icelandic students in Copenhagen from the 1770s onwards (Sigurðsson, 1990). Iceland had then been part of the Danish realm from the end of the 14th century. The Danish authorities felt responsibility when Iceland suffered various calamities during the 18th century, such as exceptionally cold climate with pack ice in the 1750s. In the period 1770–1780 a number of informative printed texts were distributed in Iceland, even for free, by proponents of the Enlightenment, supported by the Danish government or gentry. The two first substantial arithmetic textbooks, written in Icelandic by young Icelanders in Copenhagen, Olavius (1780 and Stefánsson (1785), were published with such financial support. We shall examine the reception of one of them, *Greinileg vegleiðsla til talnalistarinnar* [A clear guide to the number art] by Ó. Olavius of 1780.

The period of cold climate in the mid-18th century was followed by volcanic eruptions and earthquakes in the 1780s which caused death of livestock and consequent famine. It was called the Haze Famine due to poisonous gases from the volcanos. The population sank form about 50,000 inhabitants below 40,000. The two Latin schools were closed down temporarily. From 1802 until 1930 there was only one school at that level in the country. These conditions and the bankruptcy of the Danish Kingdom in 1813 following the Napoleonic wars, hindered results of the efforts of the Enlightenment proponents. For example, no other mathematics textbooks written in Icelandic were published until the 1840s.

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In the 1860s, another cold period with pack ice set in, continuing into the 1890s, resulting in large-scale emigration to North-America. However, small-scale industrialization and mechanization of fishing vessels grew in the same period, gradually improving living conditions. From the mid-19th century, a movement had risen towards independence, and enhancement of education was a factor in that effort. From 1869 until the Great War, a stream of arithmetic textbooks for beginners by at least eight authors were published. We shall examine a reaction to one of the later arithmetic textbooks of that period, *Reikningsbók* [Arithmetic] by S. Á. Gíslason, published during 1911–1914.

In both textbooks, those by Olavius and by Gíslason, the authors emphasized mental and unconventional arithmetic as an aid to written arithmetic but were received with some mistrust. The arguments for this mistrust will be examined.

This paper is a study of reactions of influential persons to these books. The question is on what grounds they founded their opinions. Both persons may be considered as authorities in educational matters for their time. The study is based on one single source each.

Background

Education frame

The two Latin-schools, which became only one in the year 1802, laid their main emphasis on Latin and religious studies as they educated the clergy until the mid-19th century, but most of the time they taught very little arithmetic. Students were given Stefánsson's (1785) arithmetic textbook but they studied arithmetic only if they so wished and were then guided by older students, not the teachers (Helgason, 1907–1915). Other schools did not exist until the 1870s, apart from scattered privately run primary schools. By the King's order of 1791, children were to be able to read, otherwise they could not be confirmed. Knowledge of arithmetic was not widespread, even among the clergy who had the duty to oversee education provided by the homes.

Law no. 2 of 1880 on education in writing and arithmetic required homes to teach children the four arithmetic operations in whole numbers and decimals. Primary school legislation was enacted by law no. 59 of 1907 when the communities were to overtake the responsibility of providing education; in schools in urban areas, and by itinerant teachers in the more rural areas. Both legislations gave sparks to arithmetic textbook publications.

Unconventional arithmetic and mental arithmetic

Mental arithmetic has been part of popular culture in many societies through the centuries. In a summary of researches made by Threlfall (2002), it emerges that most calculations in adult life are done mentally. Mental work develops insight into the number system, also termed "number sense", and mental work develops problem solving skills.

Ancient Egyptian multiplication and the related Russian peasant multiplication are widely known. They primarily involve creating repeated doubles of the multiplicand and adding up those relevant to fill up the multiplier (Seppala-Holtzman, 2007). (An example of this method is multiplying 13·21 as (8 + 4 + 1)·21. The multiplicand 21 is doubled as many times as necessary, here three times, and the individual results are used until the powers of 2 add up to 13). The widespread use of doubling in societies where access to school education and/or writing material was limited, indicates that doubling was considered more accessible than general multiplication. In extant Latin translations of Al-Khwarizmi's arithmetic textbook, *Kitāb al-hisāb al-Hind* of the 9th century, such as *Dixit Algorizmi* and *Liber Alchorismi*, doubling and halving were counted among seven or even nine arithmetic operations (Allard, 1992, p. xxxi).

