MATHEMATICAL NARRATIVE: FROM HISTORY TO LITERATURE

Yi- Wen SU, Wann-Sheng HORNG, Jyun-Wei HUANG, Yuh-Fen CHEN
Department of Mathematics, University of Taipei, No.1, Ai-Guo West Road, Taipei, Taiwan
yiwen@uTaipei.edu.tw
Department of Mathematics, National Taiwan Normal University,
Taipei Municipal Heping High School,
New Taipei Municipal Mingder High School

ABSTRACT

Mathematical narrative is a concept that has gradually been attracting the attention of school teachers and mathematics educators, especially in recent years. The term 'mathematical narrative' in this workshop proposal refers to a form of narrative that is used to communicate or construct mathematical meaning or understanding. By introducing certain metaphors, a narrator may induce or promote learners' or listeners' mathematical understanding. This teaching tool can be used, for instance, to encourage further reflection upon some mathematical concepts or for alternative remedial learning, especially after conventional classroom teaching. Moreover, this can be applied in liberal arts courses at the university level. One straight forward way to introduce mathematical narrative into teaching is to use stories, arguments, problems, and their solutions from history, but one can also use literary metaphors to improve understanding. In this workshop, two school teachers and two mathematics educators will share their experiences in using mathematical narrative in their teaching, and also exchange teaching ideas with other participants in the workshop. The four presenters and the topics they will be discussing are described below.

Presenter 1: **Yi-Wen Su**, Department of Mathematics, University of Taipei.

The starting point of HPM is always history, and so is the beginning of this workshop. The first presenter is also the organiser of this workshop, and she shall share her experiences on how to design an activity to help improve undergraduate mathematics majors' reading comprehension using pre-modern mathematical arguments. The main material in this activity is the original proof of Heron's formula by Heron, and also another procedure that is equivalent to Heron's formula found in a thirteenth-century Chinese mathematical text written by Qin Jiushao.

Presenter 2: **Jyun-Wei Huang**, Taipei Municipal Heping High School.

The second presenter shall discuss his teaching in an elective mathematics course in senior high school in Taiwan (10^{th}–12^{th} grade). The core of this course is the use of the five most beautiful mathematical equations. The teacher tries to integrate mathematics discourse and mathematics writing in his teaching, and guides students in creating artistic works such as mathematical fictions, poems, games, and movies. In this way, he can help his students discover the beauty of mathematics and enhance their motives in learning.

Presenter 3: **Yuh-Fen Chen**, New Taipei Municipal Mingder High School.

The third presenter mainly teaches junior high school in Taiwan (7^{th}–9^{th} grade). She will share many activities designed by the team of mathematics teachers in her school, including publishing a school mathematical magazine, engaging students in mathematical games they created, decorating the school with 'mathematical tablets' that have their origins in the mathematical writings from Japan's Edo period, and encouraging students to write popular science novels after some mathematical reading.

Presenter 4: **Wann-Sheng Horng**, Department of Mathematics, National Taiwan Normal University.

Last but definitely not the least, history is transformed into narratives. The final presenter will introduce his experience in combining literary metaphor in teaching for undergraduate liberal art courses in mathematics. The main example he shall use is a questionnaire he employs to help students reflect upon mathematical concepts. He uses a metaphor quoted from Eli Maor's popular work, *The Pythagorean Theorem: A 4,000-Year History*. Maor uses the metaphor of ‘democracy’ to talk about a theorem related to chords in a circle. The presenter’s questionnaire guides students to reflect upon many aspects of Euclidean geometry and helps many university students, who are relatively weak in mathematics, to gain a new perspective about the discipline.

1 Reading and mathematical narrative

Reading ability and lifelong learning are closely related. In recent years, many countries increasingly put more of their educational efforts on students’ reading ability. To understand how students read, these countries have participated in international reading evaluations and comparisons such as the Programme for International Student Assessment (PISA) and Progress in International Reading Literacy Study (PIRLS), the results of which have served as reference criteria according to which all countries can improve the teaching of reading. In Taiwan, the Ministry of Education continues to promote reading education and develop practical programs for teaching reading to help students develop an interest in reading. The Ministry seeks to promote reading not only in (Chinese) language and literature education but also in various disciplines, such as mathematics reading, social studies reading, and science reading (Executive Yuan, 2010). This article attempts discuss methods to add historical reading material in mathematics teaching, to help students have a cultural perspective on mathematics.

