ANTHYPHAIRESIS, THE "ORIGINARY" CORE OF THE CONCEPT OF FRACTION

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ABSTRACT

In spite of efforts over decades, the results of teaching and learning fractions are not satisfactory. In response to this trouble, we have proposed a radical rethinking of the didactics of fractions, that begins with the third grade of primary school. In this presentation, we propose some historical reflections that underline the "originary" meaning of the concept of fraction. Our starting point is to retrace the anthyphairesis, in order to feel, at least partially, the "originary sensibility" that characterized the Pythagorean search. The walking step by step the concrete actions of this procedure of comparison of two homogeneous quantities, results in proposing that a process of mathematisation is the core of didactics of fractions. This process begins recording the act of comparison by a pair of natural numbers, and is realized in the Euclidean division. Classroom activities ensure that children perceive the Euclidean division as the icon of their active process of learning. The Euclidean division becomes the core of many feedback loops along which the teaching process is developed.

1 Introduction

The scientific literature we have seen for a period of about fifty years, reports that, in spite of the efforts both in research and in practice, results of teaching and learning fractions are not satisfactory and difficulties are widespread and persistent.[1]

Some causes of these difficulties, widely discussed in scientific literature, are: (a) The multi-faced structure of rational numbers.[2] This feature is well represented in the scheme of the "sub-constructs of the construct of rational number" proposed by Kieren.[3] (b) Another obstacle: research and teaching practice have proposed teaching-learning processes that are often based on an unique pattern of action. This approach implies some difficulties in transferring the acquired knowledge to other situations. In teaching practice, the situation of division, associated to the part-whole substructure, is used in most cases.[4] It is important to

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[1] Among the many quotes, we choose the following two, distant in time. "The concept of fraction has manifested itself in education as a refractory one" [Streefland, 1978]. "It is now well known that fractions are difficult concepts to learn as well as to teach" [Tunç-Pekkan, 2015].

[2] "Rational numbers should be a mega-concept involving many interwoven strands" [Wagner (1976) in Kieren, 1980].

[3] The five Kieren's sub-constructs are: part-whole, quotients, measure, ratios, operators. Other possible subconstructs are: proportionality, point on the number line, decimal number and so on.

[4] The scientific literature has largely confirmed that the situation of division has limited effectiveness [Nunez & Bryant, 2007].

stress that this teaching choice may also develop inhibitions.[5] (c) The natural numbers bias causes confusion between the features related to natural numbers and those related to fractions.[6] In scientific literature there are two attitudes. The one supports continuity between natural numbers and fractions.[7] The other considers the universe of fractions as a new universe, with its own rules and properties.[8]

The prolonged poor results in teaching and learning fractions have produced different conducts. Sometimes the purpose of teaching is to master the rules for calculating with fractions, without making an explicit connection between calculation and conceptual understanding.[9] Another option has been to postpone the teaching of the fractions.[10]

In Italy we have experienced a paradoxical situation: while the unsatisfactory results highlighted by research are widely acknowledged in middle and in high school, on the contrary, in primary school teachers mostly percept the teaching and learning fractions as easy.[11]

2 Historical reflections

These considerations, along with the direct experience in teaching, led us to a radical rethinking of the didactics of fractions. The resulting unusual project begins with the third grade of primary school and is characterized by five key points. (1) It is a process of familiarization with fractions rather than a process of teaching and learning them; this point, taken from Davydov[12], must be interpreted within the ZPD.[13] (2) From the didactic point of

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[5] "Not only 'part of a whole' diagrams are possibly misleading, but, more seriously, it will be argued later that their use may well inhibit the development of other interpretations of a fraction …". [Kerslake, 1986]. The development of inhibitions has received not adequate attention both by researchers and by teachers.

[6] "The natural number bias is known to explain many difficulties learners have with understanding rational numbers." [Van Hoof, Verschaffel & Van Dooren, 2015].