Arithmetic textbooks of the 19th and 20th century have generally taught written procedures to perform arithmetic operations which may have made mental arithmetic redundant in the minds of students as well as their teachers. The procedures have provided a sense of security which no less applies to the widespread use of pocket calculators in modern everyday life.

Mental arithmetic was, however, still considered important in the mid-20th century. The mathematician Gustave Choquet, a guest-speaker at the Royaumont seminar, emphasized that children must know how to do simple and rapid mental calculations, and they must be accustomed to finding very quickly an order of magnitude for a total of a product, that is to use estimation (OEEC, 1961, p. 66).

Researchers have attempted to assess methods to train strategic flexibility of students in mental arithmetic. Threlfall (2002) proposed that mental calculation is an interaction between noticing and knowledge, and …

> … if the aim of teaching mental calculation is flexibility, what children and adults actually do to calculate efficiently should not be distilled into general descriptions of methods or 'strategies' and promoted as holistic approaches to calculation, offered as models to emulate, or taught as procedures to learn. Rather it suggests that solutions to problems would be better approached as specific examples of how particular numbers can be dealt with, how numbers can be taken apart and put together, rounded and adjusted, and so on. (Threlfall, 2002, p. 45)

Threlfall's conclusion was that flexibility cannot be taught as 'process skill' but will rise consequentially through the emphasis on considering possibilities for numbers rather than by focusing on holistic 'strategies'.

Conventional arithmetic of proportional problems

For centuries, proportional problems were solved by a method called in Latin *Regula Trium*, the Rule of Three. The method consists of finding the fourth proportional to three known quantities. It is traced back to Italian merchants in late medieval times, described in arithmetic books, the Italian *libri d'abbaco* (Van Egmond, 1980). The way of thinking in the Rule of Three can be found in ancient Indian works, e.g. by Brahmagupta (597–668) and Bhaskara II (1114–1185) (Tropfke, 1980, pp. 359–361).

In *Arithmetica Danica*, the oldest arithmetic textbook known to have been available in Iceland, the solution method is described. Four numbers are to be arranged so that the first and the third are of the same kind and the second of the same kind as the fourth. When one wants to know the fourth number, the second and the third numbers are to be multiplied together. The product is then to be divided by the first number to give the sought after fourth number (From, 1649, pp. 77–78). This method echoes in the textbooks by Olavius (1780, pp. 176–177) and Stefánsson (1785, pp. 132–137).

E. Briem wrote an influential arithmetic textbook, first published in 1869. It was used in the emerging lower secondary schools from the 1870s into the 1910s. The search for the unknown in direct Rule of Three was to find a number that is as many times greater or less than the middle term, as the rear term is greater or less than the front term in the previously mentioned sequence of the three known proportional numbers. This was to be found by multiplying the middle term by the rear term and then divide by the front term (Briem, 1898, p. 63–65).

A Clear Guide to the Number Art of 1780 by Olavius

The author

The first comprehensive printed mathematics textbook in Icelandic was *Greinileg vegleiðsla til talnalistarinnar* [A clear guide to the number art] by Ólafur Olavius (1780). Olavius (1741–1788) studied philosophy in Copenhagen during 1765–1768. He was one of the founders of the Icelandic Enlightenment society, in 1779, called *Lærdómslistafélagið*, the Society of the Learned Arts, established by students and intellectuals in Copenhagen. From the beginning of printing in the 16th century, the sole printing press in Iceland had exclusively printed religious books. Olavius brought a new printing press to Hrappsey Island in 1773. For about a year he oversaw there

the first printing of secular books in Iceland. Then he left, presumably due to disagreements about finances. After that he travelled around the country during the summers of 1775–1777 under the auspice of the Danish King in order to write an economic description of Iceland, published in 1780. Olavius stayed in Denmark until his death; during 1779–1788 as customs officer in Skagen. Thus he stayed only for one year, 1773–74, and three summers in Iceland after his Latin school graduation in 1765 (Ólason, 1951, pp. 72–73).

