On French heritage of Cartesian geometry in Elements from Arnauld, Lamy and Lacroix

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Abstract

When Descartes wrote La géométrie in 1637, his purpose was not to write "Elements" with theorems and proofs, but to give a method to solve "all the problems of geometry". However, in his Nouveaux Éléments de Géométrie in 1667, Antoine Arnauld included two important Cartesian conceptions. The first one is the systematic introduction of arithmetical operations for geometric magnitudes and the second one is what he called "natural order", that means Cartesian order which goes from the simplest geometric objects (straight lines) to others. This last conception led Arnauld to numerous novelties, mainly, a chapter on "perpendicular and oblique lines", and new proofs for Thales and Pythagoras theorems. In 1685, Bernard Lamy followed Arnauld's textbook in his Éléments de géométrie, in which he also introduced Cartesian method to solve problems. Our first aim is to analyze incorporations of Cartesian conceptions and Cartesian method into Arnauld and Lamy's Éléments. Our second aim is to analyze their impact for the heritage of Cartesian geometry into mathematical teaching, especially the "natural order" coming from Arnauld and the "application of algebra to geometry" coming from Lamy. In this framework, we show that the geometric teaching of Sylvestre-François Lacroix played an important role in the 19th century and beyond.

Keywords: René Descartes, Antoine Arnauld, Sylvestre-François Lacroix, Cartesian order, arithmetization of geometry

Introduction: Cartesian order and arithmetization of geometry

Towards the end of the 1620s, René Descartes wrote *Règles pour la direction de l'esprit* [Rules for the Direction of the Mind]. This text had never been achieved and published in his lifetime, but it is interesting to know that it had been read by Antoine Arnauld. In his *Rules*, Descartes criticized Aristotle's science based on syllogisms, because they can conclude with certainty but they banish obviousness (Rule X), and he gave his proper conception of science. Indeed, he wrote in Rule XII: "We can never understand anything beyond these simple natures and a certain mixture or composition of them with one another" (Descartes, 1998, p. 155). Hence,

> all human knowledge consists in this one thing, to wit that we distinctly see how these simple natures together contribute to the composition of the other things (Descartes, 1998, p. 161).

In that way, he proposed to substitute an order of simplicity of things instead of a logical order of propositions. Descartes continued to call deduction the manner by which a composite nature can be obtained from simple ones. Thus, Aristotelian and

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Cartesian deductions are different because, in the first one, propositions are deduced from others by logical rules and, in the second one, composed things are deduced from simple ones by simple operations.

Simple things and simple operations of geometry are introduced as soon as the first sentence of *La géométrie* (1637), where Descartes wrote:

> Any problem in geometry can easily be reduced to such terms that a knowledge of the length of certain straight lines is sufficient for its construction. Just as arithmetic consists of only four or five operations, namely addition, subtraction, multiplication and the extraction of roots [...] (Descartes, 1954, p. 2).

So, simple things are straight lines and simple operations are arithmetic operations. This 'arithmetization' of geometry, leans on the introduction of one line called "unit" by Descartes, by analogy with arithmetic. Indeed, this unit permits us to obtain a product of two lines *BD* and *BC*, not as a rectangle, like in Greek geometry, but as a simple line. If *AB* is the unit, then *BE* is the product of *BD* and *BC* (figure 1 left). It also permits us to divide two segments and to obtain a segment. To consider a square root of a line, has no meaning in Greek geometry, but in Cartesian geometry, if *FG* is the unit then *GI* is the square root of *GH* (figure 1 right).

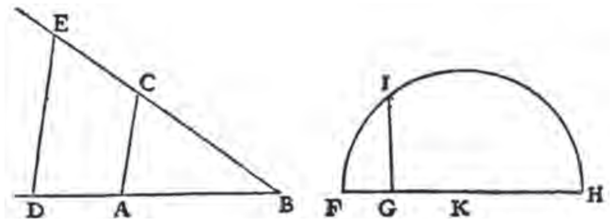


Fig. 1. Product of two segments and square root of a segment (Descartes, 1954, p. 4)

Descartes pointed out that often, it is not necessary to draw lines and it is sufficient to designate them by single letters, to which symbols of arithmetic will be applied. Moreover, thanks to the unit, it is possible to consider for instance a^3 or b^2 as simple lines, and, for instance, to consider the cube root of $a^2b^2 - b$ without taking into account the geometric meaning of this formula.

Descartes' purpose was to provide a systematic method to solve problems of geometry by deducing unknown lines from known lines. This method consists of translating problems by equations on lines and to solve these ones. In the First Book of *La géométrie*, Descartes used his method to solve, not elementary problems, but a difficult problem left to us by Pappus. In the Second Book, in accordance with his general conception and thanks to the unit line, he considered curves as composed by simple lines by means of arithmetic operations, when for a given line *AG* and for each point *C* of the curve, there exists a single equation linking *CM* and *MA*. These lines are called "geometric" and the others "mechanical". So, he did not introduce a "Cartesian coordinate system". He used his method to find normal lines *CP* to a "geometric curve" (figure 2).

