ON THE RELATIONS BETWEEN GEOMETRY AND ALGEBRA IN GESTRINIUS' EDITION OF EUCLID'S *ELEMENTS*

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ABSTRACT

In 1637 the Swedish mathematician Martinus Erici Gestrinius contributed with a commented edition of Euclid's *Elements*. In this article we analyse the relationship between geometry and algebra in Gestrinius' *Elements*, as presented in Book II. Of particular interest are Propositions 4, 5, and 6, dealing with straight lines cut into equal and unequal parts, and the three kinds of quadratic equations Gestrinius associates with them. We argue that Gestrinius followed Clavius translation of the *Elements*, but was influenced by Ramus to include algebra.

1 Introduction

Martinus Erici Gestrinius (1594–1648) was a Swedish mathematician and became a professor of geometry (then named *professor Euclideus*) at Uppsala University in 1620. He was born in Gävle as the son of the parson and in 1611 he became the student of Claudius Opsopæus, the professor of Hebrew at Uppsala. As a student Gestrinius repeatedly visited Germany and some of its universities, which made him familiar with German mathematics. In 1614 he travelled to Helmstedt and in 1615 to Wittenberg. In 1618 he received his university degree at Greifswald before he returned to Uppsala to start teaching at the university. In 1630, 1638, and 1643 he was the vice-chancellor of the university and in 1633 he was one of the university's delegates at the Parliament.

Uppsala University, founded in 1477, is the oldest university in the Nordic countries and was the only university in Sweden until King Gustav II Adolf in 1632 founded the university at Dorpat, which today is the University of Tartu in Estonia. The university had grown out of an ecclesiastical centre and had been charted through a papal bull by Pope Sixtus IV. During the turbulent times of the Reformation in the 16th century, however, there was very little activity at the university. In 1593, at *The meeting of Uppsala* – the most important synod of the Lutheran Church of Sweden – Lutheran Orthodoxy was established in Sweden, and the Duke Charles (later King Charles IX) gave new privileges to the university, which reopened in 1595.

It is known that Gestrinius lectured at Uppsala University on the significance of arithmetic for the bourgeois life, algebra, astronomical calculations, and geometry. He seems to be the first in Sweden to use the symbols + and − and he was the first Swede to use square roots. One of his greatest mathematical achievements was that he introduced algebra into Swedish mathematics. In 1637 he contributed with the textbook *In Geometriam Euclidis Demostrationum Libri Sex* – a commented edition of the first six books of Euclid's *Elements*. This is the first edition of Euclid's *Elements* published in Sweden, and it was used as a textbook at the university at least until the beginning of the 18th century. Gestrinius probably had knowledge of Campanus', Clavius' and Peletier's editions of the *Elements*. However, contrary to these three interpreters of Euclid´s *Elements*, Gestrinius included algebra in Propositions 4, 5 and 6 of Book II in his edition of the *Elements*.

In this paper we will consider the relations between geometry and algebra in Gestrinius' *Elements*. In order to do this we will in detail consider the connections between Propositions 4, 5, and 6 of Book II and the three kinds of quadratic equations Gestrinius associates to them, as well as the different ways in which Gestrinius solves the equations. We will also investigate who influenced Gestrinius' work. Even if Petrus Ramus is not mentioned in Gestrinius' edition of the *Elements*, as we shall see there are indications that he inspired Gestrinius: They use the same notation and they also classified the quadratic equations in the same way. However, an important difference is that Gestrinius explained the solutions of the equations geometrically, by utilizing diagrams.

2 Book II of Gestrinius' *Elements*

In the introductory text of Book II in Gestrinius' version of the *Elements*, he mentions the multiple uses of the contents of Book II – not only in geometry, but also for cossic algebra, geodesy and astronomy. He also mentions that he solves algebraic equations and uses surd numbers in certain propositions of Book II. Gestrinius' version of Book II contains 14 propositions divided into twelve theorems and two problems. He establishes all of the propositions in a traditional manner. He also exemplifies the propositions, using numbers.