During 1770–1786 Olavius's voluminous informative writing was published, including works about economics and the use of products of nature, all intended to improve and remedy Iceland's economy. The arithmetic textbook Clear guide was financed by an unknown sponsor. About 1300 copies were distributed for free in Iceland where the population was around 50000 and the estimated number of homes was 7000. Records indicate that several copies existed in private libraries in one county in northwest Iceland some 50 years after its publication so it must have been studied by some number of people (Jensdóttir, 1969).

The content

Clear Guide (Olavius, 1780) begins by an address in Danish to Count Schach Ratlau, Knight of the Elephant Order, a possible sponsor, and to whom the author dedicated the book. He described the miserable life of Icelanders who had no education in the art of reckoning and must do with carving dashes in a piece of wood for numbers. He wondered how people had been able to survive there through the centuries. The book was destined for use in the Latin Schools as well as for other children of the country that had desire for exercising arithmetic. It was, however, never used at the Latin Schools, which were in bad shape during the following decades due to the Haze Famine.

In his following address in Icelandic to the reader (Olavius, 1780, pp. ix–xxviii), the author revealed that he had modelled his book after the German textbook *Demonstrative Rechenkunst* by Christlieb von Clausberg (1732), republished in 1748 and 1762. The author said that he had also taken examples from Danish textbooks, one of them Arithmetica Tyronica by Chr. Cramer (1735, 1755, 1762, 1766, ...), but to a lesser degree. As none of the well knowledgeable Icelanders had written such a book, Olavius dared to present this one. Without such knowledge, one could hardly avoid loss of money or other capital in trade, e.g. with foreign merchants. He had strived to use Icelandic words for the general public who might not understand or articulate the foreign terms.

Olavius said that he had struggled to provide clear explanations as he did not know of anyone out there [in Iceland] who taught the general public anything in the reckoning art, which he said was in line with most other situations in that 'un-country',

apart from small teaching activities in the schools. Indeed, each common person who wished to learn something had to be his/her own teacher. Therefore, he must produce a great number of solved examples. Much more was to be learned from one solved problem than from ten unsolved ones.

In continuation, Olavius explained what he meant by what he called 'talnabrögð', number tricks, a term that he used to translate the German term 'Vortheilen', in English advances or benefits. These were to be used either for a quicker or easier work or for less heavy thinking. In more detail:

1. addition and subtraction could be applied in place of multiplication and division
2. multiplication applied instead of division
3. smaller numbers used than mentioned, and
4. fractions be avoided, even if working with them might sometimes be quicker and easier (Olavius, 1780, p. xx).

The author expressed his feeling that these tricks or advances could be used on one third to one half of the problems. But certainly, time had to be devoted to studying them. It was also advantageous to know more than one method to solve a problem in order to confirm its correctness.

Before giving examples of his tricks, Olavius explained how multiplication by chosen numbers could be simplified. It seems that multiplying by 8 was assumed to be done by doubling three times mentally, and even multiplying by 32 by doubling five times (p. 75). In multiplying by 11, digits for unities and tens should be added, then digits for tens and hundreds and so on. For instance, in multiplying 26748219 for 11 he wrote 9 on the farthest place on the right, then added 1+9 mentally, writing 0, 2+1, adding 1, writing 4, 8+2, etc. to reach 294230409 (p. 66). Not quite mental arithmetic, but saving writing space. He discussed what to do when multiplying by 0, both at the right end of a number, as by 0 in 80, and also internally, as the 0 in 207 (p. 67). He remarked that multiplying by 25 could be done by multiplying by 100 and dividing by 4 (halving twice); 375 is 1/8 of 3000; 1 2/3 is 1/6 of 10; 12 1/2 is 1/8 of 100, etc.

We shall look at some examples of each case of Olavius's number tricks:

1. Dissolving multiplication partly into addition or subtraction:

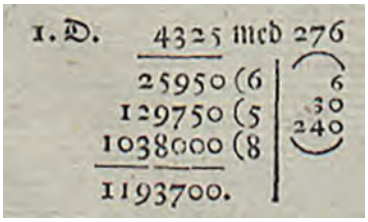

Fig. 1. Multiplication by dissolving 276 into 6 + 6·5 + 6·5·8 (Olavius, 1780, p. 71).