[7] "The major hypothesis to be tested was that children could (and should) reorganize their whole number knowledge in order to build schemes for working with fractional quantities and numbers (the rational numbers of arithmetic) in meaningful ways." [Behr, Harel, Post & Lesh, 1992].

[8] "Kieren argues that rational number concepts are different from natural number ones in that they do not form part of a child's natural environment" [Kerslake, 1986].

[9] This choice had already been confuted in the Erlwanger's seminal article: "Benny's case indicates that a mastery of content and skill does not imply understanding." [Erlwanger, 1973].

[10] "Instruction in rational numbers should be postponed until the student has reached the stage of formal operations." [Kieren, 1980].

[11] This perception is explained by the fact that the only sub-construct proposed to children is the part-whole one. This choice allows structuring a feasible proposal and building evaluation processes with satisfactory results on average. However, this type of proposal can produce some inhibitions that will burden on the later learning process of rational numbers, causing the difficulties highlighted by research.

[12] In the 60s of the last century, Davydov has experienced an interesting approach to the concept of fraction in some schools at Moscow. This approach is distinct from those favoured in Western school: Davydov researches the "objective origin of the concept of fraction" and he identifies it in the measure (as in the measure he identifies the objective origin of the concept of multiplication) [Davydov, 1991]. Following this approach, he builds a practical, unitary and coherent teaching proposal, that we have experienced in our classrooms and that has contributed significantly to the construction of our proposal.

view, fractions are a new universe, with its own rules and properties, distinct from the universe of natural numbers. (3) The process of mathematisation begins with the act of identifying the comparison between two homogeneous quantities with a pair of natural numbers, and characterizes itself as “elementary and fundamental”. (4) The measure of a quantity is defined as the comparison between the quantity and the “whole”, while the term “unit” is assigned to indicate the common unit between quantity and whole. (5) The dialogy[14] among the activities, that consists in choosing the most appropriate manipulative to introduce a specific property, should create a “polyphony” among the different activities; so each of them finds meaning in the others and gives meaning to them.

A more detailed structure of the project is presented elsewhere.[15] Here we intend to do a historical reflection, because history is “the site” where our project has arisen and has found its structure; in this site we have looked for the “originary”[16] meaning of the concept of fraction and we have found some foundational aspects that allow to rethink its didactics.

Our research has begun with a suggestion emerged thanks to the speech of Imre Toth at the conference held in Bergamo in 1999.[17] He exhorted to listen to hidden meanings that could still be kept in the Pythagorean mathematics and in the mathematics of the Platonic Academy. That's how we discovered the anthyphairesis. In the Appendix 1 we present the procedure of the anthyphairetic comparison. This method of comparison did not last long. Its crisis came with the discovery of incommensurable quantities[18] and its difficulties were contrasted by the effectiveness of the Euclidean algorithm. This latter overshadowed the anthyphairetic comparison, which was consequently forgotten.[19]

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[13] In our activity of familiarisation with the concept of fraction we provide children with “a broad base of experiences both practical and linguistic” [Nunez & Bryant, 2007] and we assess the actions and reactions of children properly guided.

[14] The word “dialogy” is taken from Bakhtin. It has been preferred to the word "dialogue" because it keeps some characteristics of the Bakhtin’s thought, as “voices”, “formative interaction”, “polyphony”, that better pinpoint the sense of our proposal. There are two ways of expressing this word: “dialogy” and “dialogism”. The second is partially compromised by the excessive use of the suffix "ism" made in the twentieth century.

[15] We presented a first version of our project at Cieaem 67 in Aosta [Alessandro, Bonissoni, Carpentiere, Cazzola, Longoni, Riva, & Rottoli, 2015]. An updated version will be presented at ICME 13 in Hamburg.

[16] The word “originary” is not an English word. Nevertheless, some authors (Roth & Radford, 2011) are beginning to use it. In this way the wealth of meaning possessed by the corresponding term “originario/originaire” that is used in continental philosophy, is recovered.

[17] “Matematica, Storia e Filosofia: quale dialogo nella cultura e nella didattica?” Bergamo, 1999, May 19.

