THE CONCEPT OF MATHEMATICAL COGNITIVE TRANSGRESSION IN EXPLORING LEARNERS' COGNITIVE DEVELOPMENT OF NONDETERMINISTIC THINKING

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ABSTRACT

The concept of 'mathematical cognitive transgression' (MCT) has been introduced by Zbigniew Semadeni (2015) to investigate mathematics development from phylogenetic as well as ontogentic perspective. The author discusses many examples of the concept of MCT but none of them is directly connected with the development of nondeterministic thinking. Careful analysis of historical development of probabilistic and statistical concepts which led to hypotheses concerning regularities of cognitive development of probabilistic thinking (Lakoma 2000) became a point of departure of my current research in an area of understanding randomness and probability – based on an overview through the lens of the concept of MCT. The aim of this article is to discuss how the concept of MCT can be helpful in recognizing main levels in learners' cognitive development of probabilistic and statistical thinking and to show that it can be useful in constructing an approach to teaching and in arranging an effective process of learning.

1 Introduction

In recent years we are witnessing and participating in civilization changes progressing at enormous pace. They are related in particular to an omnipresence of information and communication technology in every field of science, technology and social life. Mathematics is a natural support for technological environment, therefore areas of human activity are structured in a formal way, in order to be suitable for explorations based on mathematics and information technology applications. This implies strong social demand for getting mathematical knowledge than ever before.

Students, in their process of learning, must have an opportunity to develop mathematical thinking in comprehensive way, to shape their own skills to mathematical modelling of phenomena and to be able to communicate by means of mathematics language. Students must also gain readiness to self-learning and using mathematical skills in future adulthood and professional life.

Recently randomness, probabilistic and statistical methods are present in methodologies of many various scientific disciplines, i.e.: physics, biology, economy, linguistics, history, psychology, engineering, information technology, and their applications. Therefore developing nondeterministic thinking is demanded as one of fundamental aims of education at all levels.

On the other hand, observing ways of probability and statistics learning in today classroom provides with many examples of students' difficulties and epistemological problems, even at the level of higher education. Learning probability theory and statistics seems to be more difficult for students than other parts of mathematics. These difficulties have often their roots not only in nondeterministic aspects of reasoning but also in misunderstanding of other fundamental mathematical concepts.

Creating a device for recognizing some symptoms of understanding mathematical concepts and for diagnosing student's cognitive difficulties could be very helpful for a teacher in supporting a process of students' learning as well as in constructing a didactical approach to teaching, more effective and more suitable for students, from their cognitive development point of view.

A useful support in this area of investigations seems to be obtained on the one hand by exploring the historical ways of mathematical thinking and – on the other hand – by careful studying students' individual ways of mathematical reasoning.

In this paper, after presenting the concept of mathematical cognitive transgression and its examples given by Semadeni (2015), I will present current results of my research in an area of randomness and probability – based on an overview through the lens of the concept of mathematical cognitive transgression.

2 The concept of mathematical cognitive transgression

The concept of 'mathematical cognitive transgression' has been introduced by Zbigniew Semadeni (2015) to investigate mathematics development from phylogenetic as well as ontogenetic perspective. The idea to introduce this concept to mathematics education was an effect of the author's inspiration by the book of Jozef Kozielecki (2004) who defined the concept of 'transgression' in the field of social psychology.

The mathematical cognitive transgression (MCT) is defined as an act of exceeding some previously existed cognitive limitations of knowledge which makes an individual or scientific community able to gain a new knowledge and to 'open the door' to think and reason at the more advanced and mature level than it was before (Semadeni, 2015). This transition is possible to appear as a result of active actions. The author argues, that MCTs expose some momentous changes in mathematics development. They consist in overcoming limitations of possessed prior knowledge and can be recognized in the history of mathematics when following ways of reasoning of people in the past (phylogenetic aspect) – or today when observing individual's process of mathematical development (ontogenetic aspect). Usually a MCT is an effect of long-term reasoning and investigations of a person or a group of scientists or learners. Sometimes it can be revealed suddenly (Semadeni, 2015).

However, in the author's opinion, in order to be able to judge a given transition as MCT the following necessary conditions must be fulfilled: this transition concerns a broadly defined mathematical idea; the difficulty of this transition is inherent, lies in the nature of structure of this idea, and is directly linked to it how a person or community recognizes this idea; this transition strongly counts as important for development of this idea and for other concepts related to it; this transition leads from a lower level into a new level, higher and well defined; this transition is a result of an action - conscious although possible to be unintentional - of a person or scientific community to get to know something new, to understand something that was not yet possible to understand; this transition is accompanied by a kind of surprising cognitive dissonance (Semadeni, 2015).