Olavius first multiplied by 6, then the product by 5, and subsequently multiplied that product by 8. Then he added up the products by 6, 30 and 240.

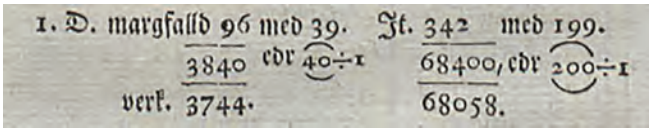

Fig. 2. Multiplying by powers of 2, then multiplying by powers of 10, followed by subtracting once (Olavius, 1780, p. 87).

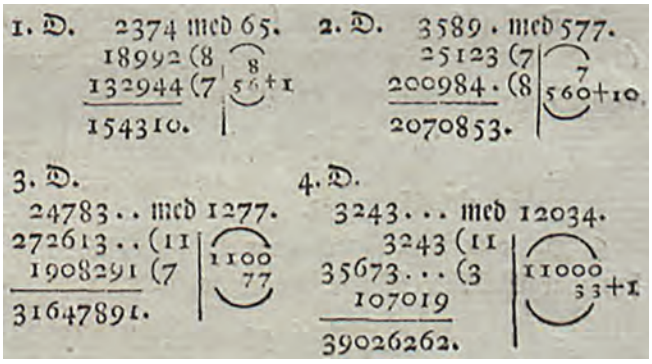

Fig. 3. Multiplying by 8, 7 and 11 (Olavius, 1780, p. 73).

We see from the examples taken in Figures 2 and 3 that the author used his easy methods to multiply by 10 and 11. He also liked to multiply by 8, while he had to multiply by 7 in three examples in Figure 3.

2. Multiplication applied in place of division, and
3. smaller numbers used than mentioned

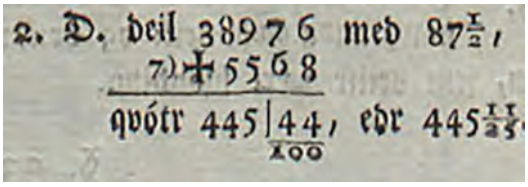

Fig. 4. Dividing by 87 ½ which is equal to 7/8 of 100 (Olavius, 1780, p. 286).

Dividing by 87 ½ can be done by noticing that 87 ½ is 7/8 of 100 so the division equals dividing by 7 and multiplying by 8, which is fortunate, and then dividing by 100. Easy!

According to his foreword, Olavius knew of criticism of that kind of tricks. The critics say, he said, that they cannot be used in all cases, they fit seldom, and the method is difficult to handle, and more incomprehensible and more confusing than the ordinary method. Olavius protested and explained that a suitable method had to be chosen for each case and there was no reason to deem methods impracticable even if they did not fit everywhere. Time and knowledge were needed. Those who read and cogitated would be rewarded with discovering much usefulness and many advantages (Olavius, 1780, pp. xx–xxii)

Who were the critics that Olavius referred to? It is unlikely that they were residents of Iceland. The book had not yet been published when the foreword was written, and Olavius did not stay in Iceland. The critics must have had its model in mind, that is Clausberg's book, or a similar one. Few Icelanders read German, and no sources are available about Clausberg's book in Iceland. The critics are more likely Olavius's pals in the Society of the Learned Arts in Copenhagen.

The critique

In the 18th century there was no platform in Iceland for educational discussion. The first Icelandic journal ever, the *Journal of the Icelandic Society of the Learned Arts* was established in 1780, the same year as the *Clear Guide* was published. The only sources available about such discussions are private letters and memoirs such as Helgason's (1907–1915).

Private letters by the reverend G. Pálsson, an ex-headmaster at one of the Latin schools, have been edited and published (Sveinsson, 1984). Pálsson was an authority in education in Iceland and a renowned teacher. He published a noteworthy primer (Pálsson, 1782), printed by the controversial printing press in Hrappsey. There he devoted three pages in an appendix to numbers and the multiplication table. For various reasons, both external and personal, he had financial difficulties that he was unable to handle even though he dedicated his primer to the most powerful persons in his neighbourhood in hope for support (Sveinsson, 1982).