[18] “Consideration of the non terminating anthyphairesis of incommensurable magnitudes would lead to serious philosophical problems and technical mathematical difficulties… . When the ratio theory, based on anthyphairesis, was abandoned for Book V – style proportion theory, the interest in anthyphairesis as a mathematical procedure would greatly diminish, and the details of its erstwhile connection with ratio would be forgotten.” [Fowler, 1979].

[19] The effectiveness of the Euclidean algorithm results from its direct operating on numbers and from its characterization as a much faster multiplicative process. The additive–subtractive feature of the Pythagorean procedure was thereby lost; an additive-subtractive feature we bring to light by retracing step by step the procedure.

Walking step by step the concrete actions of the anthyphairetic process, we have met some indications stored in it and still potentially significant: indications of historical type, as this walking allows to listen again to not secondary aspects of Greek philosophy;[20] indications of mathematical type, as it highlights a "physical" language for rational numbers[21] and it enables an unusual outlook on intrinsic reciprocity;[22] indications of pedagogical type, which have moved our project.

3 Pedagogical indications

As shown in Appendix 2, the anthyphairetic comparison produces a binary string and a pair of numbers. The binary string is a "physical" language for rational numbers; the pair of numbers is the "logos",[23] that makes "effable"[24] the comparison. So the Pythagorean statement "all is number" acquires the characteristic of a search for a "scientific" procedure in the description of nature, and the anthyphairetic procedure is a primitive form of mathematical knowledge of it. This is the indication from which our educational project has began. The objective of our slow retracing step by step this archaic procedure of comparison, was not the trivial

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[20] The additive-subtractive feature that characterizes this procedure reflects the time of thinking of the Pythagoreans. To listen to this time permits to feel, at least in part, their astonishment in front of the number; the astonishment that led them to say that everything is number. In fact, if you think you establish an unit of measure among homogeneous quantities, each of them has its own additive-subtractive structure expressed by a logos. Overcoming the lack of homogeneity of the real world by the "arché", the unit becomes the Parmenidean One, with which to compare every quantity. So every quantity has its own additive-subtractive structure in its comparison with the One; a structure expressed by a pair of numbers: everything is number.

[21] In the Appendix 2 we show how, marking the sequence of actions of the comparison by modern symbols, it is possible to build a binary string. In this way each rational number can be expressed by a binary string. While the correspondence between the set of natural numbers and the set of binary strings is the foundation of the contemporary sciences of information and computers, the correspondence between the set of rational numbers and the set of binary strings seems not to be the subject of adequate attention.

[22] Retracing the steps of the anthyphairetic comparison, the reciprocity between the two compared quantities comes evident. The role of reciprocity in teaching and learning fractions is under discussion in scientific literature [Thomson & Saldanha, 2003]. But we believe that, by retrieving the reciprocity with the characteristics that it shows in the anthyphairetic comparison, the investigation could be provided with useful new indications. For example, the intrinsic reciprocity of anthyphairesis suggests the possibility to go sometimes beyond the usual definition of quantity: " … a quantity is completely determined in mathematics when a set of elements and the criteria of comparison are determined… The comparison is usually traced back to the application of the relation "equal to", major to" or "minor to" [Davydov, 1991]. Intrinsic reciprocity might enrich the criteria of comparison and, therefore, the concept of quantity and the process of measure. We like to think that the fact of bringing reciprocity at the heart of the measure may provide insights into the challenges that complexity today presents, especially in the presence of quantities of dual nature.

[23] The Greek word "logos" that the Pythagoreans use for denoting the pairs of numbers obtained by anthyphairetic comparison, is a polysemic word that acquired different meanings in the historical course of ancient Greece: word, speech, talk, oration, discourse, ratio, logic, cause, rationale. At the Pythagoreans, it looks like a wonderful synthesis of two different meanings: the one comes from the verb "légein", that is "to bind", "to relate"; the other is contained in the meaning of "voice", "speech", that, already at that time, the word "logos" had taken. Its translation in the Latin word "ratio" has originated the wording "rational numbers".