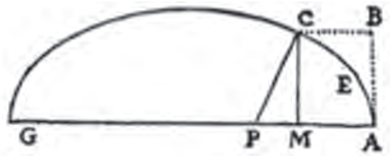


Fig. 2. Normal line to a "geometric curve" in *La géométrie* (Descartes, 1954, p. 97)

Cartesian order in Arnauld's *Nouveaux Éléments* (1667)

In 1662, the Jansenists Antoine Arnauld and Pierre Nicole wrote *La logique ou l'art de penser* [Logic or the art of thinking], in which they gave the list of the defects of Geometers[1]. The first one is "Paying more attention to certainty than to obviousness, and to the conviction of the mind than to its enlightenment" (Arnauld & Nicole, 1850, p. 331) while the fifth defect is "Paying no attention to the true order of nature" (Arnauld & Nicole, 1850, p. 335). They added about this defect:

> This is the greatest defect of the geometers. They have fancied that there is scarcely any order for them to observe, except that the first propositions may be employed to demonstrate the succeeding ones. And thus, disregarding the true rule of method, which is, always to begin with things the most simple and general, in order to pass from them to those which are more complex and particular, they confuse everything, and treat pell-mell of lines and surfaces, and triangles and squares, proving by figures the properties of simple lines, and introducing a mass of other distortions which disfigure that beautiful science (p. 335).

That means that they considered it as a defect to not follow the Cartesian order in geometry: it is the order of nature. They wrote that Euclid' s *Elements* are quite full of this defect:

> He measures the dimension of surfaces with that of lines. […] It would be necessary to transcribe the whole of Euclid, in order to give all the examples which might be found of this confusion. (Arnauld & Nicole, 1965, p. 335).

They opposed the 'method of doctrine', found in Euclid, to the 'method of invention', that is Descartes' one.

1 Translations of Arnauld, Lamy, Lacroix' s texts by Évelyne Barbin.

'Natural order' accordingly to Arnauld

In 1667, Arnauld edited *Nouveaux Éléments de géométrie contenant un ordre tout nouveau* [New Elements of geometry containing a very new order], intended for the schools of Port-Royal. The "very new order" of the title is the Cartesian order, called "natural order" by him. He wrote in his preface:

> It was a very advantageous thing to get accustomed to reduce our thoughts to a natural order, this order being as a light that clears up ones by the others [...] Euclid's Elements are so confused and muddled, that far from bringing to the mind an idea and a taste for a true order, on the contrary, they only make the mind used to disorder and confusion (Arnauld, 1667, np).

After four Books setting out the arithmetization of magnitudes, there are thirteen Books in natural order: straight and circular lines, perpendicular and oblique lines (Books V); parallel lines (Book VI); lines ended by a Circumference (Book VII); angles (Books VIII and IX); proportional lines and reciprocal lines (Books X and XII); plane figures according their angles and sides (Books XII and XIII); plane figures according their surfaces (Books XIV and XV).

To follow a natural order requires new proofs, for example for the theorems of Pythagoras and Thales; two propositions on simple lines but proven in Euclid by using triangles, more composite figures than lines. Arnauld wrote that natural order

> gives rise to find more fertile principles, and clearer proofs. And indeed, in these *New Elements*, there are nearly very new proofs, which arise from principles by themselves, and which contain a great number of new proposals (Arnauld, 1667, np).

The first important principles concern perpendicular and oblique lines, that are defined and studied without using angles.

"Perpendicular and oblique lines": new principles

Euclid defined perpendicular lines by using angles:

> when a straight line set up on a straight line makes the adjacent angles equal to one another, […] the straight line standing on the other is called a perpendicular to that on which it stands (Euclid, 1956, p. 153).

Arnauld explained that to form a more distinct definition of two perpendicular lines we can conceive that when two points of the cut line are equally distant from the cutting one, every point of the cutting line is equally distant from these two points of the cut line. For that, he introduced the notion of distance and $CA = CB$, $DA = DB$, $EA = EB$ (figure 3 left). Arnauld claimed that this statement is true because it is obvious:

> I say that the consideration of the nature of straight lines only makes us see the truth of this proposition, and that it is impossible to keep the natural order of things in Geometry without this consideration [...]. So, we have to reject the scruple we could have, to receive this proposition as obvious by itself; we cannot do anything else without muddling the natural order of things and using triangles to show properties on straight lines, that means without using more compound to explain more simple, which is contrary to the true method (Arnauld, 1667, pp. 87-88).

Here he followed Descartes' general rule of the *Discours de la méthode*: "things that we very clearly and distinctly conceived are all true" (Descartes, 1637, p. 33). The role of obviousness in Cartesian science permitted him to admit as axioms, sentences that are necessary for following the natural order.

Arnauld continued with an explanation of the manner to consider oblique lines for understanding them better. Three lines have to be conceived together with three distances: *kb* for oblique line, *kc* for perpendicular line, *bc* for the distance away of the perpendicular line (figure 3 middle). He pointed out that *bc* and *kc* can be considered as oblique lines also with perpendicular line *gc*. He insisted on this explanation because

> the consideration of these three lines [...] will help us to understand several things on oblique lines which cannot be explained by triangles, as it is a reversed order (Arnauld, 1667, p. 95).

