ENACTING INQUIRY LEARNING *IN* MATHEMATICS THROUGH HISTORY[1]

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ABSTRACT

We explain how history of mathematics can function as a means for enacting inquiry learning activities in mathematics as a scientific subject. It will be discussed how students develop informed conception about i) the epistemology of mathematics, ii) of how mathematicians produce mathematical knowledge, and iii) what kind of questions that drive mathematical research. We give examples from the mathematics education at Roskilde University and we show how (teacher) students from this program are themselves capable of using history to establish inquiry learning environments in mathematics in high school. The realization is argued for in the context of an explicit-reflective framework in the sense of Abd-El-Khalick (2013) and his work in science education.

1 Introduction

The Rocard (2007, p. 12) report emphasizes inquiry-based teaching methods as a means to increase students' interest in mathematics and science. Inquiry-based teaching aims at inviting students into the workplace of scientists and mathematicians. The idea is that if students are engaged in activities and learning processes similar to the way scientists and mathematicians produce knowledge, the students will develop a deep understanding of science and mathematics, as well as of the epistemology and more broadly the nature of these subjects. Problem solving and mathematical modelling play a major role in inquiry-based mathematics education. In such activities the focus is often on mathematics as a tool for solving problems outside of mathematics – not on the production and validation of mathematical knowledge. In the present paper we point towards history of mathematics as a means for enacting inquiry learning activities in which students gain experiences with research processes in mathematics. To be more specific, how students through history can develop informed conception about i) the epistemology of mathematics, ii) of how mathematicians produce mathematical knowledge, and iii) what kind of questions that drive mathematical research.

The analyzes take point of departure in an existing study program in mathematics at Roskilde University (RUC) in Denmark, where students get to work with history of mathematics through problem oriented project work. We present some of the students' project work and discuss in what sense these projects fulfil the intentions of inquiry-based learning in

[1] Some of the content of this paper is also presented in Kjeldsen (2014). It appears here with the permission of the journal.

mathematics as a scientific subject. We discuss and explain the realization of the problem oriented project work in history and philosophy of mathematics in the context of an *explicit-reflective framework* in the sense of Fouad Abd-El-Khalick (2013) and his work in science education. Finally, we illustrate by an example how (teacher) students[2] from this program are themselves capable of establishing inquiry learning environments *in* mathematics in upper secondary school by using history – and point to some of its benefits for high school teaching.

2 Inquiry teaching in mathematics through history of mathematics

How can students in higher mathematics education get first-hand experiences in a meaningful way with research practices in a field that is so specialized and operates with such abstract notions as mathematics? Usually, students are not invited into a research environment until they become Ph.D. students. There is not much help to gain from their textbooks. Mathematics textbooks very rarely discuss or indicate where the mathematical objects they are dealing with come from, why they look the way they do and why they are interesting. On the contrary, mathematical objects are usually introduced as timeless entities that appear in textbooks seemingly out of nowhere. Most of the mathematics that students are introduced to at bachelor level and in the first year of master programs has not been developed recently. For most students, observing how mathematicians work when they do research will not be a feasible way to gain insights into inquiry processes in mathematical research. However, students can examine such processes of how and why mathematicians have generated the ideas, constructed the concepts and produced the mathematics they read about in their textbooks by studying the history of the content matter of mathematics. In the following we will illustrate and discuss how students in the mathematics program at RUC through project work in history of mathematics are challenged to reflect explicitly upon concrete aspects of the nature of mathematics like the ones listed in the introduction.[3]

3 The PPL implementation of history in higher education in math at RUC

History is implemented in the mathematics program at RUC through Problem-oriented Project Learning (PPL). PPL is the pivotal pedagogical principle underlying the organization of *all* the study programs at RUC (Andersen & Heilesen (2015). It refers to the principle of problem-oriented participant-directed project work as it has been developed and is practiced at RUC. Common for all study programs is that in each semester the students use half of their time on regular course work. The other half of their study time they work in groups of two to eight students on a project which objective is to solve (or analyse or formulate or make solvable) a problem of the group's own choice under supervision of a professor.