[24] The word "effable" conveys, in addition to the meaning of "capable of being expressed", also the effort of the search and the wonder of the discovery: what is indescribable and could not be adequately expressed in words, becomes expressible thanks to the logos. The word "effable" hints also at the broad discussion on "commensurability" that troubled the world of the Pythagoreans.

knowledge of the procedure; according to Toth's indications, our aim was to retrace the procedure in order to feel, at least partially, the "originary sensibility"[25] that characterized the Pythagorean search.

3.1 The comparison

Consequently, differently from how we acted time ago [Rottoli & Riva, 2000], when we proposed the anthyphairetic procedure in some classrooms of a high school, in our present project this procedure is not directly used. Instead we have tried to make operating the originary feature of this basic form of mathematical knowledge: the act of comparison is a pair of natural numbers. This feature becomes the starting point for teaching the concept of fraction in primary school.

To this end, we have proposed to the children of two third classes of primary school, numerous and diverse activities of recording the act of comparing two homogeneous quantities by a pair of natural numbers. The children have worked with discrete and continuous quantities. As regards the activities with discrete quantities, the decisive choice of the teacher in order to motivate the act of comparison, has been to associate it to a game, the game of multiplication tables.[26] As regards the comparison of continuous quantities, we have made use of the activities with water, proposed by Davydov. The children have represented all the comparisons by squares or segments and by a formula of the type A;B = 13;8 : "the comparison between the quantities A and B is the pair of numbers 13;8".

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[25] The wording "originary sensibility" is taken from Roth and Radford, 2011, but it has here a different nuance of meaning. While in Roth and Radford its interpretation is "achieved as part of a categorical reconstruction of the human psyche on evolutionary grounds", here we give it the meaning of sensory experience obtained by retracing step by step the actions that constitute the process of the anthyphairetic comparison. It is this sensibility that, in our case, makes effable the "originary". According to the philosophical reflections about the "arché" (see note 20), the "originary" as substantive, is identifiable with the true substance and is referable only to the absolute One, which is "ineffable". What becomes "effable" is its adjectival transformation, that is its transformation in being, linked to the bodily-historical sensibility.

[26] The teacher prepares a special deck. A multiplication is written on each card. The class is divided into two groups. The teacher plays a card and reads the multiplication. The group who first gives the product wins a candy that is put in the basket of the group. If the answers of both groups are almost simultaneous, each group wins a candy. At the end, there is the comparison of the candies won by each group.

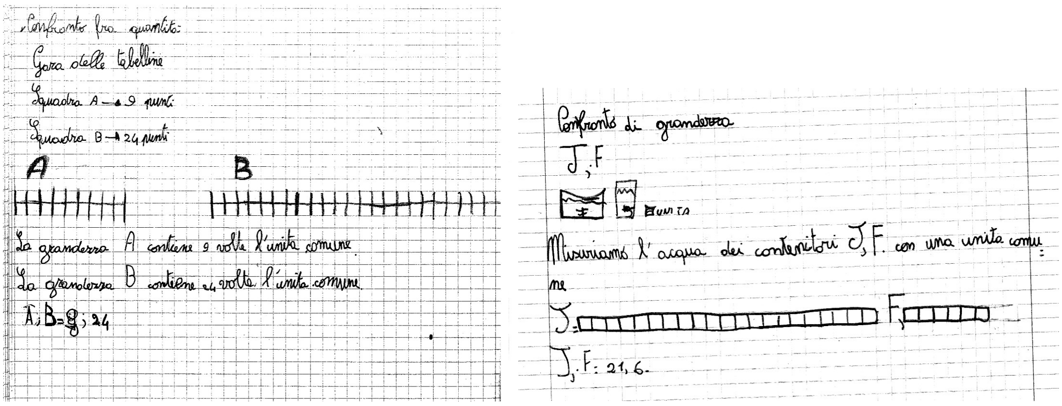

Figure 1. Examples taken from the exercise books: on the left the comparison between discrete quantities; on the rigth the comparison between continuous quantites