"Fundamental proposition on oblique lines" states that oblique lines *Kf* and *Kg* from the same point *K* to a same line *z* are longer when they are more distant of the perpendicular line *KB*. Arnauld drew *KB* = *BC*, *Cf* and *Cg* (figure 3 right). By the more exact definition of a perpendicular line (Arnauld), *Kf* = *Cf*, *Kg* = *Cg*, and by the means of Archimedes, *KfC* is shorter than *KgC*. Therefore, *Kf* (half of *KfC*) is shorter than *KgC* (half of *KgC*).

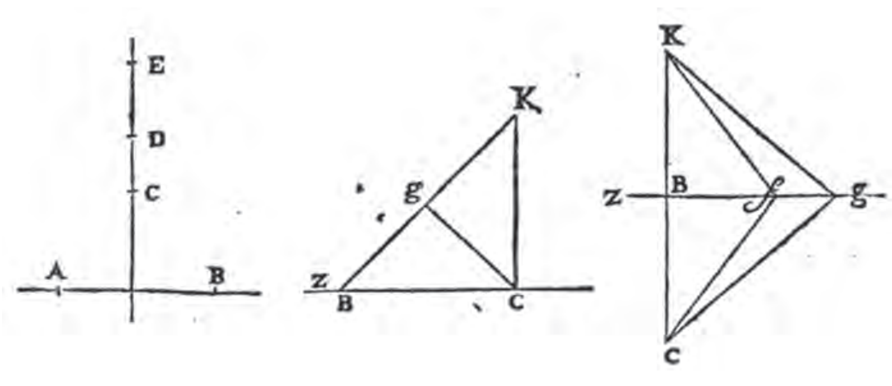


Fig. 3. Perpendicular and oblique lines in Arnauld (Arnauld, 1667, pp. 87, 95, 96)

Arnauld 's new proof of Thales' theorem

We take this proof as an example because of its long heritage in teaching. Contrary to Euclid (Prop. 2, Book VI), Arnauld did not use triangles and gave "the very natural proof, that nobody ever gave I think" (Arnauld, 1667, p. 191). He began Book X on proportional lines with the "Fundamental proposition of proportional lines": if two lines C and c are "equally inclined" in two "parallel spaces" A and E, then we have $P : p = C : c =$ B : b (figure 4 left). To prove it, he divided p in 10, 20, 500, 6000, 10000, &c. equal lines x (Arnauld, 1667, p. 191). He then considered parallels through the points of division; then c is divided in equal spaces because they have the same perpendicular. Then he wrote that the same can be done for P and C and he concluded, with taking in account the incommensurable case. First corollary of the "fundamental proposition" is Thales' theorem in a more general situation than in Euclid (figure 4 right): "several diversely inclined lines in a same parallel space are proportionally cut by parallels to this space" (Arnauld, 1667, p. 193).

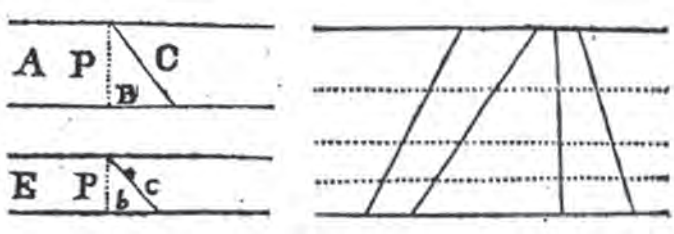


Fig. 4. Thales' theorem in Arnauld's Nouveaux Éléments (Arnauld, 1667, p. 193)

Arithmetization of geometry and method in Lamy's *Éléments*

Bernard Lamy was an Oratorian and a teacher of mathematics (Barbin, 1991). In 1676, he was banished from the University of Angers because of his Cartesian convictions. He published several textbooks and modified them all along their successive editions, like *Traité de la grandeur en général ou les éléments de mathématiques* [Treatise on magnitude in general or Elements of mathematics] with 23 editions from 1680 to 1765, *Les éléments de géométrie ou de la mesure de l'étendue* [Elements of geometry or measurement of extension] with 14 editions from 1685 to 1758, *Entretiens sur les sciences* [Conversations on sciences] with 12 editions from 1683 until 1768. In this last book, he showed himself an unfailing disciple of Descartes, Arnauld and Nicole. For instance, he wrote:

> For perceiving well, we have to wait the clarity before to consent. We have not to do it before being forced by the obviousness of the truth (Lamy, 1966, p. 86).

In his *Éléments de geométrie*, Lamy followed the natural order of his predecessor Arnauld, because as he wrote in his Preface:

> [Geometry] has to be treated with method, which is not done by Euclid. He only thought to range his propositions, in such a way that they can serve for proving ones from the others; in this he succeeded. The truth is contained in his Elements; but besides there is so much confusion [...] as Monsieur Nicole complains in the Preface of *Elements of geometry* by Monsieur Arnauld, which were printed in 1667 for the first time. It is in these *Elements* of Monsieur Arnauld that we find this natural order, which is not in those of Euclid (Lamy, 1734, p. vi).