The students document their project work during the semester through outlines and discussion-papers produced by the students. These are discussed and planned for at weekly meetings with the group's supervisor. The group writes a coherent report of 50-100 pages, in which the students state and argue for their problem and its relevance, their methodology and

[2] In Denmark, upper secondary mathematics teachers have a university degree in mathematics.

[3] See also Barnett, Pengelley and Lodder (2014) for uses of history in higher mathematics education in the USA.

its validity, their choice of theories, experiments (in the science program students often design experiments), data-collections, analyses, results and conclusions. They make a critical evaluation of their results. The students consult textbooks, research literature and experts in their field(s) of investigation. The project work is evaluated at an oral group-examination with an external examiner. All the students in a group are responsible for every aspect of the project work. They are supposed to be able to reflect explicitly and concretely about how their analyses and solutions depend on their methodology including their choices of theory and possible experiments. The students are in charge of the project work from beginning to end, supported by their supervisor and various milestones throughout the semester.

The students have opportunities to work with history of mathematics at the bachelor level and at the master level. However, we will only describe the study regulation for the master level, since it also fulfils the requirements for the relevant projects at the bachelor level. The project at the master level is defined by the theme: 'mathematics as a scientific discipline'. In the study regulation it is described in the following way:

> The project should deal with the nature of mathematics and its "architecture" as a scientific subject such as its concepts, methods, theories, foundation etc., in such a way that the nature of mathematics, its epistemological status, its historical development and/or its place in society gets illuminated.

Every mathematics student at RUC at the master level will work on a project that fulfils these requirements. In the next section we will we present two of them in more detail to illustrate how the students 1) gained experiences with inquiry processes that resemblance how mathematical research is done, and 2) developed informed conception of the production and validation of mathematical knowledge.

4 Examples of Students' History of Mathematics Projects at RUC

The list of selected titles in Figure 1 gives an idea of the variety of subjects and issues the students have dealt with in the PPL history of mathematics project over the years at RUC.

The Contribution of Galois to the Development of Abstract Algebra (1986)
Euler and Bolzano: A mathematical analysis in an epistemological perspective (1993)
The Intuitionistic Mathematics of Hermann Weyl (1995)
Hamilton's Quaternions – an assessment of their applicability (1996)
Paradoxes in Set Theory and Zermelo's 3th. Axiom (1996)
$\sqrt{-1}$ *a multiple discovery?* (1997)
Cayley's Problem: a historical analysis of the work on Cayley's problem 1870-1918 (1998)
What Mathematics and Physics did for Vector Analysis (1999)
Generalizations in the Theory of Integration (2001)
Euler's Differential Calculus compared with the modern one (2000)
Hilbert's Philosophy of Mathematics (2001)
The Solving of Equations Algebraically from Cardano to Cauchy (2002)
Fourier and the Function Concept: From Euler's to Dirichlet's concept of a function (2003)
Foundations of Mathematics – assessed using Euclidean and non-Euclidean geometry (2003)
D'Alembert and the Fundamental Theorem of Algebra (2003)
Infinity and 'Integration' in Antiquity (2004)
The Real Numbers – Constructions in the 1870ties (2004)
Physics' Influence on the Development of Differential equations (2005)
Haar's contribution to generalized Fourier theory (2005)
Holomorph Dynamics – a Historical Perspective (2010)

Figure 1. List of titles of students' history of mathematics projects and its year

In the following we will present two of these PPL-projects in more detail. They illustrate how the students through history 1) gained experiences with inquiry processes that resemblance how mathematical research is done, and 2) developed informed conception about: i) the epistemology of mathematics, ii) how mathematicians produce mathematical knowledge, iii) what kind of questions that drive mathematical research.