In order to understand the contribution that the historical reflection has given to our project, it may be interesting to compare the evocative/indicative meaning of the word "logos",[27] used in our approach, with the formalized meaning of the term "ratio" used by Lachance and Confrey.[28] In their introduction of the concept of fraction, starting from the subconstruct ratio, they look in the direction of the "broader" subconstruct, which would contain all the other subconstructs. We refer instead to history in order to "e-vocate", by an endeavour to listen to "originary sensations", and to investigate towards "indications" we receive from this listening.

3.2 The measure

[27] Leopardi, in the "Zibaldone", underlined the difference between the meanings of "word" and "term". Differently from the scientific, rigid meaning associated with "term", "word" has evocative value: "evocative" because it brings to light some meanings that have belonged to other poetic contexts. We interpret "evocative" as opening to new directions of investigation.

[28] "Using multiple contexts and experiences, we hope to guide students to first explore and understand a broad construct such as ratio and use that understanding to explore more specific instances of that construct such as fraction, decimal and percent." [Lachance & Confrey, 2002]. Certainly the search in the direction of "the broader", characterizes many investigations of the twentieth century. But we hope, by the search in the direction of "originary sensations", to be entered in resonance with one of the many echoes that come from the world of originary.

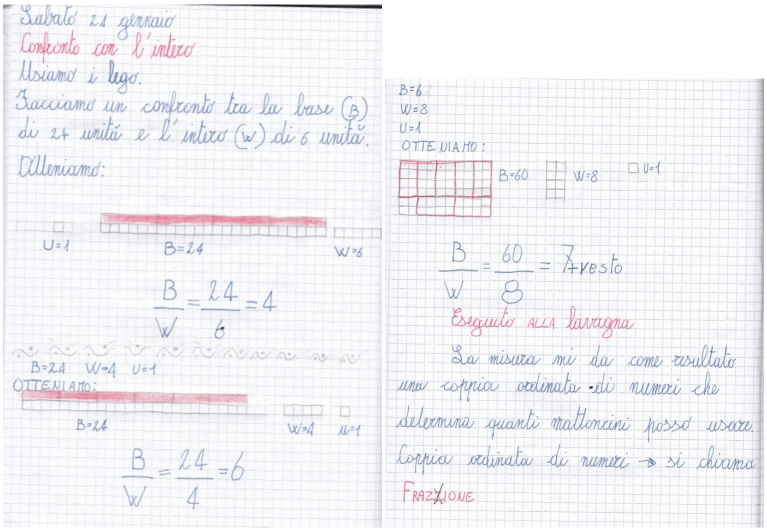

Figure 2. Examples of activities of measure

The procedure of anthyphairetic comparison highlights features that the usual process of measure leaves aside: the reciprocity between the two compared quantities and the search for a common unit. With regard to the role of reciprocity, we refer to what previously said. The search for a common unit characterizes the introduction of the concept of measure within our teaching process. Here the measure is defined as the comparison between a quantity and the "special" quantity called "whole". The term "unit" is reserved to indicate the common unit. The children yet make use of discrete and continuous manipulative: egg boxes, lego, packages of candies, picture cards; water, cakes, stripes of paper and so on. The special nature of the whole is highlighted by the special symbol "W" and by colouring. Also in this situation, all comparisons are represented by squares or segments. In order to write the comparison/measure, a special symbol is used: B/W = 25/6. This pair of numbers is the fraction: "The measure of the quantity B with respect to the whole W is the fraction 25/6".

3.3 Euclidean Division as core of the didactic activity

An important achievement of our proposal is that the children arrive naturally to write, already in the third grade, the formula of Euclidean division: Z/W= 16/5 = 3+1/5.