Book I concerns straight lines (perpendicular, oblique, parallel lines) then circular lines, while Book II concerns plane areas. Lamy added a Book V on Bodies to Arnauld's *Élements*.

Lamy's arithmetization of geometry for proving propositions

Book III gives "properties which suited every magnitude, applied to lines, planes, solids, and proven". It begins with the four operations of Arithmetic on lines, planes and bodies. Like Descartes, Lamy introduced the unit line to operate arithmetically on all kind of magnitudes and to obtain lines as results, independently of their geometric names. In this manner, geometric properties on figures can be translated by arithmetic formulas on simple things that are lines. He wrote:

> Also remark that, as it is advantageous to accustom one's spirit quickly to these kinds of calculations [...] we will delete all the figures used by Euclid and his interpreters for their proofs ordinarily [...], and it is now appropriate to make calculations with the pen in the hand (Lamy, 1731, p. 143).

Cartesian arithmetization of geometry is used by Lamy for giving new interpretations and new proofs of Euclid's Book II. He wrote in his Preface that

> it is important to get used to see without images, and to be convinced that there are truths which are conceived otherwise than with bodies (Lamy, 1731, p. iii).

We take as an example Proposition VI of Euclid's Book II, that states that

> if a straight line is bisected and a straight line is added to it in a straight line, then the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half equals the square on the straight line made up of the half and the added straight line" (Euclid, 1956, p. 385).

Euclid proved that (the area of) rectangle *ADMK* juxtaposed to square *LHGE* is equal to (the area of) square *CDFE* (figure 5 left). While Lamy considered a simple line and explained that we have to prove an arithmetic formula on lines, that is $AD \times BD + BC^2 = CD^2$ (figure 5 right). He named $AC = CB = b$, then the proposition results of the two equalities: $AD \times BD + BC^2 = 2\ bd + dd + bb$ and $CD^2 = (b + d)^2 = bb + 2\ bd + dd$

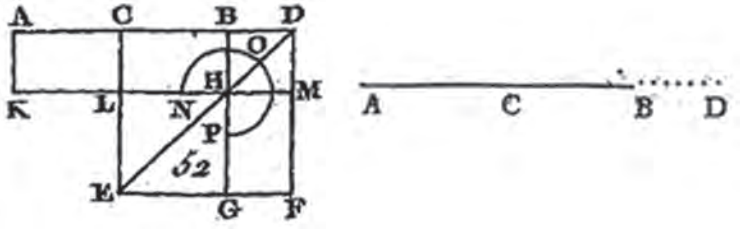


Fig. 5. Lamy's proof compared to Euclid's proof (Lamy, 1731, p. 146)

Cartesian method in Lamy's Book VI

In the last Book of his *Élements*, Lamy wrote that there exists another method than the "method of doctrine", that is the "method of invention". Then he exposed Descartes' method to solve problems (without giving his name) and he applied it to solve many elementary problems.

In Problem I, he opposed the two methods by giving two manners to find a parallel DE to a basis BC of an isosceles triangle in such a way that $DB = DE$ (figure 6). For the first manner, he applied geometric propositions to prove that BE has to be the bisector of angle ABC. While, for the second manner, he named known lines $AB = a$, $BC = d$, and unknown line $AE = x$, he translated the problem by equations and obtained $a : d = x : a - x$ and $aa = cx$ as solution. He added: "this second analytic manner is general, and not particular to this problem" (Lamy, 1731, pp. 409-410). It is clear that the same method could be applied to find DE when knowing any kind of relation between DB and DE.

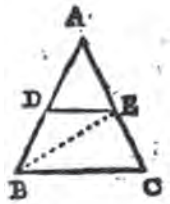


Fig. 6. Cartesian method in Lamy's Problem 1 (Lamy, 1731, p. 409)

Lamy's textbooks had been known in the 18th century by their numerous editions over a long period and by commentaries of the French philosopher Jean-Jacques Rousseau. Lamy's *Entretiens sur les sciences* and *Éléments de géométrie* were in his library and they are discussed in Rousseau' books, like in his *Confessions* (1770) (Barbin, 2003). In this last book, Rousseau opposed Euclid to Lamy:

> [Books] which combined devotion and science were most suitable for me, particularly those of the Oratory and Port-Royal, which I began to read, or rather, to devour. I came across one written by Father Lamy, entitled *Discussions on Sciences* a kind of introduction to the knowledge of those books which treated of them. I read and re-read it a hundred times, and resolved to make it my guide. I did not like Euclid, whose object is rather a chain of proofs than the connection of ideas. I preferred Father Lamy's *Geometry*, which from that time became one of my favorite works, and which I am still able to read with pleasure. Next came algebra, in which I still took Father Lamy for my guide (Rousseau, 1896, p. 238).

But, when he wrote on "application of algebra to geometry", it is clear that he did not follow Lamy on his proposal to "see without image":

> I have never got so far as to understand properly the application of algebra to geometry. I did not like this method of working without knowing what I was doing; and it appeared to me that solving a geometric problem by means of equations was like playing a tune by simply turning the handle of a barrel-organ. The first time that I found by calculation, that the square of a binomial was composed of the square of each of its parts added to twice the product of those parts, in spite of the correctness of my multiplication, I would not believe it until I had drawn the figure (Rousseau, 1896, p. 245).