In Gestrinius' edition of the *Elements*, Propositions 4, 5 and 6 of Book II are of particular interest (Gestrinius, 1637, pp. 117–129). The propositions consider straight lines cut into equal and unequal segments. Gestrinius presents traditional geometrical proofs of these propositions, as well as exemplifying them through the solution of the following quadratic equations associated with the proposition:

$$1q + 10l \text{ æ}quat. 119$$

$$1q + 20 \text{ æ}quat. 12l$$

$$1q \text{ æ}quat. 12l + 64$$

Gestrinius describes the way to solve these equations in three different ways: rhetorically, with tables, and geometrically. We will consider them in detail below.

2.1 Proposition 4 of Book II

In Gestrinius' edition of the *Elements*, Proposition 5 of Book II is formulated in a traditional geometrical manner:

Proposition 4: If a straight line is cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.

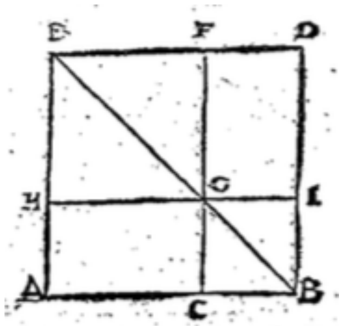

Figure 1. Geometric interpretation of Proposition 4

According to the proposition, the straight line AB is cut at the point C. The square on AB is equal to the sum of the squares on AC and CB together with the two rectangles with sides AC and CB.

Gestrinius establishes the proposition in a purely geometrical manner, similar to the traditional proof of Euclid. Thereafter he exemplifies the proposition with numbers, letting the length of the straight line be 12 and the segments 7 and 5. Gestrinius' conclusion, with modern notation, is that $12^2 = 7^2 + 5^2 + 2 \cdot 7 \cdot 5$.

Gestrinius now associates a quadratic equation of the first kind ("Æquatio Algebraica secunda primi generis") to the proposition: $1q + 10l$ *æquat.* 119. He uses the symbol q for the unknown square (quadratum), l for the unknown side (latus) and the abbreviation *æquat.* (æqualitatem) for the equality. In this equation the sum of the "maximum" and the "medium" terms is equal to the "minimum". Gestrinius proceeds by verbally describing the solution of the equation:

> Half of the medium 10 is 5, its square is 25, composed with the minimum 119 is 144, is from its side 12 the half of the medium 5 is removed it is remained the value 7 of one side.

Gestrinius tests the received value 7: $10l$ gives the value 70 and $1q$ gives the value 49, the total equals 119. This verbal solution could be summarized with modern notation as follows:

$$\sqrt{(\frac{10}{2})^2 + 119} - \frac{10}{2}.$$

Now he summarizes the solution in a table (see Figure 2):

	1q + 10l æquat. 119.
Dimid. medij	5.
□. dim. medij	25.
Minimus	119.
Compoſit.	144.
Latus compoſ.	12.
Dimid. med. ſubtr.	5
Reſid. & val. 1l	7.

Figure 2. The table summarizing the solution of the equation associated with Proposition 4

The reason for including the table is probably to show how to find the solution to the equation through a more algorithmic-like process.

Gestrinius solves the equation one more time, this time using geometry, and referring to Figure 1. Translated into English, Gestrinius' text proceeds as follows:

> This equation is applied to the present proposition and the truth of the equation will be evident. To the square AD the gnomon is IEC. Now $1q + 10l$ equals the gnomon IEC; therefore it equals 119. Moreover $1q$, which is HF, is the square of the gnomon: wherefore $10l$ is equal to the two rectangles AG, GD and the plane $5l$ is equal to the rectangle whose length $1l$ is the side of the square HF already placed, and 5 is the side of the remaining square CI. Therefore, 25 is the remaining square CI, and added to the gnomon IEG, which is 119, it makes the square AD complete to 144. From the side 12 of the square AD the side 5 of the square CI is removed, remaining 7, which is the side of the square HF and the value of $1l$.

The gnomon IEC is easily found in the equation $1q + 10l = 119$, where $1q$ corresponds to the square HF of the gnomon and $10l$ corresponds to the two rectangles AG and GD. The crucial part of the proof is the squaring of the gnomon, which is done by finding the side 5 of the remaining square CI. After this has been done, Gestrinius easily finds the side of the square HF, i.e., the square root of $1q$.

