

A MULTIDISCIPLINARY APPROACH TO TEACHING MATHEMATICS AND ARCHITECTURAL REPRESENTATION: HISTORICAL DRAWING MACHINES. RELATIONS BETWEEN MATHEMATICS AND DRAWING.

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ABSTRACT

The aim of this paper is to highlight some of the results of the interdisciplinary work carried out by the authors in the School of Architecture of Roma Tre University which unifies research and didactics showing a contemporary approach to the disciplines of drawing and mathematics. They conduct a course to develop a methodology of study that foresees three aspects: the construction of a mathematical drawing machine, the understanding of the analytic representations of the curve that the machine tracks, the consultation of historical sources. Case studies extracted from treatises on history of architecture, mainly from the nineteenth century, can show how geometrical issues have been resolved over time and interpreted in different ways. The goal is to reinforce in students the awareness of the relationship between physical objects, their representation, and the mathematical theory underlying, through the study of historical cases.

1 Introduction

This work aims to highlight some of the results of the interdisciplinary teaching implemented in the elective course "*Mathematical drawing machines: historic drawing from a parametric point of view*", born from the encounter between the courses on the Science of Architectural Representation and those of Mathematics at the Department of Architecture of Roma Tre University (Italy)¹.

The course aims to compare themes which are common to both disciplines and is active from the academic year 2014/2015. In the cultural context, typical of Schools of Architecture, these two disciplines support the training of future architects, and in the specific case of Roma Tre they are part of the study plans of all degree programs. In recent years, the teachings of both disciplines have been object to many considerations and transformations. The path of the first 10 years of teaching mathematics in the School of Architecture of Roma Tre is described in the article by Pagano-Tedeschini Lalli², then continued by the activities of "formulas.it"

¹ The two authors are researchers of Drawing and Representation (Farroni), Mathematics (Magrone) in the Department of Architecture of Roma Tre University.

² Pagano, & Tedeschini Lalli, L. 2005

maths Laboratory³. As for the teaching of the Representation of Architecture, it has been transformed with the introduction of digital and information technology. These transformations have posed a set of problems, not only in the management of the contents of drawing, but also on how to transfer knowledge to learners. Those questions were posed in all specific areas, such as Descriptive Geometry, Drawing of Architecture, Survey and Representation Techniques.

A great inspiration has come from the work of Mariolina Bartolini Bussi⁴, from her book⁵ on drawing machines, along with the Mathematical Machines Association⁶ and also from the studies by Riccardo Migliari⁷, who for years has been dealing with these issues, linking them to the evolution of the teaching of Descriptive Geometry and shape recognition through the survey of built architecture. For this reason the elective course "*Mathematical drawing machines: historic drawing from a parametric point of view*"⁸ fits into a fertile environment of research of new methodologies and teaching strategies. It aims to put the student in a position to verify the same knowledge by comparing the two points of view. The course is intended for students of all degree programs of the School of Architecture of Roma Tre University. Relying on prior knowledge that we will explain in detail later, the course offers a methodological approach to the study of some cases found in the treatises of the history of architecture, mainly from the nineteenth century, combining the analytical geometrical treatment, the hands-on approach and representation.

The study of nineteenth-century treatises leads us to investigate issues that over time have been resolved and interpreted in different ways, leading students towards a greater understanding of the relationship between "curva figura e curva oggetto" (Gay 1999) that is "(curve) shape and (curve) object". Precisely in the nineteenth century we witness a systematization of the typologies of geometric constructions, the geometric drawing procedures and instruments. In the historical texts which were selected there is a description of elements of architectural construction whose shape is taken into consideration from the point of view of its geometric genesis, along with the design procedure and drawing. Not less important in these historical texts in some cases we find the description of the methods of construction made possible through the use of the mathematical machines.

The compulsory courses in mathematics of the first three years of studies already educate students to visualization via analogical models such as drawing machines. In fact, in

³ A group of mathematicians and architects, whose purpose is the dissemination of scientific culture, educational innovation, informal teaching of mathematics. In particular the web site gathers all the courses with mathematical content of the Department of Architecture of Roma Tre University.

⁴ Full Professor of Didactics of Mathematics at Modena and Reggio Emilia University, Italy.

⁵ Bartolini Bussi, Maschietto (2006) op.cit., Tedeschini Lalli (2009) op.cit.