To add historical material into mathematics teaching, we need to first explain the term ‘mathematical narrative.’ By ‘Mathematical narrative,’ we mean a form of narrative that is used to communicate or construct mathematical meaning or understanding. It is a concept that has gradually attracted the attention of mathematics educators and school teachers in recent years. To communicate or construct mathematical meaning or understanding, a narrator usually has to introduce some metaphors to induce or promote learners’ or listeners’ mathematical understanding. This teaching tool can be used to encourage further reflections upon some mathematical concepts or for alternative remedial learning, especially after conventional classroom teaching in the primary or secondary level. Certain related approaches have been suggested by educators for the solution of underachievement in mathematics (Solomon & John O’Neill, 1998). Moreover, mathematical narrative can be applied in liberal arts courses at the tertiary level. One direct way to introduce mathematical narrative into teaching is to use stories, arguments, problems, and their solutions from history, but one can also use literary metaphors and fictions to improve understanding. Besides, mathematical narrative is also suitable for assessments.

In this article, two mathematics educators and two school teachers will share their experience pertaining to the use of mathematical narrative in their teaching. Their teaching practices are described below.

2 HPM and mathematical narrative practices

2.1 Practice for mathematics majors in the tertiary level

Why do we talk about the history of mathematics when teaching mathematics? There are many suggestions regarding this matter, from local and international circles. For example, Furinghetti and Paola (2003) explain that teaching history in the classroom not only stimulates the interest of the students in mathematics, but also makes it possible to arrange the classes according to the main mathematical contexts or concepts, thus, helping to achieve the objective of learning mathematics. By contrast, Barbin (2000) considers that the most common reason we teach the historical dimension of mathematics is to enable us to think about its raison d'être, thus, helping us to comprehend the concepts and theories of the subject. Tzanakis and Arcavi (2000) state that including history in mathematics teaching could help in the learning process, allowing teachers to learn the nature of mathematics and its development; in this way, history would enhance the understanding of teachers, who would then view mathematics as part of the cultural development of knowledge. By integrating the history of mathematics into the teaching of mathematics, educational goals can be reached, covering affective, cognitive, and cultural aspects. What role does the history of mathematics play in the current national mathematics curriculum? Fasanelli (2000) analyses 16 countries that implemented this strategy, and observed an improvement in the affective level among the students in China, Greece, Italy, the Netherlands, and Poland. An increase in the affective level can lead to greater learning motivation and interest from the students, and also inspire them through learning about the biographies of mathematicians. History in mathematics learning allows the students to compare different solutions or directions of thinking through mathematical texts, freeing them from a single way of thinking. Regarding cultural activities, students from Brazil, Denmark, France, New Zealand, Norway, and the United States, achieved a meaningful learning experience mainly through the understanding of the mathematical development of all ethnic groups, thus cherishing their own mathematical culture. In the nine-year compulsory education system of Taiwan, the importance of history for learning mathematics has also been proposed (Ministry of Education, 2008). Thus, introducing the history of mathematics into mathematics instruction is necessary.

This project is carried out with qualitative research method, based on such comprehension as retrieving and accessing, interpreting and integrating, reflecting and evaluating (OECD, 2010), by reading Heron's Formula texts, to re-produce in classrooms the lively thoughts among mathematicians of different eras. The main material in this activity is the original proof of Heron's formula by Heron, and also another procedure that is equivalent to Heron's formula found in a thirteenth-century Chinese mathematical text written by Qin Jiushao. It is hoped that, through the design of history of mathematics reading materials, the students will be able to discuss the differences between Greek and Chinese mathematics cultures. A total of 39 university students of mathematics participated in this study. These students were at 3rd and 4th grade. Each participant read the proofs of Heron's Formula and finished the worksheet which was designed by the teacher. These worksheets were collected and analysed.

This study has discovered the following:

1. Retrieving and accessing process: All students could illustrate by themselves the most beautiful part of Heron's original proof.