Semadeni (2015) discusses many examples of MCTs but none of them is directly connected with a development of nondeterministic thinking. In the next part of this paper I will present these examples of MCTs, analysed by Semadeni, which are not directly related to probabilistic concepts, but their lack can be an important difficulty for students in learning probability and statistics.

3 Examples of mathematical cognitive transgressions

Zbigniew Semadeni in his work (Semadeni, 2015) presents and analyzes several examples of mathematical cognitive transgressions. Among them I have chosen those that seem to be related with ability to construct models of random phenomena.

When we follow and analyse students' mathematical reasoning we are able to recognize often easily a lack of some sorts of MCT among students' solutions containing errors or misunderstandings. For a teacher who carefully observes a student's process of learning, there is an opportunity to pose a diagnosis what kind of support a student needs in order to be able to experience a transgression which is necessary to reason correctly at the more mature level.

3.1 The transition from processes to objects represented by symbols

From the ontogenetic point of view, there is possible to observe many examples of so called mini-transgressions in the process of children mathematical thinking. For example: the symbol ‚3+4' has two basic meanings depending on the context: as a process of adding two numbers: 3 and 4 or as an object which is a result of this process and is represented by this symbol: ‚3+4'. Children initially are able to recognize this symbol just as a task to calculate; when they become able to treat this symbol ‚3+4' as an object we can agree that they already experienced the MCT under consideration (Semadeni, 2015).

This kind of transgression appears not only in arithmetic, in algebra or mathematical analysis, but also in probability theory and statistics. For example, one of the most fundamental notions in probability theory – the distribution function which is defined as a function whose set of arguments is the set of real numbers and a function is defined for each real argument x as the probability that the random variable X has values smaller than an argument x – is treated by students as one of the most difficult notions. One of reasons of this feeling seems to be a lack of transgression 'from processes to objects represented by symbols': there are many students, even at academic level, who when considering this function:

for each real number x: $F(x) = P(X < x)$

try at first 'to solve' inequality inside parentheses and they are not able to understand this formula as an object represented by symbols.

Another example of a lack of this MCT we can meet when exploring students' solutions of probabilistic problem of 'waiting for the first success': two players shoot to the target in turns; who will reach the target as the first is a winner and the game is over; calculate probability to win for the player who begins this game and probability to win for the second player. Frequencies of getting a target in one trial are the same and equal to 0.5.

A solution of this problem leads to calculations of a sum of the following series:

for the first player: $\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\cdots$ for the second player: $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots$

Even if students are able to present the solution in the formulas above, they are very often not able to find the correct answer; they try to add numbers but they cannot see these formulas as objects and consider both series from this perspective: some of students are able to compare appropriate components and to notice that in each pair the number 'belonging to the first player' is two times greater than the number 'belonging to the second player', so the whole sum expressing a probability to win by the first player is two times greater than probability for the second player, thus probability to win for the first player is 2/3 and for the second one: 1/3.

Experiencing by a student the MCT 'from processes to objects represented by symbols' gives an opportunity to solve this problem in very simple way. The problem presented above is also connected with another example of MCT – related to the concept of infinity.

3.2 Transgressions related to infinity in its ‚increasing' aspect

The transition from large numbers, such as a thousand or million, to a consciousness that for each number it is always possible to give even bigger number is a mental process which was carefully described by Aristotle and is known as ‚potential infinity'. Aristotle also distinguished the ‚actual inifinity' which means that the process of infinitely many steps is completed, and is realized in the form of a single act (Semadeni, 2015).

Aristotle distinguished two types of the ‚potential infinity': inifnity in view of adding – increasing and infinity in view of dividing. Understanding the ‚actual infinity' is a symptom of experiencing the MCT. In the example above we can observe both kinds of the ‚potential infinity': when students try to summarize numbers and to find a sum of series, also when students try to use a geometric representation of these calculations – to present numbers as appropriate parts of a geometric figure whose area is equal to 1. Anther example of the MCT related to infinity is an ability to treat a set of all natural numbers as a single object which is a final result of an endless process of increasing the number of objects (Semadeni 2015). My observations of students' efforts when they use basic probability distributions as the models of random phenomena reveal often in students' reasonings a lack of this kind of MCT when considering sets of natural numbers.