⁶ www.macchinematematiche.org, the site offers pictures, animations and explanatory pages on many drawing machines. Much attention is devoted to conicographs, which therefore constitute a valid first approach for students considering the huge amount of teaching materials available.

⁷ Full Professor of Descriptive Geometry in Sapienza, University in Rome.

⁸ See also Farroni & Magrone (2014) where the authors talk about the very first experience of this cross-disciplinary teaching.

the obligatory course of the second year⁹, students can choose to build a machine to draw a conical curve and discuss it in the oral exam. This has further encouraged the authors to propose the course. Probably also inspired some of the students, encouraged by the typical traditional laboratory approach adopted by schools of architecture, where students often build physical models to scale.

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2 Course objectives, content and teaching strategies

The goal of this course is to lead students to acquire and make explicit the relationships, which has always existed, between the graphical and the analytical representation in the interdisciplinary approach to architectural drawing and mathematics. Specifically, the course addresses the problem of the tracking of curved objects and the process of abstraction, which is fundamental in both disciplines. This process is necessary to switch from a real object to its representation. For this purpose the authors offer the tools to expand the knowledge of plane curves, their genesis, their construction and tracking. The mathematical knowledge required as prerequisites are Euclidean geometry, which is studied in secondary schools, and differential calculus in one variable, which is acquired during the first year course. As for the drawing, the prerequisites consist of the knowledge of the elementary geometric constructions and the use of graphical instruments such as compass and ruler, which are part of the first year course called "Drawing of Architecture".

The advent of information technology, both in the setting of mathematics and architectural representation, has produced softwares that allow to easily manage both simple and complex shapes. Often the creation of a shape and its modification and transformation is implemented automatically and unconsciously. To educate the skill of creating the desired shapes with autonomy and consciousness, the authors propose the direct hands-on manipulation of the data and the subsequent identification of the parameters in the analytical representation as the main tool in learning contemporary instruments and theories dedicated to the formal project. To this end, each student is assigned a mathematical machine to design, implement and test.

During the lessons the analytical equations are related to the specific graphic constructions with ruler and compass, through the construction and use of machines such as some ellipsographs, parabolographs, and hyperbolographs. The workshop sessions require that students take back knowledge gained in the course of drawing, acquire new expertise, build prototypes of drawing machines for continuous tracking and then use them. The interdisciplinary teaching objectives are to develop in students the ability to foresee the main characteristics of the figurative project on a two-dimensional support, at the very moment of its drawing; to introduce the scientific basis for facing the digital modeling; consolidate the ability to relate shapes and formulas.

⁹ The course is "Calculus (several variables) and Geometry 2", led for the current year 2015-2016 by Valerio Talamanca and Laura Tedeschini Lalli

The choice of teaching strategies is connected to some considerations. The curve tracking highlights the problem of the continuity of the sign. Some of the methods of graphic constructions, although rigorous, allow detection of a curve by points instead of the continuous tracking. Therefore the problem arises of drawing a curve that joins the identified points and also find tools that directly allow the continuous tracking, without passing through the discretization. The mathematical machines meet this need: they draw the curves with continuity, with the exception of only a few points (only in some cases) where the tracer should be disconnected physically from the two-dimensional support in order to overcome the physical barrier of the mechanism and then proceed to the tracking of the sign. Therefore it has been decided, for educational purposes, to show the mathematical machinery presenting both continuity and discontinuity tracking during the lessons, so that students themselves can grasp this aspect.

In the early teaching units much space is devoted to conic curves, for reasons that will be explained below. A first consideration concerns the recognition of geometric loci, which is a key capability in the training of an architect. This recognition turns out to be a more immediate process when referred to conic curves. All the curves that learners are supposed to study and draw during the course are two dimensional, and can be interpreted as sections and profiles useful for the future management of three-dimensional objects (both analogical and digital) whether in the field of architectural design or in recognition of the built forms. Their detection, both graphical and analytical, leads to the ability to associate a geometric entity to a shape, in order to be able in a second moment to edit and modify it. The proposed working method aims then to start a continuous check between constructive and analytical procedure. For this purpose, the conical curves are the ideal starting point: they are familiar objects to most students, their analytic representation is not difficult.