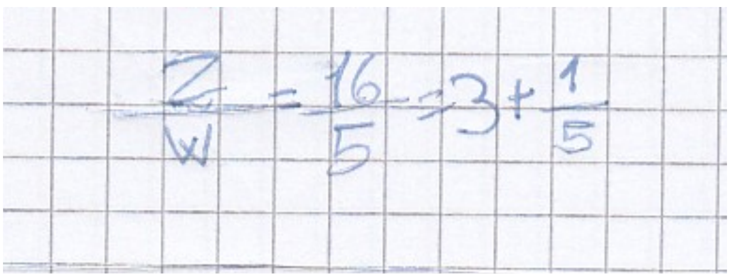

Figure 3. Euclidean division in a exercise book.

This achievement completes the process of mathematisation concerning the didactics of fraction: it starts by associating the act of comparing with a pair of natural numbers and is realized in the Euclidean division. Thanks to this approach, the Euclidean division is experienced by the children not as a formula to be memorized, but as the icon of their active process of learning.

This mathematisation, which differs from modelling,[29] characterizes itself as elementary and fundamental. It is elementary because turned to "the originary elements", but also by reason of the "lightness" [Calvino, 1988] with which the children have lived these activities: their answer has been quiet, serene, with adequate results. Il is fundamental because it determines the structure of the new universe of fractions; a structure that has the Euclidean division as core.

As the Euclidean division is the core of the universe of fractions, their teaching follows a particular proceeding: teaching is not a linear path of accumulation of consecutive learning; it is composed of many feedback loops: they start from the core, immerse in the contexts that are characterized by the different subconstructs, and return to the core. It is in this sense that we affirm that all "subconstructs of the construct of rational number", have their roots in the Euclidean division and find unity in it.

4 Concluding remarks

"In classroom activities, the children's answer was light, that is quiet, serene, with adequate results."[30]

The teachers have found many difficulties: only 10% continued the started activity. Notwithstanding common manipulative are used mostly, the teachers are asked to change their way of seeing and thinking, in order to discover objects, properties, operations that belong to the new universe. According to Thomas Kuhn[31], they are called for a change of "paradigm", for a "revolution" that upsets their habitual way of seeing and thinking fractions. This change should produce acquisition of awareness[32], reshaping the way of teaching proceeding[33], structuring the mutual interaction among teacher, class and research[34].

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[29] "Mathematisation" differs from "modelling": while "modelling applies a fragment of mathematics to a fragment of reality" [Israel, 2002], mathematisation has an "universal" meaning, because it guides the structuring of the universe of fractions as a new universe, distinct from the universe of natural numbers.

[30] Here some examples of adequate results: - Children, already in the third grade, consider the pair of numbers that form the fraction as a number (the number of packs). - Immediately and by themselves, children connect the fraction to the division. - In fourth grade children naturally know that in a decimal number (for example 3.7) 3 indicates the wholes and 7 indicates the decimal part. (d) Children can put themselves properly in front of some problems containing concepts not yet unfolded. In the experiment of the candle: Teacher: "We know that the oxygen constitutes 21% of the air; how much the water grows in the glass when the candle goes out?" "It is difficult to divide the container into 100 parts; we can divide it into ten parts." So children calculate roughly where the water comes. The topic "percentage" has not yet been presented in the classroom.

[31] We make an analogical use of the Kuhn's concept of "scientific revolution".

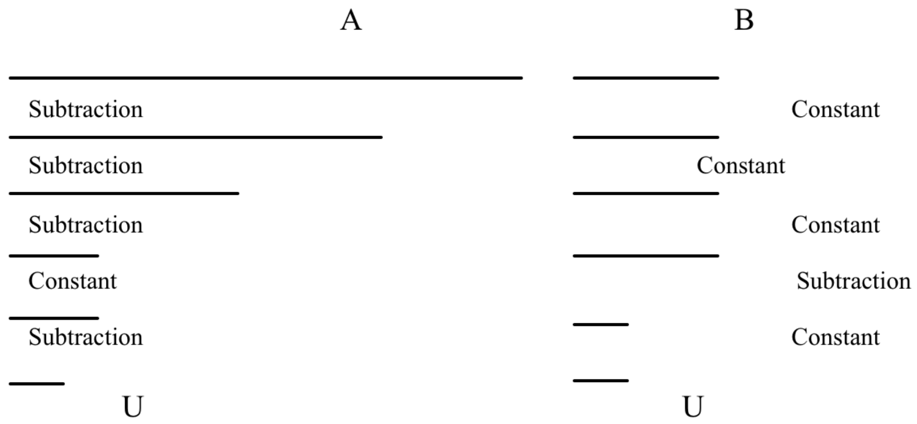
[32] A double awareness: the awareness that comes from the at least partial recovery of the "originary sensibility" that gives meaning to this mathematisation process; the awareness that some concepts are fundamental in "scaffolding" the universe of fractions.