2. Interpreting and integrating process: All students could understand Heron's original proof, only parts of the students can use Triangle area formula and The Pythagorean Theorem to derivate Qin Jiushao's formula.
3. Reflecting and evaluating process: The students would reflect over the way mathematicians solve these problems. And this includes the idea that mathematicians can use various approaches to solve the same problem and the idea that Greek mathematics and Chinese mathematics exhibit different styles of argumentations and expressions.

2.2 Practice in the higher secondary level

In this part, we discuss the teaching in an elective course of mathematics in the senior high school level in Taiwan (10^{th}-12^{th} grade). This was a six-week (twelve-hour) course. In order to show different characteristics with other schools' mathematics courses, the course focused on 'polyhedron and geometry' and the allegedly five most beautiful mathematical statements:

$$e^{i\pi}+1=0,$$

$$V-E+F=2,$$

There are exactly five regular polyhedra,

There are infinite many prime numbers,

$$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$$

Since the last equation is too hard for 10^{th} -grade students, this equation was only showed but not discussed in depth the course.

At the beginning, some historical stories and snippets about geometry were told to students, then, the primary sources in Euclid's *Elements* were used to introduce the origin of the five Platonic polyhedra. The teacher also used the software GSP to draw and introduce Platonic polyhedra *and* Archimedean solids and their developed views. Then, the teacher asked students to design and create models of Platonic polyhedra *and* Archimedean solids, and then they used these models to guess and discuss the Euler formula '$V-E+F=2$'. After that, the teacher designed a historical package about the mathematician Euler, his life and great works. Finally, the teacher guided students to understand how to prove that there are just only five Platonic polyhedra.

After that, the teacher let students watch the movie *The Professor's Beloved Equation* (*Hakase no aishita sûshiki*, 2006) which is about the formula '$e^{i\pi}+1=0$', and discussed the mathematics they learned from this movie and the reflections about this movie. The teacher also told some historical stories and snippets about prime numbers, perfect numbers and friendly number pairs that appeared in the movie. Students in the class also discussed the correspondence between the main roles in this movie and the constants in the '$e^{i\pi}+1=0$'. Through this course, the teacher combined the abstract mathematics with the image of literature and combined amusement with mathematics learning.

In the third part of the course, the teacher introduced the mathematics history and related mathematic culture about ancient Greece, ancient China and Wasan, the traditional mathematics of Japan in the Edo period and also introduced Greek text *Elements*, Chinese text *Nine Chapters* and some Wasan texts. The teacher used the *reductio ad absurdum* to prove that there are infinite many prime numbers, discussed many kinds of proof about the Pythagorean Theorem in each ancient text, introduced the proofs of many kinds of area formulae, that of sphere volume and some intuitive arguments by graphs in the *Nine Chapters*, and finally the teacher introduced the Wasan mathematician, Seki Takakazu, and his interesting works, along with the mathematic culture of Sangaku (mathematical tablets in shrines and temples) and some interesting mathematical problems written on the Sangaku.

The final part of the course was the assessment, which was held in the form of an achievements exhibition. Students could read a popular mathematics book and show their reflections, and they could also design and display creative works about mathematics or about all kinds of geometry models and report their ideas and motivations to all students in the class. Some students wrote poems about mathematics, and one of whom used an ancient mathematics problem in *Sun Tzu Suanjing* to create a poem. Another student designed a mathematics puzzle which combines the awareness of environmental protection with a mathematics problem.

Finally, students showed their reflections and gains in a questionnaire survey about this course. The teacher summed up and analysed the answers of all students, then concluded that there were four major aspects they mentioned in the questionnaire:

1. History and culture of mathematics
2. Learning motivation and the interests in mathematics
3. Diversity in mathematics
4. Mathematical knowledge

Table 1. The average satisfaction degree of this course scored by students

	The subjects of each week	Average score (Max 7)
Week1	The five Platonic *polyhedrons and* Archimedean solids	5.68
	The g history of *polyhedrons*	5.59
Week2	Design the regular *polyhedrons*	6.11
	The Euler formula about *polyhedrons*	5.89
	The most beautiful mathematical equations	6
	The popular mathematics books and movies about mathematics	6.11
Week3	The developed views of *polyhedrons*	5.95
	The intersection of mathematics and aesthetics	5.98
Week4	The Housekeeper and the Professor	6.55
Week5	Geometry in ancient China, ancient Greece, and Wasan	5.61
	Pythagorean theorem and the proofs	5.55
Week6	Achievements Exhibition	5.34

Table 1 shows the average satisfaction level of this course scored by students in a 7-point scale. From this table, we can see that all items are above the average score 3.5. This shows that students were satisfied with this course in the learning process.