2.2 Proposition 5 of Book II

In Gestrinius' edition of the *Elements*, Proposition 5 of Book II states:

> *Proposition 5*: If a straight line is cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.

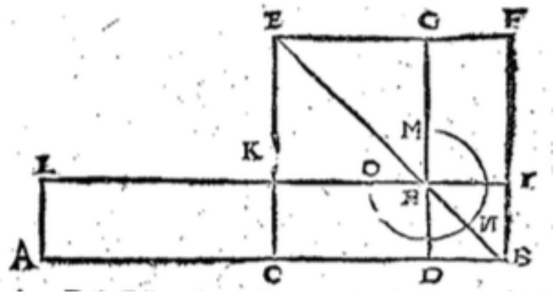

Figure 3. Geometric interpretation of Proposition 5

The straight line AB is cut into equal parts at C and in unequal parts at D. The sum of the rectangle AH (the rectangle contained by the unequal segments) and the square KG (the square on the straight line between the points of section) equals the square CF (the square on the half).

After establishing the proposition in a geometrical manner similar to the traditional proof of Euclid, Gestrinius exemplifies the proposition with numbers. He lets the length of the straight line AB be 12. It is divided in equal parts at C such that $AC = 6$ and $CB = 6$, and in unequal parts at D such that $AD = 10$ and $DB = 2$. The length of the straight line between the point of section is $CD = 4$. He concludes, translated into modern notation, that $6^2 = 10 \cdot 2 + 4^2$.

Gestrinius now associates a quadratic equation of the third kind ("Algebraica Æquatio secunda tertij generis") to the proposition: $1q + 20 = 12l$. In this equation the sum of the "maximum" and the "minimum" terms is equal to the "medium". Gestrinius proceeds by describing the solution of the equation verbally:

> Half of the medium is 6, its square is 36 from which the minimum 20 is subtracted, remaining 16, if its side 4 is added to half of the medium 6 it will make 10, if withdrawn it will make 2. Therefore $6 + 4$ and $6 - 4$, i. e., 10 and 2, will be the values of $1l$.

In modern notation this verbal solution could be summarized to:

$$\frac{12}{2} \pm \sqrt{(\frac{12}{2})^2 - 20}\,.$$

Just as in Proposition 4 Gestrinius now summarizes the solution in a table (see Figure 4):

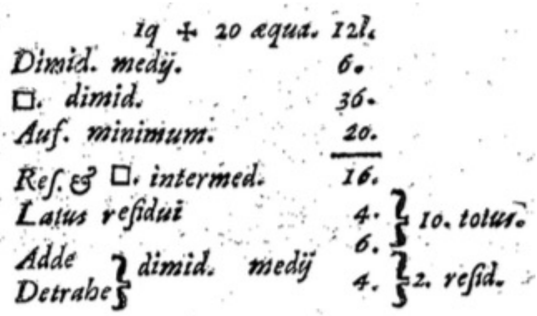
1q + 20 æqua. 12l.
Dimid. medij. 6.
☐. dimid. 36.
Auf. minimum. 20.
Ref. & ☐. intermed. 16.
Latus residui 4. } 10. totum.
6.
Adde } dimid. medij 4. } 2. resid.
Detrahe

Figure 4. The table summarizing the solution of the equation associated to Proposition 5

If an equation of the third kind has a positive root it always has one more. This makes it difficult for Gestrinius to give a geometrical solution that is well connected to the equation. Despite this fact, Gestrinius uses geometry to solve the equation one more time:

> The diagram makes this equation visible. For if AB 12 is divided into unequal segments AD 10 and DB 2, then AH 20 is the plane made up of the unequal segments AD and DB. And therefore if this plane is reduced from 36, the square of the half segment CB, i.e., from the square CF, there remains 16, the square KG of the intermediate segment CD, whose side 4 together with the half 6 gives 10, the longest segment AD, for the value of $1l$. The same subtracted from 6, leaves 2, the smallest segment DB, which in the same manner is the value of $1l$. And both operations depend however on this proposition by addition and subtraction. Since the plane of the whole and the longest segment, is equal to so many times the longest segment, as there are unities in the whole. And similar is the plane of the whole and the smallest segment equal to so many smallest segments, as there are unities in this. As in the same example the plane 120 out of 12 and the longest segment 10 is equal to $12l$, which contains 100, the square of the longest segment 10, and 20 the plane of the unequal segments. Thus the plane 24 out of the whole 12 and the smallest segment 2 is in a similar way equal to $12l$, which contains 4, the square of the smallest segment, and the plane 20 of the same unequal segments, as the demonstration of the operation shows. And therefore there will be a free choice to use sometimes addition and subtraction, sometimes just one of them, as it will be manifested by the algebraic problems.

To be able to tackle the problem, Gestrinius had to give the roots of the equation beforehand. One problem is that the diagram only corresponds to one of the solutions of the equation ($1l = DB$). Since Gestrinius did not use a minus sign he could also not easily find a gnomon that can be squared. For example, the gnomon MNO could be described with the expression $12l - 1q$. To square this gnomon, the "square on the straight line between the points of intersection", i.e. 16, has to be added. Then the square on the half is equal to $20 + 16 = 36$, i.e., the half is 6, which gives $6 - 4 = 2$ as a root. A similar diagram would give the root $6 + 4 = 10$.

2.1 Proposition 6 of Book II

In Gestrinius' edition of the *Elements*, Proposition 6 of Book II states:

> *Proposition 6*: If a straight line is cut into equal segments and a straight line is added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.

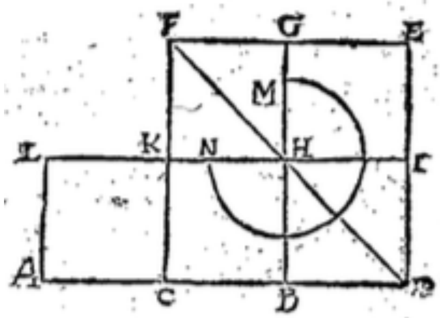

Figure 5. Geometrical interpretations of Proposition 6

According to the figure, AB is the straight line and it is bisected at C and the straight line BD is added. The proposition states that the rectangle AI (the rectangle contained by the whole with the added straight line and the added straight line) together with the square KG (the square on the half) equals the square CE (the square on the straight line made up of the half and the added straight line).

Gestrinius proves also this proposition in a traditional geometrical manner before exemplifying it with numbers. He lets the length of the straight line AB be 12. It is divided in equal parts at C such that $AC = CB = 6$. The added straight line is $BD = 4$. He concludes that since the whole and the added straight line equals 16 and the half together with the added straight line equals 10, it follows from the proposition that $10^2 = 16 \cdot 4 + 6^2$.

Gestrinius now associates a quadratic equation of the second kind ("Secunda Æquatio Algebraica secunda generis") to the proposition: $1q = 12l + 64$. In this equation the "maximum" term is equal to the sum of the "medium" and the "minimum" terms. Gestrinius proceeds by describing the solution of the equation verbally:

> Half of the medium is 6, its square is 36, add the minimum 64 which gives 100, whose side is 10. To this add half of the medium 6, the total will be 16 which is the value of $1l$.

He concludes that $12 \cdot 16 + 64 = 16^2$. In modern notation this solution could be summarized in:

$$\sqrt{(\frac{12}{2})^2 + 64} + \frac{12}{2}.$$

Just as in the other propositions Gestrinius now summarizes the solution in a table (see Figure 6):

	1q æquat. 12l + 64.
Dimid. medij.	6.
□. dim. med.	36.
Minimus	64.
Totus compositus	100.
Latus	10.
Add. dimid. med.	6.
Val. 1 lat.	16.

Figure 6. The table summarizing the solution of the equation associated to Proposition 6

Since an equation of the second kind only has one positive root, it is easier to illustrate the solution with the help of the diagram in Figure 5. Gestrinius uses geometry to solve the equation in this way:

> The proposed equation is applied in this way: The minimum is the plane made up by the composed and the added, and the square of the half of the middle form is the square of the half; and therefore the sum of these is equal to the square of the half and the added, whose side if the half is added is the side of the great square. As in the same example $1q$ æ$quat.12l + 64$. The plane of the composed and the added AD, DB, i.e., rectangle AI, is 64, added to the square of the middle CB, i.e., square KG, which is 36, gives 100. This number 100 is equal to the square CE, out of the half and the added CD. Its side is 10, together with 6, the half AC, gives the side of the great square.