[33] As mentioned above, the central importance of Euclidian division requires that didactic proceeding is structured in the form of feedback loops.

Appendix 1: Anthyphairesis

The anthyphairesis of two homogeneous quantities is a method of repeated removing and consists in subtracting the smaller of the two quantities from the larger one; after each removing, the larger quantity is replaced by the excess, while the smaller one stays unchanged. The process continues until the excess obtained by removing, is equal to the unchanged quantity.[35]

If you consider, for example, two segments, you can compare them in the following way:



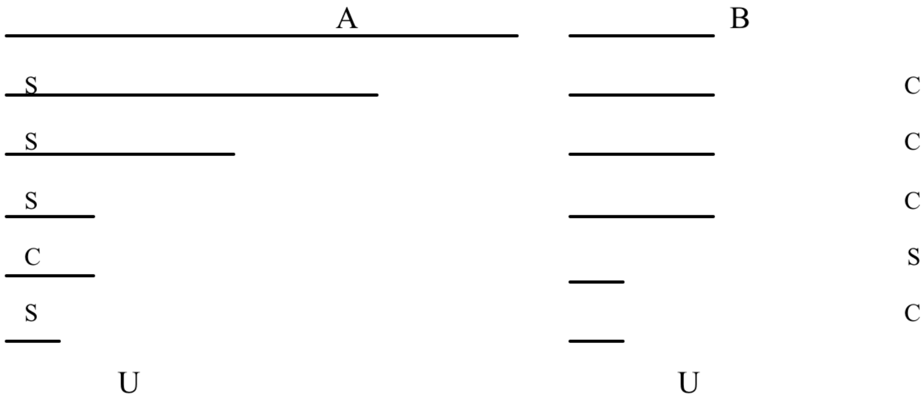
The result of the comparison (if the process terminates) is the unit U, common to both segments. Each segment contains the common unit a certain number of times: A = nU; B = mU.

[34] In the interaction between teacher and class, teacher listens to and is guided by the class in the development of teaching proposal; in the interaction between teacher and researcher, the effectiveness of the didactic path is continually rethought.

[35] The word anthyphairesis comes from the ancient Greek and its etymology is the following: anti–hipo–hairesis / reciprocal–sub–traction. Aristotle uses for the same procedure the term antanairesis: anti–ana–hairesis / reciprocal–re–traction. [Zellini, 1999]

Appendix 2: Logos

Our going step by step along the ancient procedure of anthyphairetic comparison, is characterized by the modern attitude of using symbols to indicate the action taken in each step. For example, the previous comparison between the quantities A and B looks like this:



The following chart uniquely represents the comparison:

S	C
S	C
S	C
C	S
S	C

To get how many times the quantities A and B contain the common unit, it is enough to turn upside down the chart and to retrace it, reading “S” as “Sum” rather than “Subtraction”:

1 U			1U
	S	C	
2 U			1 U
	C	S	
2 U			3 U
	S	C	
5 U			3 U
	S	C	
8 U			3 U
	S	C	
11 U			3 U
A			B

So A = 11U and B = 3U

The comparison has produced the chart, which may be expressed by the binary string SSSCS, and the pair of numbers (11;3). The binary string is a “physical” language for rational numbers. The pair of numbers is the “logos”.

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