"Application of algebra to geometry" as a subject in textbooks

"Application of algebra to geometry" exists as a subject in textbooks from the beginning of 18th century. But it is possible that Rousseau took the expression into the *Encyclopédie* of Denis Diderot and Jean d'Alembert, where it appeared as an item. This item had been enriched in the *Encyclopédie méthodique* (1784), where d'Alembert wrote:

> Application of algebra or analysis to geometry
>
> It is in the Geometry of M. Descartes that we find the application of Algebra to Geometry, as well as excellent methods to improve Algebra itself: with that this great genius render an immortal service to Mathematics [...]. He was the first to learn how to express the nature of curves by equations, to solve problems of geometry with these curves; then to prove theorems of geometry with the help of algebra calculation (D'Alembert, 1784, p. 92)

Guisnée and Malezieu's textbooks in 18th century

Nicolas Guisnée was 'Royal teacher of mathematics'. In 1705, he published his *Application de l'algèbre à la géométrie* [Application of algebra to geometry], where he followed Descartes and Lamy by giving

> methods to prove all theorems of geometry by algebra, and to solve and to construct all geometric and mechanical problems (Guisnée, 1705, p. ii).

After an introduction of 65 pages on algebraic calculus, Section I gives general principles to apply algebra to geometry and Section II concerns problems where solutions are given by determined equations of first or second degree, for instance a problem to inscribe a square in a given triangle (Guisnée, 1705, p. 38).

Other sections are ranged according to the types of equations obtained for solving problems, so curves are essentially considered as solutions of problems. Section VIII gives method to solve indeterminate problems of straight lines, circles and conics. Like in Descartes, coordinates of a point of a curve are straight lines not defined in a Cartesian system of axis. Guisnée treated conics and their propositions by algebra, but also mechanical curves, like spiral and cycloid, where coordinates are not straight lines but curved lines (Guisnée, 1705, p. 327).

Nicolas de Malezieu edited a new edition of *Éléments de géométrie de Monsieur le Duc de Bourgogne* [Elements of geometry of Sir the Duke of Burgundy] in 1722, where he used Arnauld's order and devoted a part on "perpendiculars and oblique lines". This edition contains a part on introduction of application of algebra to geometry, where Malezieu explained the

> great advantage of algebraic calculus […] that permits to prove all theorems and to solve all problems with as much great facility as there are difficulties with the manner of the Ancients (Malezieu, 1722, p. 2).

His introduction to algebra covers around 80 pages. Then, he only gave some applications, for instance, to determine the area of a triangle from its sides, to obtain the equation of an ellipse from its Apollonius' definition, or to prove an Archimedean theorem on sphere and cylinder.

Lacroix's Elementary treatise *on application of algebra to geometry*

In 1798, Sylvestre-François Lacroix edited a *Traité élémentaire de trigonométrie et d'application de l'algèbre à la géométrie* [Elementary treatise of trigonometry and of application of algebra to geometry] for the Collège des Quatre-Nations, that had an 11th edition in 1863. After a historical part, where he referred to works of Descartes, Euler and Cramer, Lacroix explained that "there is no reason to imitate them and, on the contrary, one has to take an opposite way to the one they followed, because one has to tend towards a very different purpose" (Lacroix, 1803, p. x). For him, the question is:

> What a treatise on application of algebra to geometry has to contain, when it is intended for students who have to practice physico-mathematics, for young people who study to enter in Polytechnic school for instance? It is clear that we have to insert all that is necessary to understand the most recent and complete books that treat physico-mathematics, or lessons that are given in Polytechnic school. (Lacroix, 1803, p. xi).

Like Monge for descriptive geometry, Lacroix researched an "élémentation" for the application of algebra to geometry that put forward links between ideas (Barbin, 2015). Indeed, he wrote:

> all that does not increase the power of methods or does not shorten the chain that links results between them has no to enter in elements (Lacroix, 1803, p. xii).

He explained that he wants to show the double point of view of the application of algebra to geometry, as a means to combine theorems of geometry and as a general means to deduce properties of extent from a little number of principles. He pointed out that:

> This branch of mathematics, considered in general, contains, not only the research of properties of the extent by the means of the algebraic process, but it has also to show how we can represent all what means any algebraic expression by these properties, to reduce construction of figures to operations continually; and to come back from these last ones to the first ones (Lacroix, 1803, p. 73).