3.3 Transgression from reality perceptible by senses to ideas perceived by the intellect

Semadeni (2015) presents this kind of MCT from the phylogenetic perspective: he evokes historical mathematical reasoning based on the concept of symmetry, Platon's distinguishing reality and ideas, and Euclid's considering abstracts only. On the other hand, when we take into account the ontogenetic perspective, we can give several examples of students' ways of thinking in which problems with distinguishing empirical observations of considered phenomena and reasoning related to their mathematical models are evident. It seems that a lack of this kind of MCT is one of the fundamental problems when students learn probability and statistics. These problems can appear even in very unexpected moments. For example, students at academic level (management field of studies) had to solve the following probabilistic task: to calculate an expected value of discrete random variable X which has three values: -1, 0, 1 with appropriate probabilities: 0.1, 0.4, 0.5. The following solution (Fig.1) reveals these problems: a student at first recognized this task as theoretical one (in a model), calculated the expected value of this random variable, but afterwards he decided to treat the values of the random variable as statistical data with frequencies equal to given probabilities and to calculate an arithmetic mean, but when he tried to obtain it, he reminded that these values could be equally frequent and he divided the result obtained into 3 (three values).

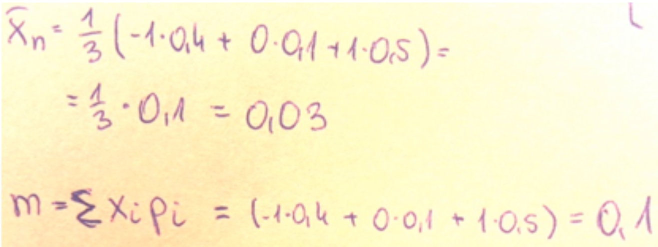

Figure 1. Student’s incorrect calculation of the expected value of discrete random variable.

Another student’s solution of the problem above is an example of sophisticated combination between the notion of arithmetic mean and the notion of expected value (the sum of probabilities is eual to 1, like number n – total amount of data in the formula of arithmetic mean (Fig.2), even the symbol of expected value is borrowed from arithmetic mean).

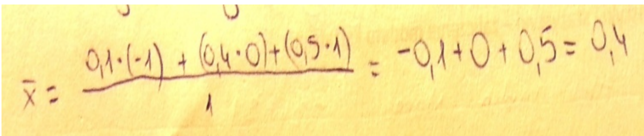

Figure 2. Student’s calculation of the expected value of discrete random variable.

It is worth to notice, that this kind of reasoning is not rare among students, and strongly needs to be taken into account by teachers.

3.4 Transgression from axioms understood as certainties to the axioms understood as assumptions

Among several examples of MCTs, Semadeni (2015) has chosen the MCT which has a great importance in a field of probability theory. Semadeni argues, considering this transgression from the phylogenetic point of view, that until the discovery of non-Euclidean geometry mathematicians understood the term ‚axiom‘ as a ‚certainty‘, ‚self-evident truth‘, that means a statement which doesn’t require to be proved. This kind of axioms were fundamental for mathematical theories until the early twentieth century, such as Euclidean geometry or arithmetic of real numbers, arithmetic of natural numbers. In the twentieth century axiomatic theories appeared in which ‚axioms‘ are understood as ‚assumptions‘ – ‚postulates‘.

Kolmogorov's theory of probability belongs to this kind of axiomatic theories (Semadeni 2015). In the process of probability learning experiencing this transgression by a learner is very important for constructing his mental structure of probability concepts and models.

4 Searching for mathematical cognitive transgressions in the area of probability and statistics

Careful analysis of historical development of probabilistic and statistical concepts led to hypotheses concerning regularities of cognitive development of probabilistic thinking (Lakoma 2000). Following the opinion of Ian Hacking (1975) the history suggests that the concept of probability has a dual nature. Hacking distinguishes two aspects of this concept. One of them - epistemological – is implied by the general state of our knowledge concerning a considered phenomenon – related to the degree of our belief, conviction or confidence, which arise in connection to an argument – related to this phenomenon – and are supported by this argument. The other aspect – aleatory – is related with the physical structure of the random mechanism and with the admission of its tendency to produce stable relative frequencies of events. The first aspect gives the basis for 'chance calculus' and the other – for the 'frequency calculus' (Lakoma, 2000). The phylogenetic analysis shows that both these aspects became inseparable and pierced each other starting from the time of Pascal and Fermat. Before that time they were developed independently. Conscious sticking them together, subtle contrasting and verbalising, done by Pascal, Fermat and also Huygens, has caused an act of illumination in the way of probabilistic thinking. The history points that in order to acquire the probability concept it is necessary to make conscious its dual nature (Lakoma 2000), thus there is possible to pose hypothesis that making conscious the dual nature probability is an example of MCT. In time of Huygens, Pascal and Fermat there is also possible to notice that before that historical period, all probabilistic problems, which appeared, could be characterised by finite probability spaces. The first problem, which was solved, using an infinite probability space, was found in the work of Christiaan Huygens (1657). The history suggests another possible example of MCT: transgression from finite probability space to infinite probability space. However in this case there is a need of more careful explorations: distinguishing between discrete infinite case and continuous case modelled by continuous random variables.