Even though Gestrinius does not mention any gnomon to be squared, it is involved in the solution. The root $1l$ is equal to "the whole with the added straight line", i.e., AD. The rectangle AK is equal to the rectangle HE, so the gnomon composed by the rectangle CH, the square BI and the rectangle HE is equal to the rectangle AI. The square KG, which equals to 36, completes the square.

3 Influence on Gestrinius

In the introduction of his *Elements*, Gestrinius mentions four post-classic mathematicians: Johannes Kepler (1571–1630), Campanus of Novara (1220–1296), Jacques Peletier (1517–1582), and Christopher Clavius (1538–1612). Kepler did not publish his own edition of the *Elements*, but he is mentioned in connection to the treatment of the five regular polyhedra and his Platonic solid model of the solar system. However, Campanus, Peletier, and Clavius published versions of the *Elements.* We will now explain the connection between these three versions of the *Elements* and the connection to Gestrinius' edition, as well as other possible influences on Gestrinius. In particular, we argue that Gestrinius was influenced by Petrus Ramus.

3.1 Campanus, Peletier, Clavius, and Commandino

Campanus' Latin version of the *Elements,* based on Arabic translations, was completed between 1255 and 1259 (Corry, 2015). In 1482 it appeared as the first printed edition of the *Elements*, and thus became the standard source until the 16th century, when other editions based on direct translations from the Greek were published. Campanus' treatment of the *Elements* contains many modifications, additions and comments, as he was making efforts to present the text in an as self-contained form as possible. In Book II Campanus remained within a geometric context, but he also stated that the first 10 propositions of Book II were true for numbers as well as for lines.

Gestrinius mentions the two 16th century mathematicians Peletier and Clavius in connection with Proposition 16 in Book III and their debate regarding the angle of contact. Peletier's edition of the *Elements* was published in 1557 with the title *In Euclidis Elementa Geometrica demonstrationum, Libri sex*. Its title is very similar to Gestrinius', but Peletier's edition is a work on geometry and includes no algebra. Clavius' edition of the Elements was published in 1574 with the title *Euclidis Elementorum Libri XV. Accessit liber XVI*. Book II of Clavius' *Elements* includes many problems solved numerically. The wording of the propositions and the proofs of Book II of Clavius are very similar to Gestrinius', but Clavius did not include any algebra.

One of the most important Latin translations of the *Elements* is the 1572 version of Frederico Commandino (1509–1575) (Heath, 1956). Commandino followed the original Greek more closely than Clavius. Gestrinius mentions Commandino when he in Book V besides the traditional 25 propositions included 9 extra propositions in an *Appendix Ex Commandino*. However, Commandino in his *Elements* of 1572 only included 33 propositions of Book V, compared to the 34 included by Gestrinius, as well as by Campanus, Peletier and Clavius.

In 1632 Gestrinius lectured on geometry at Uppsala University. In the lecture notes we can see that the wording of the propositions is very similar to those of Clavius, indicating that Gestrinius followed Clavius. In the lecture notes Gestrinius also gave the same numerical examples, as he would later present in his *Elements*. However, algebraic applications are missing in the lecture notes. Even if Gestrinius followed Clavius' *Elements*, his idea of including algebra into Book II cannot come from Clavius. As we will argue below, there are indications that Gestrinius got this idea from Petrus Ramus.