Lacroix began to solve elementary problems, for example to inscribe a square into a triangle, like Guisnée. He generalized Lamy's Problem I (figure 6) by finding *DE* equal to a given line *MN* (Lacroix, 1803, p. 91). Then he exposed "Descartes' fundamental idea" to represent curves by equations between two undetermined lines, by pointing out the role of the unit line:

> Descartes, the first one, by remarking that figures and forms determine relations of magnitudes between straight lines, reached to apply algebra to theory of lines in general, and by this discovery, mathematics changed their face. If we conceive, for instance, that from all points of any line *DE*, we lead perpendiculars *PM*, *P'M'*, *P"M"*, etc; to a straight line *AB*, given by its position, and that from *A*, we measure distances *AP*, *A'P'*, *AP"*, etc., each of these lines and their corresponding perpendiculars will be linked in such a manner that we can deduce one from the other (figure 7). […] Nothing prevents us to imagine that lines *AP*, *PM*, are related to a common line, taken as a unit, and from that, they can be represented by numbers or letters. If this relation, between *AP* and *PM*, between *AP'* and *P'M'*, etc. can be expressed by an

algebraic equation, this equation will characterize the line *DE* (Lacroix, 1803, pp. 105-106).

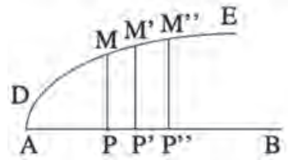


Fig. 7. Lacroix's figure to represent a curve by an equation (Lacroix, np)

Lacroix began with equations of a straight line and a circle. He constructed each conic corresponding to equations of second degree, then obtained properties of them by algebra. Later, he solved other problems, like duplication of a cube and trisection of an angle. He introduced the "Cartesian system of axis" only in an Appendix of 30 pages, devoted to curved surfaces and double curved lines. So, we can understand his teaching on application of algebra to geometry as an elementary way to combine objects and to solve problems, in the spirit of the "élémentation" of the new schools after the French Revolution (Barbin, 2015)

Application of algebra to geometry in the 19th century

From the beginning of the 19th century, textbooks on analytic geometry applied to curves and surfaces of 2nd order (those of Biot, Le Français, Boucharlat) had been edited for candidates for the École polytechnique. In those books curves and surfaces are determined by the means of a Cartesian system of axes. But other textbooks maintain Lacroix's branch application of algebra, like the one of Jean-Guillaume Garnier in the second edition of *Géométrie analytique ou application de l'algèbre à la géométrie* [Analytic geometry or application of algebra to geometry] (1813). He devoted a first Chapter on geometric constructions before defining points and lines in a Cartesian system of axis. Pierre Louis Marie Bourdon's *Application de l'algèbre à la géométrie* [Application of algebra to geometry](1825) has three sections: Section 1 gives "a first method to solve questions of geometry by calculation", while Sections 2 and 3 are devoted to "analytical geometry with two (three) dimensions". Georges Ritt's *Problèmes d'application de l'algèbre à la géométrie* [Problems of application of algebra to geometry] (1857), intended for students of collèges, contains 122 elementary problems of constructions (see Moussard, 2015). These kinds of problems appear in French textbooks for students of collèges (14 years aged) until the years 1960. We find, for instance, Lamy' problem I (figure 6), where it is asked that *DE* = *BD* + *CE* (figure 8) in a textbook of 1962 (Lebossé & Hémery, 1962b, p. 91).

133. Problème de géométrie. — *Soit un triangle* ABC *dans lequel* BC = 13 cm, AC = 15 cm, AB = 11 cm; *par un point* D *du côté* AB, *on mène la parallèle à* BC *qui coupe* AC *en* E. *Déterminer* D *de façon que :* DE = BD + CE.

Pour placer le point D il suffit de connaître la longueur BD. Posons BD = x (en centimètres), et calculons EC. D'après le théorème de Thalès (fig. 4) :

$$\frac{BD}{CE} = \frac{AB}{AC} \quad \text{ou} \quad \frac{x}{CE} = \frac{11}{15}$$

d'où

$$CE = \frac{15x}{11}$$

Calculons DE, en utilisant la similitude des triangles ADE et ABC. Nous avons :

$$\frac{DE}{BC} = \frac{AD}{AB} \quad \text{ou} \quad \frac{DE}{13} = \frac{11-x}{11}$$

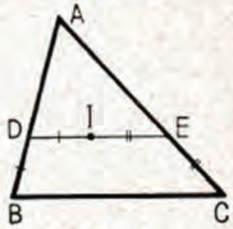


Fig. 4.

Fig. 8. Lamy's Problem I in textbook of 1962 (Lebossé & Hémery, 1962b, p. 142)

'Natural order' from Lacroix to 20th century

Lacroix wrote in his *Essais sur l'enseignement* [Essays on teaching] (1805) that Euclid introduced a kind of disorder and he continued by writing on Arnauld's textbook:

> Arnaud (of Port-Royal) […] undertook to correct this defect in his *New Elements of Geometry*, edited for the first time in Paris in 1667. This book is, I think, the first one where the order of propositions of Geometry corresponds to the one of abstractions, by firstly considering properties of lines, then those of surfaces and at last those of bodies. [...]. We could almost remark his idea to prove directly on lines, that parallels led by points taken at equal distances on sides of an angle, also cut the other side at equal distances [Proof of Thales's Theorem], proposition that those who followed Arnauld's order took as basis of the theory of proportional lines (Lacroix, 1805, pp. 289-291).