Pálsson said in a letter, dated October 5 1780 – February 20 1781, to his successor, the headmaster of one of the two Latin schools, that he had read the foreword of the *Clear Guide* and was not impressed but had ordered the book. He seized the phrase 'non-country' and others similar, and deemed that as an opinion of mean people with foreign taste. He mentioned the term 'number art' [talnalist] in the title which he found project affectation, while it indeed refers to the German title of

von Clausberg's (1732) *Rechenkunst*. Pálsson would have preferred the term 'number wisdom' [tölvísi]. In a letter dated August 11 1781, Pálsson recounts that he had not yet seen the book, and in a letter dated January 29–March 25 1782 he is still patiently waiting for the book. Actually, he would have preferred that an arithmetic book by a colleague minister, available in a manuscript, had been printed, not this one. In his letter dated April 24–May 31 1782, Pálsson recounts that he saw the book externally but his guest, who had read it, had not liked its style. Pálsson expressed the opinion that it was neither prudence not correct didactics that the author expressed himself so affective as Pálsson had seen in the foreword (Sveinsson, 1984, p. 372–373, 383, 387–388, 394–395).

Why should Pálsson, well knowledgeable in arithmetic, have had such negative opinion of the first substantial printed textbook in arithmetic of 374 pages, written in Icelandic, distributed for free; a book that he had not read when he wrote his letters? According to the letters, Pálsson was repelled by Olavius's descriptions of the misery in Iceland. Financial complications, due to Olavius's import of the printing press, where Pálsson was involved, may also have contributed to his negative feelings.

No other sources have been found available as yet about others' opinions of the content of the book, while the reverend Pálsson's unfavourable review has echoed in historical texts until today (Guttormsson, 1990). Another matter is that the book may have been ambitious for the Icelandic community, where, quoting the author: "each common person who wished to learn something had to be his/her own teacher".

Already in 1785, another arithmetic textbook in Icelandic was published by another proponent of the Society of the Learned Arts, the district governor, later governor of Iceland, Ólafur Stefánsson (1785). That book was immediately legalized for the Latin schools (*Lovsamling for Island*, 1855, p. 244) and was presented to the students for free. Olavius's *Clear Guide* was never used in the schools. However, due to the bad shape of the country and the schools and that the teachers did not teach mathematics (Helgason, 1907–1915), Stefánsson's book was also of little use in the schools. But both books were listed in private libraries half a century later so that they may have been used for self-instruction in the homes (Jensdóttir, 1969).

Arithmetic by S. Á. Gíslason

The author

Sigurbjörn Á. Gíslason (1876–1969) studied theology at the Icelandic School of Theology for a tertiary degree, to become eligible for teaching at secondary schools, as he could not afford to go abroad to study mathematics at a university. Gíslason

became a successful mathematics teacher in Reykjavík during 1897–1945, and wrote a six-volume series of an arithmetic textbook, *Reikningsbók* [Arithmetic], for students from the age of seven into secondary colleges. Gíslason managed to take a year-long study trip to Denmark in the early 1900s. He may have become acquainted there with Pestalozzi's pedagogical theories which were well known and favoured in Denmark at that time (Hansen, 2009).

Gíslason lived in Reykjavík all his adult years. In early 1900s, Reykjavík was a fast growing town with a large primary school, the sole Icelandic grammar school (the previous Latin-School), a commercial college, a teacher college, a secondary school for girls, and an engineering college. Gíslason taught at all these schools for some periods of time except at the grammar school; the longest period at the engineering school. His books covered the then current syllabi of these schools except the grammar school where Danish textbooks were used.

The content

In his foreword to the first volume, Gíslason emphasized that mental arithmetic was the main issue in all general and simple exercises. The students were to gradually get used to writing down the exercises with correct symbols and explain orally why they do the particular operation as they do. Rote learning was worth nothing but regrettably was commonly practiced. Gíslason followed up his vision by starting most sections through volume six by exercises in mental arithmetic.