Students' favourite parts of this course were the movie, the developed views of polyhedra, the course of designing and creating models of polyhedra, and the introduction of the popular mathematics books and movies, the score of which are all above 6.

For most senior high school students in Taiwan, mathematics is the most abstract and most difficult subject. But this elective course of mathematics can serve as a supplement to the regular course of mathematics. Through more diversified teaching methods, including explaining and discussing, working in small teams, designing models, watching the movie and the achievements exhibition, and through the intersection of mathematics and aesthetics, literature, art, history, culture…etc, the teacher designed and combined various teaching materials in order to help students experience the interesting and useful aspects of mathematics and also expand more extensive learning views. In this way, we hope to help students discover the beauty of mathematics, and enhance their motives in learning.

2.3 Practice in the lower secondary level

Junior high school students in Taiwan seldom do any extracurricular reading about science and mathematics. According to our survey in one rural secondary school in Taiwan, although 38% of the students have extracurricular reading habits (which is already low enough percentage), less than 1% read popular science or mathematics. This may not represent the general situation in Taiwan, but it does tell us that we need to do something to enhance students' interests in mathematics. Some teachers in that school began to promote popular mathematics reading and writing from students' perspectives.

In a span of more than five years, the teachers in that school tried to use several approaches to enhance students' interests in mathematics and their aptitudes in reading and writing about mathematics. Their strategies are listed below.

1. **Introducing popular mathematics publications in class**. Mathematics teachers classified popular mathematics publications into four categories and introduced them to students, helping them to find their interests in reading. The four kinds are: visual attractions, personal experiences, curiosity orientations, and narrative creations.
2. **Introducing popular mathematics publications with social media and their website**. They created an internal website for teachers and students in their school to discuss popular mathematics and mathematical knowledge.
3. **Creating reading environments in classrooms and campus**. They made an 'ema pavilion' outside of their mathematics laboratory. 'Ema' is a kind of small wooden plaques for Japanese Buddhists and Shinto worshippers to write their wishes. They put mathematical problems on the ema, and students could try to answer them, put them own solutions for public examination, and win rewards.

4. **Organising annual events: 'mathematical week'**. Students could use their mathematical knowledge to solve problems in personal and inter-class challenges. This gave them motivations and opportunities of self-learning.
5. **Publishing school magazine *Mathematics Fast Food***. This magazine has been published for more than 15 years. It has columns such as 'Teachers Column', 'Student Creations', 'Mathematics in Life', and 'New Horizons in Mathematics'. Students could appreciate the omnipresence of mathematics through this magazine.

With these approaches, teachers in this school successfully encouraged students in their mathematics reading and learning. Students wrote their reflexions about their reading, drew cartoons, created poems and mathematics riddles, and some even wrote novels. Some students had bigger projects such as making polyhedral lanterns and Escher-style tessellations. Although students did sometimes have some difficulties in reading pre-modern texts or even modern publications, these teachers' efforts have made mathematics more accessible to students, and created smiles on students' faces when they read and write about mathematics.

2.4 From history to literature – a practice in liberal-arts mathematics

In his *The Pythagorean Theorem: A 4000-Year History*, Eli Maor explains a paramount criterion for a theorem or the proof thereof to be called beautiful is symmetry. His example is the proposition that 'the three altitudes of a triangle always meet at one point'. Analogously, he also mentions the same truth holds for the medians and the angle bisectors. 'This statement,' he comments, has certain 'elegance to it, with its sweeping symmetry: no side or vertex takes precedence over any other; there is a complete democracy among the constituents.' Yet, the most interesting proposition he raises to me is as follows: If through a point *P* inside a circle a chord *AB* is drawn, the product $PA \times PB$ is constant – it has the same value for all chords through *P*. The literary metaphor he uses to illuminate the meaning of the proposition is as insightful as convincing: 'Again we have the perfect democracy: every chord has the same status in relation to *P* as any other.'