4.1 Generalizations in the Theory of Integration: An Investigation of the Lebesgue Integral, the Radon Integral and the Perron Integral

The 75 page long project report Generalizations in the Theory of Integration: An Investigation of the Lebesgue Integral, the Radon Integral and the Perron Integral was written by two students who were curious about the "need" in mathematics for further integrals than the Riemann integral. In the textbook of their first analysis course they had stumbled over a footnote in which it was pointed out that there are other types of integrals e.g., the Lebesgue integral. The students soon realized that there are many integrals, e.g. the Denjoy, the Perron, the Henstock, the Radon, the Stieltjes and the Burkill integral. They wrote:

All these integrals are most often described in the literature as *generalizations*, and sometimes as *extensions*, of either the Riemann or the Lebesgue integral. This gave rise to questions such as: What do these integrals do? Why have so many types of integrals been developed? Why is it always the Lebesgue integral we hear about? What is meant by generalizations in this respect? In what sense are the various integrals generalizations of former definitions of integrals? Are the generalizations of the same character? (Timmermann and Uhre, 2001, p. 1, italic in the original)

The students chose to investigate these questions by looking into the history of mathematics. Supported by literature from historians of mathematics, the students traced and read mathematical papers and books of Lebesgue, Perron and Radon in which they developed, or worked with mathematical ideas that motivated them to develop the integrals that bear their names. The students analysed and compared the sources with respect to 1) the motivation of the mathematicians, 2) why they created these generalized integral concepts, 3) the differences and similarities between the characteristics and scope of the generalizations.

The students based their investigations of Lebesgue's motivation and development of his integral concept on some of his notes in *Comptes Rendus de l'Académie des Sciences de Paris* and his thesis *Intégrale, Longueur, Aire*. The students found that Lebesgue's search for a new integral concept primarily was motivated by the (lacking) symmetry in Riemann's integral concept of what Lebesgue called 'the fundamental problem of integral calculus', i.e. to find a function when its derivative is known. The students based their argument on Lebesgue's own words and on their analysis of the first part of his thesis, where the students found that the theorems were organized in a hierarchy of theorems. (p. 33-34). They concluded that for bounded functions, the Lebesgue integral is an extension of the Riemann integral, but that the Lebesgue integral is not the same as the Cauchy-Riemann integral (improper integral) for unbounded functions. Lebesgue's integral is based on generalization of the measure concept. For Perron, the students found that he introduced a new definition of the integral, which he found to be more elementary than Lebesgue's. It was based on the concepts of upper and lower adjoint functions for a bounded function *f* defined on an interval [*a*, *b*]. The adjoint functions can be interpreted as approximating anti-derivatives of *f* . Perron's integral is then based on the supremum and infimum of their values at *b*. The students concluded that Perron with "elementary" meant the avoidance of measure theory and they discuss his integral concept with respect to this. They argued that Perron's integral is more abstract than Lebesgue's e.g. the intuitive conception of the integral as an area or a measure is lost, and the definition does not provide a method for constructing the integral. However, as they wrote, it has the didactical advantage that the definition is based on the anti-derivative. Regarding Radon's motivation for generalizing the integral concept, the students concluded that he needed it for his work with integral equations. These insights into mathematicians' motivations for defining a new or extending and existing integral concept made them realize that "There have been different motivations associated with the development of integral concepts [..]. This illustrates that […] there may well be several reasons involved in the creation of new mathematics. This also illustrates that mathematics does not necessarily evolve along a beaten path and that perhaps it is only in hindsight that a certain approach appears to be the most natural." (p. 57)

On the one hand, by focusing on understanding the mathematicians' motivation for generalizing the integral concept when reading the historical sources, the students became engaged with mathematical inquiry analogue to some research processes in mathematics as they are carried out by working mathematicians. The students' work was guided by historical and philosophical questions. They answered these questions through analyses of the mathematical content, definitions, theorems, proofs and techniques that were stated, treated and worked out in the sources they consulted. In this way, the students gained first-hand experiences with research processes in the production of mathematical knowledge by studying the masters, so to speak.[4] On the other hand, by focusing on analysing and identifying the characteristics of the various generalizations of the integral concept, the students came to reflect upon their inquiry investigations from within an epistemological framework. This part of the students' investigations structured their critical reflections about the function, assessment and significance of the generalizations of the integral concept.