Last but not least the focus is on the concept of parameter: the cartesian equation of a conic curve contains some parameters, whose change affects the shape of the curve. The curve drawn with a machine, puts the student in the position to lead actively the plotting and the corresponding analytical representation, by setting and then changing the initial data. Tracking a curve with the machine whose setting is reflected in the respective equation, and further manipulate the machine by observing the change of the curve, educates the process of abstraction that relates the shape with the analytical equation. This process goes through the drawing machine. This acquisition of awareness, implemented on known curves, characterized by simple equations, can be transferred at a later time to the study of more complex curves.

The first sessions are devoted to “thread conicographs”, because their mode of operation embodies the geometric locus corresponding to the curve and therefore constitute the ideal tool for transmitting its comprehension.

The reading of some passages from the *Trattato teorico e pratico dell'arte di edificare* (Theoretical and practical art of building Treatise) by Giovanni Rondelet, 1832¹⁰ has also been of help to show learners how, in the nineteenth century, a systematization of knowledge

¹⁰ Rondelet 1832 op. cit.

and know-how, was carried out, in other words how theoretical knowledge was verified in the practice of building. In fact, Rondelet reports, (Third book, Stereotomy, first section, first part) a few paragraphs dedicated to the useful curves in architecture. In particular in *Cenno sulle curve che possono servire alla superficie interna delle volte*¹¹, the graphical and analytical-geometrical description *Delle curve chiuse* e *Delle curve aperte*¹² can be found by referring to suitable instruments for plotting. The decision to propose this to learners lies on the setting of the Treatise itself. Actually it highlights that the geometric drawing is a fundamental tool for the art of building, in the light of the theories of Monge¹³.

In the text of Rondelet considerable theoretical sections of analytic geometry can be found. For example, in the section mentioned above, the ellipse is described as a projection of the circle, as a geometric locus of the plane, and at least three graphic methods to identify the curve by points are provided (see fig. 1); furthermore two machines for the continuous tracking are described: the “thread” machine (see fig. 2) and the one that embodies the graphical method called “the paper strip” (see fig. 3). Again, various graphical methods are described to show how to draw ovals, i.e. the polycentric curves, of different typology depending on the numbers of centers, and finally the cassinoide curve and the cycloid are mentioned. Only after an extensive theoretical treatment of the above curves, Rondelet exposes the motivations that can push an architect to choose either one or another:

La sensibile differenza che offrono queste curve fra loro, considerate come curvature di volte, influisce molto sulla loro solidità. La teoria d'accordo coll'esperienza prova che nelle volte schiacciate, più è curva l'arcatura del mezzo, minore è la sua spinta [...] d'onde risulta che se si ha in vista la solidità conviene scegliere una curvatura che si avvicini più alla cicloide che alla cassinoide. Nondimeno quest'ultima, che è più aperta, presenta in certi casi una forma più aggradevole che si accorda meglio coi piedritti a piombo; ma essa agisce con più forza ed esige una maggior grossezza di sostegni. L'ellissi, la cui curvatura è media, unisce la solidità alla regolarità, e perciò dev'essere sempre preferita; tanto più che ha la proprietà di poter servire per tutte le altezze di volte, mentre la cassinoide ha dei limiti, e la cicloide non conviene che ad un caso solo¹⁴...

The significant difference between these curves, regarded as vaults bending, greatly affects their solidity. The experience agreeing with theory shows that in flattened vaults, the more curved the arch in the middle part, the lower its pressure [...] whence it results that if you look for the strength, you should select a curve more similar to the cycloid than to the cassinoid. Nevertheless, the latter, which is more open, presents in some cases a more pleasant shape that harmonizes better with the pillars; but it acts with more force and requires a greater thickness of support. The ellipses, which has an average curvature, combines solidity to regularity, and therefore must always be preferred; furthermore the ellipses has the property of being able to serve for vaults of

¹¹ Hint on the curves that can serve to the inner surface of the vaults

¹² About closed curves, about open curves

¹³ In this regard it is interesting to see the work plan of the treatise by Rondelet, that here we chose to leave out.

¹⁴ J.B. Rondelet, 1832, op. Cit. Third book, Stereotomy, first section, first part

all heights, while the cassinoid is limited, and the cycloid is not convenient except for one case [...]