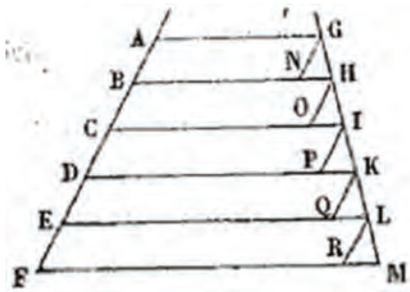
As we saw already, Arnauld's perpendicular and oblique lines is necessary to follow a 'natural order', in particular to avoid using of triangles in proofs of propositions concerning simple lines. So, it seems an obligatory way for all authors who adopted Arnauld's order. That is why we will examine the presence or not of this theory in textbooks in the 19th and 20th centuries. In the same manner, we can research the presence or not of a new proof of Thales' theorem.

Perpendicular and oblique lines in Lacroix's Éléments de Géométrie

Lacroix's *Éléments de géométrie for the École centrale des quatre-nations* (1799) contains two parts. Section I of first part treats lines only: straight and circular lines, perpendicular and oblique lines, theory of parallel lines, polygonal lines, especially inscribed in circular lines. Section II treats areas of polygons and circles. The second part concerns planes and bodies. But Lacroix introduced angles and triangles before the part on perpendicular and oblique lines, and so he reversed Arnauld's 'natural order'. We can understand Lacroix's order as a way to break with the Euclidean order, adopted by Legendre in his Éléments de géométrie of 1794, and to put forward an order of simplicity of ideas that have to be combined (Barbin, 2007). This order corresponds to the "élémentation" promoted in *École normale de l'an III* (Barbin, 2015). Indeed, Lacroix wrote in the "Preliminary discourse" to his textbook:

> The method of geometers is not the only cause of the certainty of their results, this certainty mainly comes from the nature of ideas they have to combine. […] It is less in the method than in the simplicity of the first ideas and their obviousness in which the certainty of the reasoning consists (Lacroix, 1811, pp. xiv-xvvi).

Thales's Theorem is treated as in Arnauld. Lacroix proved that if parallels *AG*, *BH*, etc. cut two straight lines *AF* and *GM* in such a way that *AB*, *BC*, etc. are equal then *GH, HI*, etc. will be equal. Then he proved that "three parallels *AG*, *DK*, *FM*, always cut two straight lines, *AF* and *GM*, in proportional parts, such that *AD* : *DF* :: *GK* :: *KM*" (Lacroix, 1811, p. 83) (figure 9). He discussed the case where the lines are incommensurable in a footnote.



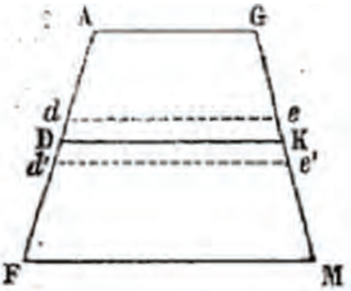


Fig. 9. Thales' theorem in Lacroix 's *Éléments de géométrie* (Lacroix, 1811, np)

Perpendiculars and oblique lines as a part of textbooks

Perpendicular and oblique lines appeared in many kinds of textbooks, where sometimes it seems as a kind of obligatory passage. For instance, in the First lesson of his *Géométrie et méchanique des arts et metiers* [Geometry and mechanics of arts and crafts] (1826), intended for the Conservatoire Royal des arts et métiers, Charles Dupin wrote that Arnauld's "fundamental proposition on oblique lines" has many applications for artists and mechanical workers (Dupin, 1826, pp. 28-29). Vincent and Bourdon, who taught in upper grades of Lycées, introduced many geometric novelties in their *Cours de géométrie élémentaire* [Elementary Course on geometry](1844, 5th ed.), like radical axes, theory of poles and polars, and elements of descriptive geometry. So, here figures are considered in space. But their Chapter I begins with a so-called "Theory of perpendicular and oblique lines" that uses the notion of distance (Vincent & Bourdon, 1844, pp. 33-37). Theorem I states that a perpendicular led from an exterior point *O* to a straight line AB is the shorter distance of the point to the line and is proven by rotation around *OC* (figure 10). Theorem II is Arnauld's "fundamental theorem", proven by a motion of folding around *OC*. The notion of distance is later used to define bisector of an angle and parallels. Triangles only appear afterwards.

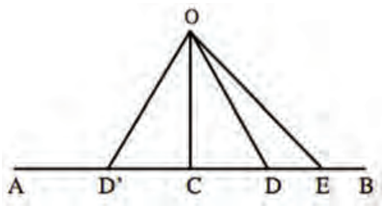


Fig. 10. Arnauld's theorems in textbook of 1844 (Vincent & Bourdon, 1844, np)