4.2 The Real Numbers – Constructions in the 1870ties

The project on the constructions of the real numbers was carried out by a group of six students. They wrote a report of 56 pages in which they answered the following question:

Why did a need for a construction of the real numbers emerge around the 1870ties?

These students were puzzled when they realized that mathematicians had worked with the real numbers on an intuitive foundation for years and years before it (the foundation) 'all of a sudden' became a problem that needed to be solved. The students explained their own motivation in their report as follows (Wandahl et. al., 2004, p. 1-2):

4 A warning needs to be issued here: Reading sources does not in itself guarantee that students will gain experiences with research processes in mathematics. For this to happen, they need to study historical processes.

> Basically we had the idea that one mathematician saw that it was a problem that a construction of the real numbers was lacking, solved it and presented it [the construction of the real numbers] as a solved problem. However, we quickly discovered that this was not the case. There was not just one mathematician who believed that there was a problem, but several. This indicated that the lack of a construction of the real numbers had become a problem in connection with developments of mathematics up to then.

The students' motivation and their research question (the problem that guided their project work) are concerned with a fundamental aspect of the nature of mathematics as a scientific subject, namely what do mathematicians wonder about? How do the problems that mathematicians struggle to solve emerge? How are they connected with ongoing research in mathematics? When, why and by whom are they deemed so important, that they become essential problems of the field in need of a solution? These questions structured the students' analyses of the mathematical sources and their reflections of how mathematical knowledge was generated in this particular episode in the history of mathematics.

To place the discussion in the historical context, the students studied the paper by Bernard Bolzano from 1817: *Rien analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegensetztes Resultat gewären, wenigstgens eine reelle Wurzel der Gleicheung liege* in which he proved the mean value theorem, and discussed methodological and epistemological issues in mathematics. The students considered Blozano's work as an indication that such concerns were present in mathematical research during the nineteenth century. In order to answer their own research question for the project work, they chose to investigate the work of Dedekind and Cantor. They realized that these two mathematicians had different reasons for constructing the real numbers with Dedekind realizing a need for a rigorous construction in a teaching situation, whereas Cantor ran into the problem of the construction of the real numbers in his work with trigonometric series.

In order to place their analyses of the historical sources in a broader historical context, the students discussed the various conceptions of the arithmetization of analysis in the 1870'ties based on the work of historians of mathematics. They realized that there were various conceptions of mathematics among mathematicians in the 1870'ties. With reference to Epple (2003) they discussed their analyses and conclusions with respect to the traditional conception, the arithmetical conception and the formal conception. This categorization made the students realize that what counts as a validation or knowledge in mathematics depends on these conceptions. They quoted Dedekind: "For myself this feeling of dissatisfaction [with geometrical arguments] was so overpowering that I made the fixed resolve to keep meditating on the question till I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis" (p. 48).

The students' were invited into the mathematical 'workbench' of Bolzano, Dedekind and Cantor among others in this project work. Through their work with following the historical development, they became engaged with mathematical inquiry. The requirement for the "mathematics as a scientific subject"-project and the specific problem-orientation provided opportunities for the students to reflect upon their inquiry investigations of the mathematical content of the sources from within an epistemological framework of how this particular mathematical knowledge was generated, validated and discussed at the time.

The two project reports illustrate how the students acquired informed conceptions about the nature of mathematics coming from different perspectives: In the first project, the students obtained concrete experiences with processes of forming new concepts in mathematics with reference to already existent concepts through generalizations. They investigated how different processes of generalization can be distinguished with respect to whether the new concept, that is generated, is an abstraction of the already existent one or whether the new concept is an extension in the sense, that it contains the concept it is a generalization of. In the second project, the students experienced how problems of foundational and epistemological nature arose with respect to the foundation of the real numbers, how this generated new research in mathematics and gave rise to discussions and debates among mathematicians – aspects which are important parts of mathematical inquiry, but rarely occur in traditional teaching. However, as the examples illustrate, such aspects can be brought forward and made explicit objects of students' reflection through history of mathematics.