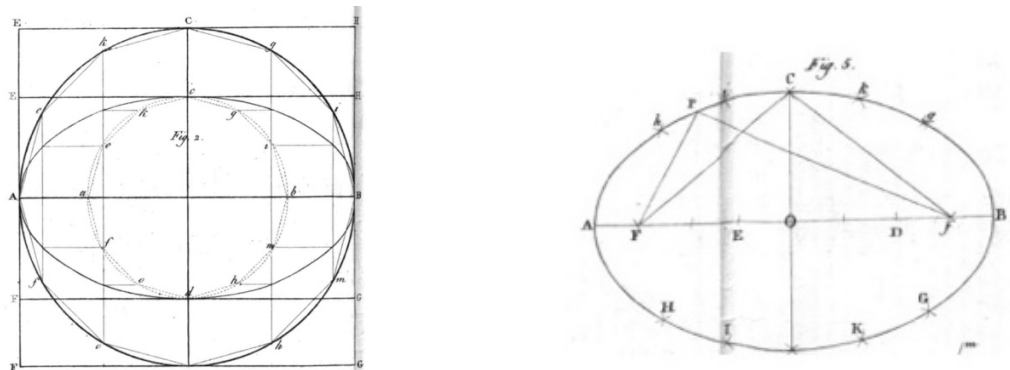


Figure 1-2. Rondelet G.B. (1832) "Trattato teorico e pratico dell'arte di edificare", Plate XIX.

On the left representation of the graphic method to draw an ellipse by points, called the "concentric circles". On the right the so-called "gardener's ellipse", for drawing this curve in a continuous way; in the same Figure, reference is made to a graphical method to track an ellipse by points.

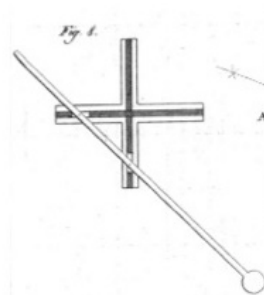


Figure 3. Rondelet G.B. (1832) "Trattato teorico e pratico dell'arte di edificare", Plate XIX
ellipsograph of Proclo

Since the beginning of the course a bibliography and webliography is suggested to the learners so that they can immediately begin to direct themselves autonomously on the topic for the final exam. To this end it is required that each student builds a drawing machine and understands how it works; knows the equations of the curve that is represented by the mechanism; studies a geometrical problem based on a historical text applied to architecture involving the studied curve. After the first in-depth sessions on conic and related machines, the second part of the course starts, which is dedicated to individual insight. The individual progresses are often discussed with all the group so that everyone can contribute with questions and interventions. It is through these group discussions and the ongoing confrontation with the instructors, as well as individual work, that the historical research is narrowed and refined allowing the arrival to the final product.

3 History as a source of case studies

The proposed study methodology proceeds on three routes: the construction of the machine, the understanding of the analytic representations of the curve that the machine is used to track and the shape-formula comparison along with the consultation of historical sources. Geometrical problems that in the course of time have been interpreted in different manners

and case studies can be extracted from treatises. The aim is to develop in students an awareness of the relationship between physical constructed reality, representation, and underlying mathematical theory, right through the study of historical cases. The nineteenth-century machines make this issue clear and explicit, which on the other hand may be less obvious to the future architect who designs and studies the forms through the computer. Here are some case studies taken into consideration because they highlight how "the accuracy of the geometry" comes to the "aid" in the definition of some architectural shapes in the nineteenth century.¹⁵

3.1 The "Pillet"¹⁶ machine

This case study shows an example of an "ellipse-hyperbol-parabolograph" (fig. 4, 5, 6), a mechanism which encloses in a single object the three thread machines to draw conic curves. A first prototype was designed and built as part of the work of a post-graduate fellowship¹⁷. In order to get the students accustomed the theoretical/practical approach, during the course they make diagrams and graphical representations of the machine and a prototype whose beam was 60 cm long, suitable to draw, for example, an ellipse whose major axis can be about 45 cm.