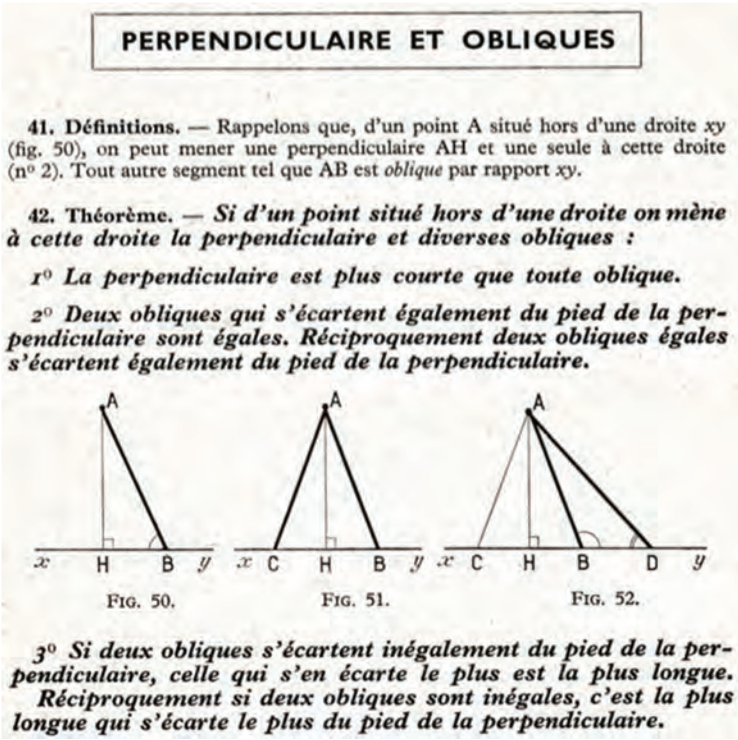
Perpendicular and oblique lines" in curricula of Collèges

In 1838, 'Properties of perpendicular and oblique lines' appeared in curricula for the 3th grade of Collèges (14 years old students), just after angles and before parallels and triangles (Rendu, 1846, p. 645). But in 1859, it appeared just after angles and triangles and before parallel lines. It is used to prove Thales's theorem like in Lacroix. We find "perpendicular and oblique lines" in *Leçons nouvelles de géométrie élémentaire* [New Lessons on elementary geometry] of Amiot (Amiot, 1865) for Lycées, in textbooks of Combette (Combette, 1895) and Hadamard (Hadamard, 1898) and, after the Reform of 1902-1905, in Vacquant and Lépinay's *Cours de géométrie élémentaire* [Course on elementary geometry] (Vacquant & Lépinay, 1909). It was a part of teaching in Collèges until the Reform of modern maths. During all this period, perpendicular and oblique lines had been treated after triangles, so the spirit of 'natural order' seemed to be lost. But, proof of Thales's theorem by Arnauld and Lacroix remained for a long time, until proofs were banished from the curricula.

More recently, perpendicular and oblique lines again appeared as a chapter of *Formes et mouvements, perspectives pour l'enseignement de la géométrie* [Forms and motions, prospects for the teaching of geometry] edited in 1999 by the team CREM in Belgium. It is the first theme before angles and triangles, of what is called "natural geometry" (CREM, 1999).

Conclusion: The Heritage of Cartesian Science in Elementary Teaching of Geometry

Cartesian science can be characterized as a new manner for conceiving deduction, which is to deduce composite things from simple things. As we have shown, two consequences are taken up in *Éléments* written by Arnauld and then Lamy in the 17th century. Firstly, Arnauld introduced a "natural order", corresponding to the order of simplicity of geometric objects, and a theory of "perpendicular and oblique lines", that is necessary to conform to this order. Secondly, Lamy developed an 'arithmetization of geometry', using the most essential feature of Cartesian geometry, that is the systematic introduction of a unit line, to translate theorems in formulas and problems in equations on simple lines.



PERPENDICULAIRE ET OBLIQUES

41. Définitions. — Rappelons que, d'un point A situé hors d'une droite *xy* (fig. 50), on peut mener une perpendiculaire AH et une seule à cette droite (nº 2). Tout autre segment tel que AB est *oblique* par rapport *xy*.

42. Théorème. — ***Si d'un point situé hors d'une droite on mène à cette droite la perpendiculaire et diverses obliques :***

1º La perpendiculaire est plus courte que toute oblique.

2º Deux obliques qui s'écartent également du pied de la perpendiculaire sont égales. Réciproquement deux obliques égales s'écartent également du pied de la perpendiculaire.

3º Si deux obliques s'écartent inégalement du pied de la perpendiculaire, celle qui s'en écarte le plus est la plus longue. Réciproquement si deux obliques sont inégales, c'est la plus longue qui s'écarte le plus du pied de la perpendiculaire.

Fig. 11. "Perpendicular and oblique lines" in 1962 (Lebossé & Hémery, 1962a, p. 142)

We can find traces of Arnauld and Lamy's conceptions in elementary textbooks until 20th century (figure 11). But, we have to remark that Arnauld's order had been followed despite that, from 19th century, new methods shown that it was fruitful to consider lines and figures in space at once. In the same way, Lamy's arithmetization had been followed despite that, from 18th century, the introduction of the Cartesian system of axes provided a unified theory to conceive geometric objects.

To understand these two contradictions, we have to take into account the role of Lacroix, who adopted Arnauld and Lamy's conceptions at the turn of 18th and 19th centuries, as an answer to a demand of "élémentation" at work in new schools created after the French revolution. They constituted two Cartesian ingredients that remained particularities of French elementary textbooks until the 20th century.

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