5 The significance of an explicit-reflective framework

Research from science education has documented that "while inquiry might serve as an ideal context for helping students and teachers develop informed NOS (nature of science) views, it does not follow that engagement with inquiry would necessarily result in improved understandings." (Abd-El-Khalick, 2013, p. 2089). As Abd-El-Khalick and Lederman (2000) have argued for the development of NOS, an explicit-reflective framework is required in order to achieve such understandings. By 'explicit' they mean that some specific learning objectives related to students' understanding of NOS must be included in the curriculum; by 'reflective' they refer to instructions aimed at helping students to reflect upon their experiences with learning science from within an epistemological framework.

The problem orientation of the project work at RUC together with the study regulation of the 'mathematics as a scientific subject' guaranties that the students work with a specific problem that address (aspects of) the nature of mathematics, in such a way that it is anchored in the subject matter of mathematics (its concepts, methods, theories, foundation etc.). Through this anchoring, the students' reflections become contextualized and concretized. The problem orientation together with the curriculum description provides structured opportunities for the students to reflect upon learning experiences from within an epistemological framework. The students gain their 'mathematics as a science' learning experience from the concrete attachment of their problem within the subject matter of mathematics. The aspects of the nature of mathematics that their problem is addressing provide the epistemological context in which the students examine their 'mathematics as a science' learning.

It goes without saying that teachers in order to be able to enact such learning environments must themselves have informed conception of the nature of mathematics and (processes of) mathematical research. The question now is whether students from the PPL program in mathematics at RUC are capable of designing and implementing such learning environments. In order to discuss this we use the notions of teaching *with* and teaching *about* the nature of mathematics in the sense of Abd-El-Khalick for teaching *with* and *about* NOS:

> Teaching *about* NOS refers to instruction aimed at enabling students to achieve learning objectives focused on informed epistemological understandings about the generation and validation of scientific knowledge and the nature of the resultant knowledge. […] Teaching *with* NOS entails designing and implementing science learning environments that take into consideration these robust epistemological understandings about the generation and validation of scientific knowledge. (Abd-El-Khalick, 2013, p. 2090).

If we adopt this framework to mathematics, teaching *about* the nature of mathematics means to teach towards learnings objectives of the following kind: To make students able to investigate and analyze the methods mathematicians use to generate knowledge, to discuss, criticize and asses the epistemological status of this knowledge, to investigate and analyze the role of proofs in mathematics, to investigate and discuss mathematical objects' ontological status, to investigate and discuss the relationship between mathematics and other sciences, to discuss and critically asses the distinct nature of mathematics, as well as its development historically and in interaction with culture, society and other sciences.

In order to couple the development of students' informed conceptions about the nature of mathematics with inquiry teaching in mathematics in a meaningful way, mathematics teachers must 1) have knowledge and informed conceptions of and *about* the nature of mathematics themselves, 2) come to understand how inquiry is in fact conducted in mathematical research, 3) be able to design inquiry based learning environments *in* mathematics teaching, and 4) be able to teach *with* the nature of mathematics in the sense explained above in the case of NOS.

6 Teaching with the nature of mathematics through history in an inquiry learning environment in upper secondary school in Denmark

The nature of mathematics (NOM) is explicit in the mathematics curriculum for upper-secondary school in Denmark. One of the learning goals for mathematics is that the students should be able to demonstrate knowledge of how mathematics has developed in interaction with the historical, scientific and cultural development. Another learning goal is that students should demonstrate knowledge of the identity and methods of mathematics.