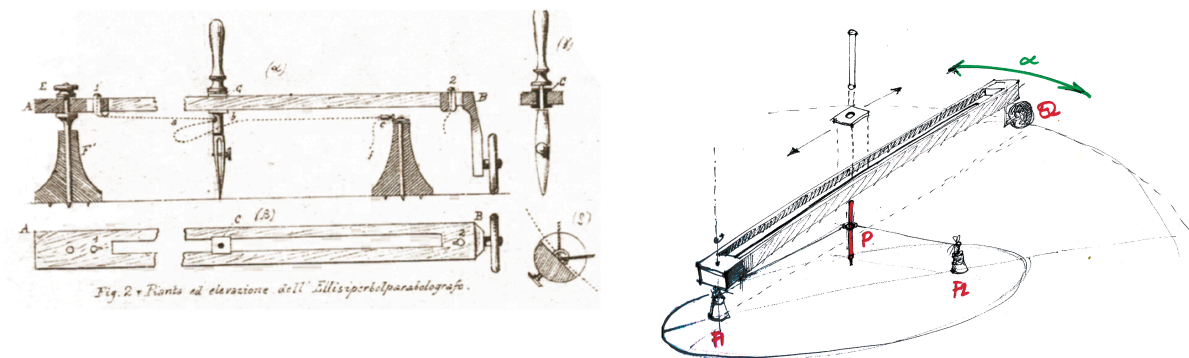


Figure 4. Plate from L'architettura pratica Disegni degli edifizii rispondenti ai bisogni moderni, (1891): scheme of the Pillet Machine. Figure 5. Hand made sketch for the construction of the first prototype of the Pillet machine, by Enrico Mele.

¹⁵ Codazza (1844) op. Cit.

¹⁶ Jules Pillet (1842-1912) was a professor of descriptive geometry at Ecole Normale des beaux Arts in Paris. The first time the authors learned about this machine was from the book of Migliari (1991).

¹⁷ The post graduate fellow was Enrico Mele, who is also the author of the sketch in fig. 5.



Figure 6. Prototype of ellipse-hyperbol-parabolograph, beam length 150 cm, made by the student Osvaldo Liva

Then one student chose this to be his product for the final exam, and then continued to deepen the knowledge of the machine, creating a model which measured 150 cm, therefore suitable to draw an ellipse with major axis length of about 120 cm¹⁸.

3.2 Tracking the entasis¹⁹ curve

The entasis curve of the columns and its tracking is a case study which foresees various solutions including the use of the Conchoid of Nicomedes.

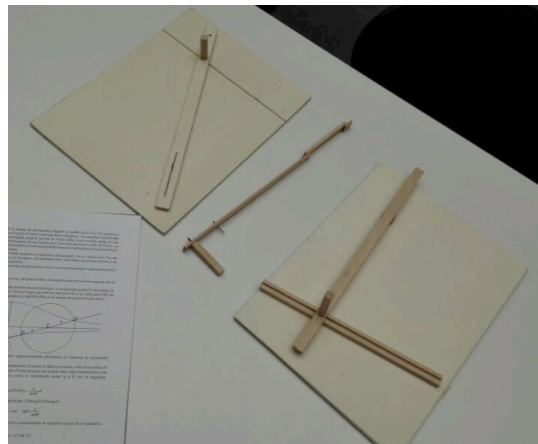
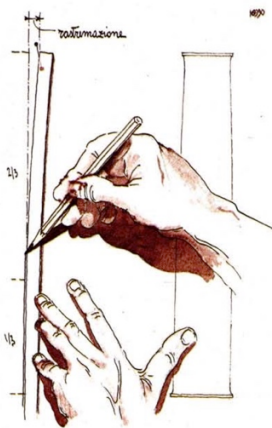


Figure 7. Migliari (1991) “Il disegno degli ordini e il rilievo dell'architettura classica: Cinque Pezzi Facili”. Sketch by Riccardo Migliari, in which the author reports the method for tracking the entasis by means of a “flexible wooden rod”. Figure 8. Picture of the drawing machine to track the conchoidal curve, design and implementation by the student Valerio Del Ferraro.

The conchoid curve was introduced to the students as a source of different insights: it can be used to resolve the trisection of the angle, one of the three Greek unsolved problems of antiquity. The curve possesses two branches, which leads to consider why we should use the

¹⁸ Work performed by the student Osvaldo Liva

¹⁹ The entasis indicates a narrowing of the shaft of the column, starting from one-third of its height, upwards.