The content of the first four volumes matched the curriculum prescribed by law no. 59 of 1907. Every child, who reached the age of 14 years, should have learned:

> §4. in arithmetic the four operations in whole numbers and fractions, and be able to use these in order to solve simple problems coming up in daily life, e.g. to calculate the area and volume of simple bodies; he/she should also be skilled in mental arithmetic with small numbers.

Volumes 5 and 6 were more advanced. Vol. 5 contained equations, proportions, percentages, and interests. The Rule of Three was treated in equations as proportions. Vol. 6 continued with powers, exponents, square root and cubic root, stocks, compound interests, and logarithms. Most sections in Gíslason's Arithmetic began by exercises in mental arithmetic. No special technique was presented, only simple exercises, chosen to throw light on the concepts being worked on.

The exercises in themselves were quite trivial. As a sample of exercises for 10–12 year-old children, questions were posed about how much a person earned a day if the salary for a 6-day work week was given. The amounts to be divided were 18, 9 and 42 crowns. More interesting for the modern reader is to see that a man-worker

had double the salary of a woman and a member of parliament had more than twice as much as an ordinary worker, both men.

Wondering about the ultimate goal of these exercises in mental arithmetic, a guess is that it was developing number sense, while written arithmetic was certainly also necessary in a country with growing trade, import and export. Gíslason taught at colleges where accounting and learning how to produce proper bills were important. Another aspect is that textbooks, paper and pencils were still expensive for families with small income so some training in mental arithmetic was useful in daily life.

In 1929, three arithmetic textbook series were legalized[1] for use at primary school level (Elíasson, 1944). One of them was Gíslason's Arithmetic, volumes 1–4, and another was a textbook series by Elías Bjarnason (1927–1929). The final topic for primary level was addition of fractions with different denominators. We shall compare Gíslason's and Bjarnason's presentations of finding a common denominator.

Bjarnason presented the following procedure, shown in fig. 5, for finding the Least Common Multiple, LCM, for the denominators of the fractions ½, 8/9, 5/16 and 7/24.

Gerum ráð fyrir, að finna eigi samnefnara brotanna ½, ⅞, 5/16 og 7/24. Allir nefnararnir eru skrifaðir í röð áfram og strik dregið undir röðina, þannig:

2) 2 — 9 — 16 — 24
2) 9 — 8 — 12
2) 9 — 4 — 6
3) 9 — 2 — 3
3 — 2 — 1

Fig. 5. Finding the Least Common Multiple of four fractions (Bjarnason, 1957, p. 32).

Bjarnason explained that the denominators were to be lined up and then be divided repeatedly by the lowest factor found to divide into any two or more. The other factors were to be pulled down. When no more could be done, the factors were multiplied to become $24 \cdot 32 = 144$. Prime numbers and prime factors were not mentioned.

Gíslason presented two methods, one similar to Bjarnason's method, and another one which included prime factoring and was easier to argue for. The example

1 No information is available about legalizing textbooks and what it meant. Legalizing is presently only known to have been done twice: Stefánsson's arithmetic textbook was legalized in 1786, and a list of textbooks in all subjects in primary school was announced legalized in 1929.

he took was to find the Least Common Multiple, LCM, of 216 and 400, and simultaneously, Greatest Common Factor, GCF, of the two numbers, see fig. 6.

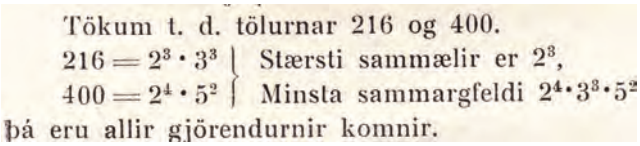
Tökum t. d. tölurnar 216 og 400.
$216 = 2^3 \cdot 3^3$ | Stærsti sammælir er 2^3,
$400 = 2^4 \cdot 5^2$ | Minsta sammargfeldi $2^4 \cdot 3^3 \cdot 5^2$
þá eru allir gjörendurnir komnir.

Fig. 6. The GCF and the LCM of 216 and 400 (Gíslason, 1913, p. 11).