Observations of students' solutions of probabilistic problems lead to important conclusion that students in their initial naive trials evoke the concept of symmetry of the random mechanism under consideration. This naive approach – using a model of even chances leads sometimes to surprising arguments. For example, I will present the solution of the following problem: The height of teenagers is modelled by random variable X which has a normal distribution $N(165, 15)$, that means the expected value is 165 and standard deviation 15. Calculate probability that two of randomly chosen teenagers will have their height above the mean. Figure 3 shows the following tree model of this problem: student (management field of studies) distinguished two steps of the random experiment, at each of them he distinguished three possibilities: height above the mean, height less than the mean, height equal to the mean. To two of them he assigned equal probabilities 82/165: 'below' and 'above' and to the third one: 'equal'- probability 1/165. The solution was led according to the rule of adding at multiplying probabilities. (Fig.3). It seems that the discrete model, and in particular the model of even chances (certainly except of the third possibility) was so strong as an device to model random phenomena that possibilities to model random phenomena by continuous random variables was rejected in this case.

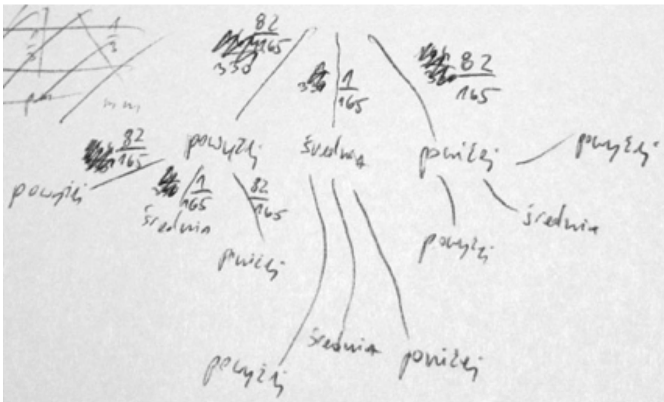

Figure 3. Student's incorrect solution of the example modeled by normal distribution.

4 Final remarks

The overview of probabilistic reasoning – in phylogenetic context as well as an ontogenetic - brings many examples of situations which reveal a presence of mathematical cognitive transgressions necessary for students to experience in the way of their process of developing mathematics. It is symptomatic that exploring students' ways of mathematical thinking there is possible to recognize a lack of some MCTs. Incorrect solutions often are not a result of inattention, they have roots which reveal deep misunderstanding of the fundamental mathematical ideas. Probability and statistics are these parts of mathematics which are especially sensitive of a lack of MCTs coming from different areas of mathematics and are necessary to use in tasks unexpectedly for a student. Therefore probabilistic and statistical problems can serve also as a tool to recognize the level of students' progress in mathematics and possible gaps necessary to fulfil.

REFERENCES

Hacking, I. (1975). *The emergence of probability*. Cambridge: Cambridge University Press.

Kozielecki, J. (2004). *Transgressive society, chance and risk* [in Polish: *Społeczeństwo transgresyjne, szansa i ryzyko*]. Warsaw: 'Zak'.

Lakoma, E. (1992). *Historical development of Probabilit*y [in Polish: *Historyczny rozwój pojęcia prawdopodobieństw*]. Warsaw: CODN-SNM.

Lakoma, E. (2000). Stochastics teaching and cognitive development. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education. The ICMI Study* (pp. 74–77). Dordrecht: Kluwer.

Semadeni, Z. (2015). Cognitive transgressions as an essential feature of the development of mathematics [in Polish: Transgresje poznawcze jako istotna cecha rozwoju matematyki]. In R. Murawski (Ed.), *Philosophy of mathematics and computer science* [*in Polish: Filozofia matematyki i informatyki*] (pp. 65 –90). Cracow: Copernicus Center Press.