For years Prof Wann-Sheng Horng has used the metaphor to examine if students who attend his liberal study course at National Taiwan University, 'Mathematics and Culture: An approach of reading mathematical fiction,' benefit by thinking of mathematics in terms of literary metaphor. So at the first class of the course, he would encourage the students to answer a questionnaire with the following queries:

• Recall junior high school mathematics; try to prove the following theorem:

If through a point P inside a circle a chord AB is drawn, the product $PA \times PB$ *is constant.*

Hint: Through the point *P* draw any other chord *CD*, prove that $PA \times PB = PC \times PD$.

(1) Eli Maor comments on the proposition in his *The Pythagorean Theorem: A 4000-Year History* as follows:

'*[Again] we have perfect democracy: every chord has the same status in relation to P as any other.*'

Question: Does the metaphor have any relation to the content of the proposition? Try to explain briefly.

(2) How does the argument of the proof benefit your understanding the metaphor? Try to explain briefly.

(3) On the contrary, does his metaphor enhance your understanding of the proof and its meaning? Try to explain briefly.

(4) Is there any similar task in your past experience of learning mathematics? If yes, in what kind of situations (say classroom)? Explain.

• Instead of the above questionnaire I also give an alternative version like the following:

Recall junior high school mathematics; try to prove the following theorem:

If through a point P inside a circle a chord AB is drawn, the product $PA \times PB$ *is constant.*

Hint: Through the point *P* draw any other chord *CD*, prove that $PA \times PB = PC \times PD$.

(1) Eli Maor comments in his *The Pythagorean Theorem: A 4000-Year History* that '*[Again] we have perfect democracy: every chord has the same status in relation to P as any other.*'

Regarding the metaphor Maor has employed for the geometrical proposition, please try to explain his popular writing approach.

(2) There are two main aspects of mathematical activities: interesting and useful. Which aspect is more likely to engage you in mathematics without assessment pressure like examination? Try to explain your reason.

In his presentation, he uses the questionnaires, among others, given in the liberal study course as both the analysing and explanatory tool in order to show how college students can be enhanced in their understanding of mathematics by means of literary metaphor. The results show that students who were relatively weak in mathematics had gained different views about mathematics.

3 Concluding remarks

From the above practices, we found that by introducing students to the history of mathematics, they have learned to realise the relation of mathematics to human activities, which have inspired the development and creativity of cultures. We hope that this article can act as a reference in providing this experience to the HPM community.

REFERENCES

Barbin, E. *et al*. (2000). Integrating history: Research perspectives. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education* (pp. 63-90). Dordrecht: Kluwer Academic Publishers.

Fasanelli, F. *et al*. (2000). The political context. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education* (pp. 1-38). Dordrecht: Kluwer Academic Publishers.

Furinghetti, F., & Paola, D. (2003). History as a crossroads of mathematical culture and educational needs in the classroom. *Mathematics in School, 32*(1), 37-41.

Ministry of Education (2008). *Grade 1-9 Curriculum Guidelines*. Taipei: Ministry of Education.

Executive Yuan (2010). Taiwan PISA 2009 report. Retrieved from http://www.ey.gov.tw/eycc/News_Content.aspx?n=DF52F83A5B7D2A47&sms=114B66117B4BF117&s=D0C45A32CD5F9B3B.

OECD (2010). PISA 2009 Assessment framework: Key competencies in reading, mathematics and science. Paris, France: OECD Publishing.

Solomon, Y., & O'Neill, J. (1998) Mathematics and narrative. *Language and Education, 12*(3), 210-221.

Tzanakis, C., & Arcavi, A. *et al*. (2000). Integrating history of mathematics in the classroom: An analytic survey. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education* (pp. 201-240). Dordrecht: Kluwer Academic Publishers.