Gíslason then remarked that all the prime factors were included in the multiple of the LCM, $24 \cdot 33 \cdot 52$, and the GCF, 23, in total $27 \cdot 33 \cdot 52$; and in general that

$$\text{LCM}(a, b) \cdot \text{GCF}(a, b) = a \cdot b.$$

Thus Gíslason's book offered flexibility in important arithmetic procedures.

Gíslason's *Arithmetic* became quite widespread. In foreword for the second edition of volume 1 in 1913, the author reported that it had been reprinted in 4000 copies. That is a large number in a country where the total population was about 87,000. This may have applied to the sum of copies of the five volumes then published.

Gíslason's *Arithmetic* survived into the 1930s as a legalized textbook series for primary level. In 1938, Bjarnason's series (1927–1929), which then was published in a revised edition, and was more in favour of training procedures, was chosen for free distribution in primary schools (Sigurgeirsson, 1987). This hindered publication and distribution of other textbooks.

The critique

In the early 1900s, teaching was becoming a growing profession. A private teacher training college operated during 1897–1908, and a state-run teacher training college was established in 1908. Teachers had their own journal, *Skólablaðið*, as a platform to express their views. During 1909–1916, its editor was Jón Thorarinsson (1854–1926). He had been the headmaster of the private teacher training college and became the first state secretary of education in 1908.

Gíslason's textbook series was mentioned several times in that journal. In 1912, a teacher praised that the exercises were expressed in words, not only by numbers. Another advantage was a great number of exercises. A special advantage was the great emphasis that the author laid on mental arithmetic in volume one. In the teacher's point of view, several exercises in volumes two and three were too difficult for the assumed age of pupils. Volume four was the best one. Still another advantage was that the books contained various pieces of knowledge: the map of Iceland, America in the eyes of Christopher Columbus, poetry by a beloved poet,

etc. However, the author used the old measuring systems too much in comparison to the new metric system (Jónsson, 1912).

In 1916, another review of Gíslason's series appeared, saying that it was good in many respects, but too large and far too expensive for small schools and itinerant schools. Another arithmetic textbook (Brynjólfsson and Arason, 1914) collected the whole material in one volume for a much lower price (Hjartarson, 1916)

In 1916, the Secretary of Education, Jón Thorarinsson, refused when asked to recommend Gíslason's series for a grant from the National Budget. In his reply, he mentioned that others had given volumes 1–4 good reviews. He remarked, however, that in volume 5 (which actually was not intended for compulsory school level but for the colleges) he found the mental arithmetic exercises far too difficult and randomly arranged. He remarked in particular an exercise in mental arithmetic where the students were asked to compare 4 ‰ of 9000 crowns and 4 ½ ‰ of 8000 crowns (Gíslason, 1912, p. 53). Furthermore, the series was too expensive. He concluded by saying that the best arithmetic textbook among the many that were being published at that time would win in their competition and that it was not right to use national funds to support one over the other (Skjalasafn Fræðslumálaskrifstofunnar 1976-C/2).

Thorarinsson was undoubtedly the person most knowledgeable about educational matters in the early 1900s. He had devoted his early life to studying education in Denmark, Germany and England and had been headmaster in the first teacher training college in Iceland. Understandably, he had concerns about the price of school books and his remark on the win of the best book was well grounded. However, he failed to see the simple solution of the aforementioned problem: to find 1 ‰ of by thinking something like "of one thousand 4 crowns, of nine thousand $9 \cdot 4 = 36$ crowns"; and similarly $8 \cdot 4\ ½$, also 36, as had been pointed out in the introduction to the chapter on percentages. It is likely that Thorarinsson learned Briem's method, to create the sequence of front term, middle term and rear term: 1000 – 4 ½ – 8000, multiply middle term and rear term, $4\ ½ \cdot 8000$ to gain 36000, and then divide by the front term, 1000, seemingly a complicated process, instead of using the primitive method of finding 1 ‰ of 8000 to begin with. Then the continuation should be easy.

Another aspect is that students studying the textbooks had seen similar tasks before and were presumably better prepared than the Secretary of Education who may just have taken a look at the book and tried to recall a method he learned long before.

One should not underestimate that money was important in the poor society that Iceland was in the beginning of the twentieth century when it had not yet gained sovereignity and was still far away from independence. Anyhow, no harm was

done, Gíslason sold his books quite well and they survived until the late 1930s when Bjarnason's textbook was chosen for free distribution and no competition existed any more.