In this section we shall see an example of how a mathematics (teacher) student from RUC used her knowledge about the nature of mathematics that was formed within the explicit-reflective framework which was unfolded and illustrated above, to teach with and about the nature of mathematics in a Danish high school. She used her knowledge and experiences from her own mathematics education at RUC to enact an inquiry learning environment that invited her 17 year old high school students (11^{th} graders) into inquiry processes that bore some resemblance with authentic mathematical practices. She created the learning environment by having the students read excerpt from original sources from the development of the function concept. By setting up explicit learning goals for the students that addressed historical and mathematical questions related to the development of the function concept, and by having the students analyze the original sources to answer these questions, she designed a learning environment in which she taught both *with* and *about* the nature of mathematics. Altogether she spent 13 lessons of 50 minutes each with the students in the classroom – and the students were expected to spend an equal amount of time working on the tasks at home.

The intentions for the students' learning outcome reflect the explicit-reflective framework (Petersen, 2011):

> The students will acquire an understanding of what is involved in the modern definition. of a function, and the role of the concept of domain in the modern function concept.
>
> The students will come to reflect upon the concept of a function – what is a function?
>
> The students will acquire an understanding of our modern function concept as a result of a historical developmental process.
>
> The students' will come to reflect upon the role of proofs.[5]
>
> The students will gain insights into the role played by the human actors [former mathematicians] in the development of the function concept.

The teaching module was designed in two steps: In step 1 the students were divided into four so-called basic-groups that had specific but different tasks. Group 1 worked with various historical definitions of a function, group 2 worked with the debate of the motion of a vibrating string primarily between Leonhard Euler (1707-1783) and Jean le Rond D'Alembert (1717-1783), group 3 situated the main actors (Euler and Peter G. L. Dirichlet (1805-1859)) in their respective societies and the time in which they lived, and group 4 worked on the modern concept of a function. Each group received a working sheet from the teacher with explicit requirements for their work. Group 1, e.g., was given Danish translations of an extract from Euler's (1748) book *Introductio in Analysin Infinitorum* with the definition of a function, and an explanation of how Euler later in 1748 extended his original function concept, and an extract from Dirichlet's paper (1837) *Über die Darstellung ganz wilkürlicher Functionen durch Sinus- und Cosinusreihen*. Based on these three texts the students received the following task (Petersen 2011, Appendix B):

> Explain what a function is according to Euler's original definition of a function in *Introductio in Analysin Infinitorum*, Euler's extended definition of a function and Dirichlet's definition of a function. Describe how these three definitions of a function are different from each other and in what ways they are similar. Explain what the

[5] This learning objective is linked to the investigation of whether history can be used to exhibit meta-discursive rules of mathematics and make them explicit objects for students' reflections, which was also a part of the experiment (see Kjeldsen & Petersen, 2014).

> principle of the generality of the variable is all about and the relationship between this principle and the principle of the generality of the validity of analysis.[6]

The task was followed by eight questions which were meant both as a help for the students to 'de-code' the task and to make the requirements and the expectations of the teacher for the students' work more explicit, and to qualify the students' reflections and structure their work:

> (1) What are the central concepts in Euler's definition of a concept? (2) Which principle characterizes a variable according to Euler, and what is this principle called? (3) What is the principle of the generality of the variable all about? (4) What are the similarities between the principle of the generality of the variable and the principle of the generality of the validity of analysis? Consider why both principles have been given names that contain the word "general".[7] (5) How does Euler's extended concept of a function differ from his original concept, and what are the similarities? (6) Find three ways in which Dirichlet's concept of a function differs from Euler's definition. (7) Explain from text 3, what Dirichlet must have thought about the generality of the variable. (8) On page 10 there are four pictures. Which of these pictures are graphs of functions according to Euler's definition in *Introductio in Analysin Infinitorum*, Euler's extended definition and Dirichlet's definition respectively? (Petersen 2011, Appendix B)