parametric equation, rather than cartesian one. After the detailed mathematical study the work of the student continues with the analysis of the practical application in architecture, comparing it to the different procedures for tracking the entasis curve, in order to produce an inventory of the different methods. The student in charge of the project, proceeded to read and interpret the original source. Then he built two prototypes of drawing machines: one to draw the conchoid of Nicomedes, and a second one, which is cited by Peter Nicholson (in the text of 1867 "Carpenter's New Guide") which consists of a simple flexible wooden rod (see fig. 7,8). The setting of the work has helped the student to develop a critical attitude so that after reading the original sources he also wished to become acquainted with the state of the art of contemporary studies (with the help of the bibliography provided during the course), like those of Riccardo Migliari (Migliari 1991). Later, elaborating critically all the information, he proceeded to verify the working principles of the two prototypes he already made. This case study gave the authors the opportunity to and underline to their students the importance of understanding the relation between the “geometric object and the real object”:

Vige sempre un'incoltabile differenza ontologica tra il *cerchio* del muratore e quello de 'l *geometra*; mentre il primo conta il numero dei mattoni circonfrenti come tre volte quello che sarebbe occorrente per il diametro, il matematico mette oggi nel rapporto tra diametro e perimetro, in luogo della malta del muratore, un miliardo di cifre decimali e dimostra che il suo lavoro contabile non finisce lì. Il profilo concoide rifinito dallo scalpellino nell'entasi delle colonne non è propriamente la stessa concoide considerata da Nicomede per risolvere la mitica duplicazione del volume dell'altare cubico del tempio di Apollo [...] (Fabrizio Gay (1999), pag 76 op. Cit.)

An increasingly unbridgeable ontological difference forever endures between the *circle* of the mason and that of the *geometer* (in this case we think that this term refers to the person who is in charge of controlling the project according to science, so someone who has a deep knowledge of theoretical geometry T.N.); while the mason counts the number of bricks to form a circle as three times those which would be needed for the diameter, the mathematician puts a billion decimal places today in the relationship between diameter and perimeter, instead of the mason mortar, and he also proves that his accounting job does not end there. The conchoidal profile of the column entasis, finished by the stonemason, is not exactly the same conchoid considered by Nicomedes to solve the legendary duplication of the volume of the cubic altar of the temple of Apollo [...]

3.3 The study of a nineteenth-century gear

This study led to the deepening of epicycloid curves. This case differs from the others described because the application is not architectural, but instead consists of a nineteenth-century engineering product. The gear shown in fig 10 led the graphical depiction which was part of the work for the student's final examination. At the theoretical level the student has studied the equations of epicycloids curves; then she designed and built some gear wheels, with holes to insert the tracer to draw different kinds of epicycloids. The first prototypes were cut from cardboard, while the conclusive gears were made of plexiglass and cut with a

Computer Controlled Cutting (CNC) machine of the “model and prototypes laboratory” of the Architecture Department of Roma Tre University.

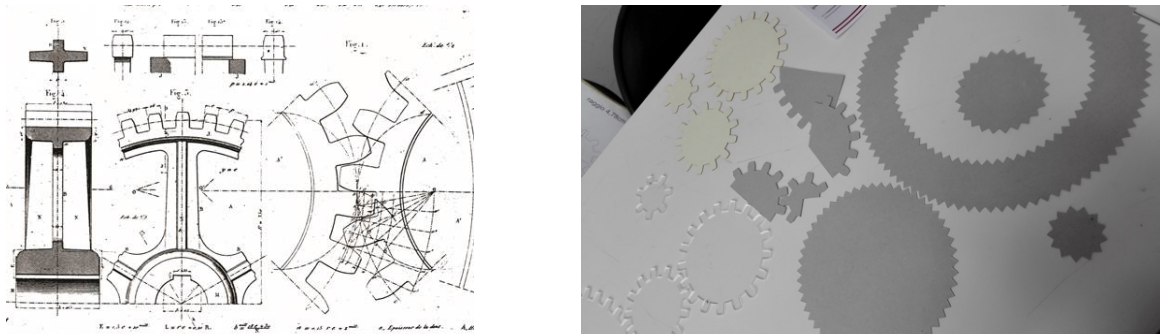


Figure 9. Plate from Armengaud, J.E, & Armengaud, C. A (1854) “The practical draughtsman's. Book of industrial design, and machinist's and engineer's drawing companion: forming a complete course of Mechanical, Engineering, and Architectural Drawing” showing the gears. Figure 10. gears in cardboard and plexiglass, design and implementation by the student Matilde Panasci

4 Conclusions and future developments in teaching and research.

The study of ancient drawing machines, as part of the training program for students of architecture, allows them to grow culturally and scientifically. If the use of the machines and the tracing of curves is accompanied by a continuous analytical verification, it strengthens the ability to recognize the geometry of curved objects, and to identify the key points of each curve. Furthermore the contextualization of the mathematical graphical problem, namely the study of the historical and cultural framework of the machine and the verification of the original sources, leads to the acquisition of analytical and critical skills in solving the issues that students may encounter in the professional world. From now on the objective is to widen the case studies and to create a collection of prototypes of different machines together with a bibliographic and iconographic filing of the original sources and of the built products.