3.2 Ramus

Petrus Ramus (Pierre de la Ramée, 1515–1572) is not mentioned in Gestrinius' *Elements*, but there are strong indications that Gestrinius knew of Ramus' work and was inspired by it. Ramus was the most important French algebraist before François Viète (1540–1603). As a mathematician Ramus is most famous for his ideas on the practical use of mathematics and the influence of his book on the education of mathematics. In his *Algebra* from 1560, he explains that "Algebra is a part of arithmetic" and he mentions that the two parts of algebra are "numeratio" ("arithmetical" handling of algebraic expressions) and "æquatio" (solving equations). In particular, Ramus showed how to treat quadratic equations. His notation was simple, with l for the unknown (latus=side) and q for the square of the unknown (quadratus=square). Instead of an equality sign he used the abbreviation "æqua." We recognise this notation from Gestrinius. Just as Gestrinius did, Ramus classified the quadratic equations into three types ("canon"), which he presented through the following examples (Ramus, 1560, pp. 13–19):

$$Canon\ primus: 1q + 8l\ æqua. 65$$

$$Canon\ secundus: 6l + 40\ æqua. 1q$$

$$Canon\ tertius: 1q + 21\ æqua. 10l$$

As we can see, Gestrinius and Ramus used the same notation, and also classified the quadratic equations in the same way. Just as Gestrinius, Ramus also illustrated the algebraic solutions of the equations with the help of the geometry in Propositions 4, 5, and 6 of Book II in the *Elements*, but he did it in an algebraic context, not using diagrams as Gestrinius did. Thus, Ramus used the three propositions to illustrate the three equations, that is, he used geometry to illustrate algebra. As we have seen, Gestrinius, on the other hand, used the equations to illustrate the propositions, that is, he used algebra to illustrate geometry.

In 1545 Ramus also published a version of Euclid's *Elements* (Heath, 1956). However, it only includes definitions and propositions, but no comments. Ramus also only included 25 propositions in Book V and not 34 as Gestrinius, as well as Campanus, Peletier and Clavius, did. This indicates that Ramus, in this sense, was not following the traditions of Campanus, Peletier and Clavius.

4 Concluding remarks

Gestrinius was the first Swedish mathematician to publish an edition of Euclid's *Elements* and to present algebra in a printed text. However, he was not the first mathematician to use algebra in a geometrical context. William Oughtred (1573–1660) was one of the first to exemplify theorems of classic geometry using algebra (Stedall, 2002). He demonstrated all of the 14 propositions of Book II of Euclid's *Elements* in his *Clavis mathematicæ* from 1631 with his analytical method, which means that he used algebra. He used Viète's algebraic notation, as presented in Viète's symbolic algebra, or the *Analytical Art.* Inspired by Diophantos' work during the end of the 16th century, Viète used capital letters instead of abbreviations as symbols for the unknown and known entities. This made it possible for him to present a general equation and to give a general method of solving it. Nevertheless, Gestrinius did not use Viète's symbols, and it is possible that he did not even know about them. Therefore it seems unlikely that he, at least in 1637, had knowledge of Oughtred's *Clavis*.

Many years before Oughtred's *Clavis*, Thomas Harriot (1560–1621) had written the propositions of Book II algebraically, but this text was never printed (Stedall, 2000). Also Harriot was influenced by Viète, but he used his own symbols, similar to Descartes'. Another important contribution was made by Pierre Hérigone (1580–1643) when he in 1634 published his *Cursus Mathematicus*, where he replaced the rhetorical language of Euclid's *Elements* with a symbolic language (Massa-Esteve, 2010). Hérigone's aim was to introduce a universal symbolic language for dealing with both pure and mixed mathematics.

We do not suggest that Gestrinius was an extraordinary mathematician even though he was the first professor of mathematics at Uppsala University who brought the subject to a more scientific study. Primarily he was an educator, and his importance lies in the fact that he transferred known mathematical theories to the following generations of Swedish mathematicians. His edition of the *Elements* was used at Uppsala University, as well as Clavius' edition of the *Elements*. For example, the Swedish mathematician and mathematics teacher Anders Gabriel Duhre (1680–1739 (possibly 1681–1739)), most well-known for his textbooks on algebra and geometry, most likely studied Gestrinius' version of the *Elements*. In his book on geometry, Duhre also connected geometry to algebra and proves the propositions of Book II of Euclid's *Elements* using algebra in Descartes' notation as well as in the notation of Wallis and Oughtred. Also Samuel Klingenstierna (1698–1765), the most well-known Swedish mathematician during the 18th century, learned Euclid by reading Gestrinius' edition of the *Elements*. Therefore, Gestrinius keeps an important position in the Swedish history of mathematics and mathematics education.

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