In step 2 new groups were formed in such a way that each group had at least one participant from each of the four basic-groups. This meant, at least in principle, that all the knowledge that had been developed in the basic groups was present in each of the new groups. In contrast to the basic-groups, the new groups, also called the expert-groups, all worked on the same assignment. They were asked to write an essay about a fictional debate between mathematicians, where one part is claiming that mathematical concepts are static, timeless entities that exist independent of humans and society. In opposition to this, the other part reinforces that mathematical concepts develop over time, that they are the results of research processes. The students received a made-up invitation from the journal NORMAT to contribute to this debate by submitting a paper for the journal on this issue. The students were required to argue for their own opinions based on the collected work that had been done in the basic-groups. The teacher had formulated four issues that the students had to address and discuss in their paper: Euler's, Dirichlet's and our current concept of a function; the two meta-rules i.e., the generality of the variable and the general validity of analysis, the raison d'etre behind domain, range of image and proofs; sociological factors that had influenced the development of the function concept; and human factors.

The analysis of data that was collected during the teaching shows that the high school students realized that the concept of a function, they read about in their textbook, was the

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[6] These principles refer to historical meta-rules of mathematics (see footnote 5).

[7] These questions address the issue of what Sfard (2008) calls meta-discursive rules in mathematics. For a discussion of this aspect of the teaching module see Kjeldsen and Petersen (2014).

result of a historical development, and that they also gained more specific knowledge about some of the key elements of this development. The students became immersed in mathematical inquiry processes by tracing (parts of) the path of the masters, as illustrated by the following quotes from one of the essays written in the expert-groups. The quotes show that these students became aware of at least one source for mathematical research questions as well as of discussions among mathematicians that relate to the validation of mathematical knowledge:

> The reason why Euler began to work with [Euler-]discontinuous functions was because of a debate between contemporary mathematicians. The debate concerned the fact that the functions the mathematicians worked with could not describe a vibrating string.
>
> […] the development of the concept of a function was among other things due to human attitudes and interpretations, which were important factors. For example, some of Euler's contemporary mathematics colleagues were of the opinion that Euler's extended function concept should not be used because it went against the principle of mathematics. They thought it was cheating. This meant that Euler's extended function concept never came to be used as intended, and a new function concept was developed by Dirichlet. (Petersen, 2011, Appendix C)

We will not go further into the results regarding the learning of the high school students, for this we refer to Kjeldsen and Petersen (2014). Here we will restrict ourselves to pointing out that the teacher's knowledge *about* the nature of mathematics and her informed epistemological conception about the production and validation of mathematical knowledge enabled her to design and implement the course described above in high school. A course where she, through history of mathematics, taught both *with* and *about* the nature of mathematics, i.e., she taught in a way that immersed the high school students in mathematical inquiry processes and made them capable of forming epistemological understandings of how mathematical knowledge is generated and validated. It is also noteworthy to keep in mind that mathematical argumentation and reasoning is very much present in the core of this teaching experiment. The ability to teach *with* and *about* the nature of mathematics in inquiry based teaching with history provides an opportunity to strengthen proofs and deduction in high school mathematics.

7 Conclusion

The problem-oriented project work in the master's program in mathematics at Roskilde University provides an explicit-reflective framework in the sense of Abd-El-Khalick (2013). The analyses of the two student projects on generalizations in the theory of integration and the construction of the real numbers respectively, demonstrate how the students in this program through working with mathematically rich and thick episodes from the history of mathematics, can develop informed conception about the epistemology of mathematics, of how mathematicians produce and validate mathematical knowledge, and what kind of questions that drive mathematical research, while experiencing an inquiry learning environment that to some extent gives them insights into authentic mathematical practices. In the course of their problem-oriented project work they came to reflect upon and criticize the way mathematicians generate and validate mathematical knowledge, i.e., inquiry processes in mathematical research were made explicit objects for the students' reflections within an epistemological framework. The analysis of the design and implementation of the teaching experiment in high school exemplifies how the RUC program in mathematics enables its graduates to use their integrated understanding of history and philosophy of mathematics to enact inquiry learning *in* mathematics through history.

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