This type of teaching has led to an experimental development and exchange between teaching and research in which both complement each other. In fact, the results obtained in the first edition of the course have been studied in depth by a student awarded with a post-graduate fellowship aimed at drawing up bibliographies related to the relationship between geometry and construction in architecture. This scientific product was then proposed for the second edition of the course in the form of lectures and as available material for in-depth study.

The construction of the historical machines highlights how each one is specialized in displaying certain properties. For example, in the case of ellipsographs, thread machines can be used to show the geometrical locus, while some linkages-type machines lead to the study of geometric transformations. So new study paths are created. The change of scale, experienced in the tracking of the conic curves with the Pillet machine, suggests reflections on how to manage curves at large. Also curved objects on an urban scale can be taken into consideration in order to introduce the use of ancient survey instruments for the collimation of points, useful for the definition of the designed curve. It has come to our attention that the

structure of the study course allows to test the previous achievements and explore new perspectives.

REFERENCES

- L'architettura pratica. Disegni degli edifizii rispondenti ai bisogni moderni, (1891), Anno II, Fascicolo VII 26-27. Torino : Tip. Camilla e Bertolero.
- Armengaud, J. E, & Armengaud, C. A. (1854). *The practical draughtsman's. Book of industrial design, and machinist's and engineer's drawing companion: forming a complete course of Mechanical, Engineering, and Architectural Drawing. Translated from the French of M. Armengaud, the Elder and MM. Armengaud, The Younger, amoureaux*. Rewritten and arranged, with additional matter and plates, selections from and examples of the most useful and generally employed mechanism of the day by William Johnson. New York: Stringer & Townsend.
- Bartolini Bussi, M. G. C., & Maschietto, M. (2006). *Macchine matematiche : dalla storia alla scuola*. Milano : Springer Verlag Italia.
- Codazza, G. (1844). *Nozioni teorico pratiche sulle centine delle volte*. Pavia: Tipografia Bizzoni.
- Cresci, L. (1998). *Le Curve Celebri*. Padova: Franco Muzzio Editore.
- Farroni, L., & Magrone, P., (2014). Mathematical drawing machines: Historic drawing from a parametric point of view. The case of conic curves. *Libro de actas del V Congreso Internacional De Expresión Gráfica. XI Congreso Nacional de profesores de expresión gráfica en ingeniería, arquitectura y áreas afines Egrafía 2014, Rosario, Argentina, I*, 130-137.
- Gay F. (1999). *Intorno agli omografi: strumenti e modelli per la geometria descrittiva*, Quaderni IUAV.12.2000 DPA. Venezia : Istituto universitario di architettura di Venezia.
- Migliari, R. (1990). Il disegno degli ordini e il rilievo dell'architettura classica: Cinque Pezzi Facili. *Disegnare: Idee, Immagini*, 2, 49-66.
- Nicholson, P. (1808). *Carpenter's New Guide being a complete book of lines for carpentry and Joinery*. London: printed for J. Taylor.
- Pagano, A., & Tedeschini Lalli, L. (2005). Università di Roma Tre, 1995-2005: Architecture and Mathematics. *Nexus Network Journal*, 7, 89-97.
- Pillet, J. (1887). *Traité de géométrie descriptive, ligne droite et plan, polyèdres, surfaces. Texte et dessins*. Paris: Librairie Ch. Delagrave.
- Rondelet, G. B. (1832). *Trattato teorico e pratico dell'arte di edificare. Prima traduzione italiana sulla sesta originale con note e giunte importantissime, per cura di Basilio Soresana, seconda edizione*. Mantova: Fratelli Negretti.
- Tedeschini Lalli, L. (2009). Mathematical machines: A laboratory for mathematics, Interview to Mariolina Bartolini Bussi by Laura Tedeschini Lalli. *Nexus Network Journal*, 11, 317-324.