Discussion

The two arithmetic textbooks of 1780 and 1911–1914 may be considered progressive for their time. Both emphasised unconventional and flexible methods, involving mental arithmetic. Olavius (1780) explained a great number of strategies built on thorough knowledge of arithmetic facts. For a reader who is well trained in arithmetic it arouses admiration and inspires flexibility. It may have repelled less experienced readers, but Olavius made it perfectly clear that it was not intended for beginners.

One would for example have expected the Rev. Pálsson to admire the many ideas presented in the book, but his standpoints about the book were expressed before he had had opportunity to read more than its foreword. One must conclude that his opinions were based on negative attitudes unrelated to the book's content. One reason could be that Pálsson felt that Olavius was in his foreword patronizing Icelanders in their misery. Another reason could be malice due to financial difficulties concerning the printing press that Olavius imported in Pálsson's county. Olavius was young and enthusiastic to bring the first printing press to print secular books, while Pálsson was an elderly resident in the area, trapped in his own financial difficulties. We do not know the exact reasons for their disagreements but Olavius left and some mistrust remained.

It is somewhat surprising that Thorarinsson did not appreciate the mental arithmetic exercise in Gíslason's (1911–1914) series that he took as an example nor the collection of exercises. Possibly, his thinking was fixed in methods laid down centuries ago on the Rule of Three which prescribed fixing the given numbers in a certain order and continue according to a fixed procedure.

Threlfall (2002) proposed that mental calculation be thought of as interaction between noticing and knowledge, and concluded that flexibility cannot be taught as 'process skill' but will rise consequentially through the emphasis on considering possibilities for numbers.

The prominent persons, Pálsson and Thorarinsson, who reviewed the arithmetic textbooks briefly with other aims than training their own personal skills, did not notice the advantages of the strategies presented in the books. This is in accordance with the opinion of Threlfall who did not recommend focusing on holistic 'strategies' in mental calculation. It is not very likely either that Icelanders were receptive to strategies as presented by Olavius in the 1780s, and hardly under the Haze Famine.

In the second case, Thorarinsson's mind did not operate in a flexible way which he cannot be blamed for. Problems that descend upon people, more or less unprepared, can create feelings of insecurity. It was, however, unfortunate, as he was a powerful person and an authority in educational matters, that he did not take into account the cumulative nature of mathematics. A whole series of books can hardly be judged by one exercise in its fifth volume, not even intended for the compulsory school level. He must have been familiar with Jónsson's more balanced review which was published in a journal of which he was editor.

Concluding remarks

On what grounds did Pálsson and Thorarinsson express their opinions? It is regrettable that they did not focus on the advances of the arithmetic textbooks that they expressed their views about. Instead of bringing out their mathematical merits, they discussed extra-mathematical matters, such as language use and prices, certainly valid viewpoints but very onesided. But did their opinions exert any influence on the distribution of the books?

Pálsson wrote personal letters to his successor headmaster in one of the two Latin schools. In his time, the only platform available for discussion was private letters. Pálsson's view may have had some influence in that school and on other colleagues in the vicinity, but ultimately the book by the district governor was legalized as the prescribed arithmetic textbook for the two Latin schools and there was no place for another one. Thorarinsson's letter is a reply to a query from the author about a grant for his publications. It is not likely that many others knew about his reaction. Thus, the reactions to the two books had hardly any widespread influence.

A documented report exists that Gíslason's book was considered far too expensive by the opinions of Thorarinsson and Hjartarson (1916). Iceland was taking its first steps into modernity, it was not yet independent, and frugality prevailed. It is not known if Gíslason was disappointed that his application for a grant was refused. At least his series continued to sell and was chosen for legalization in 1929. It was not chosen for free distribution in 1937 which was not unnatural considering that it had been written a quarter of a century earlier. Two other series were chosen, written about fifteen years later than that by Gíslason. Similar aspects concerning prices may have prevailed in the 1930s during the Great Depression as in the pre-Great-War times.

One can, however, state that both books served their purpose and were read as widely as may be expected in their milieu.

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