MATHEMATICAL KNOWLEDGE FOR TEACHING TEACHERS

The case of history in mathematics education

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ABSTRACT

Teaching mathematics in school has been researched by many, with Ball, Thames, and Phelps (2008) and their practice-based theory of mathematical knowledge for teaching (MKT) primary among them. However, the work of teaching mathematics in teacher education has been much less researched. An emerging theory of mathematical knowledge for teaching teachers (MKTT; Zopf, 2010) is of particular interest in our current work. This paper deals with part of a *Didactics of Mathematics* course given to future mathematics teacher educators at the Danish School of Education, and asks the question of how to develop these future teacher educators' MKTT in relation to history of mathematics in mathematics education. We share the key components of the theoretical constructs underlying our work and illustrate these by means of the students' own mini-project reports, which address cases or topics ranging from analysis of the inclusion of history in mathematical textbooks, to the development of an activity for pupils – or for student teachers – which include original source materials.

1 Introduction

The paper addresses the question of how to introduce future mathematics teacher educators to the discussion of history in mathematics education, and how to prepare them theoretically for a potential use of history of mathematics in their own future practice. The "answer" presented to this question is one by example, since the paper reports on a concrete design and implementation of a course. The theoretical framework adhered to in the paper is a further development of the practice-based theory of mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008) into an emerging theory of mathematical knowledge for teaching teachers (Zopf, 2010) [1]. At the core of this theory is a particular conception of teaching – being a plausible conception of the professional practice of teachers – and the work of teaching can further be defined through the mathematical tasks that teachers do in order to facilitate students' learning of mathematics (Hoover, Mosvold, & Fauskanger, 2014). Tasks of teaching can be seen as a decomposition of the work of teaching, and mathematical knowledge for teaching can thus be described as the "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball et al., 2008, p. 399). Whereas many have investigated the work of teaching mathematics in school, researchers have much less investigated the work of teaching (future) mathematics teachers. Illustrative examples of students' reports from the course will be displayed and discussed after presenting the educational setting of the course and the design of the six sessions related to the topic of "history in mathematics education." Hence, the aim of this paper is to employ aspects of this evolving framework for MKTT in order to describe the future teacher educators' development of MKT and MKTT in relation to the use of history of mathematics in mathematics education.

2 Mathematical knowledge for teaching (teachers)

In the following, and before we present some recent attempts to investigate what can be referred to as mathematical knowledge for teaching teachers (MKTT), we first describe some foundations of MKT. Far too often, descriptions of MKT in the research literature are limited to a presentation of "the egg" and the sub-categories of MKT that are depicted in it (figure 1).

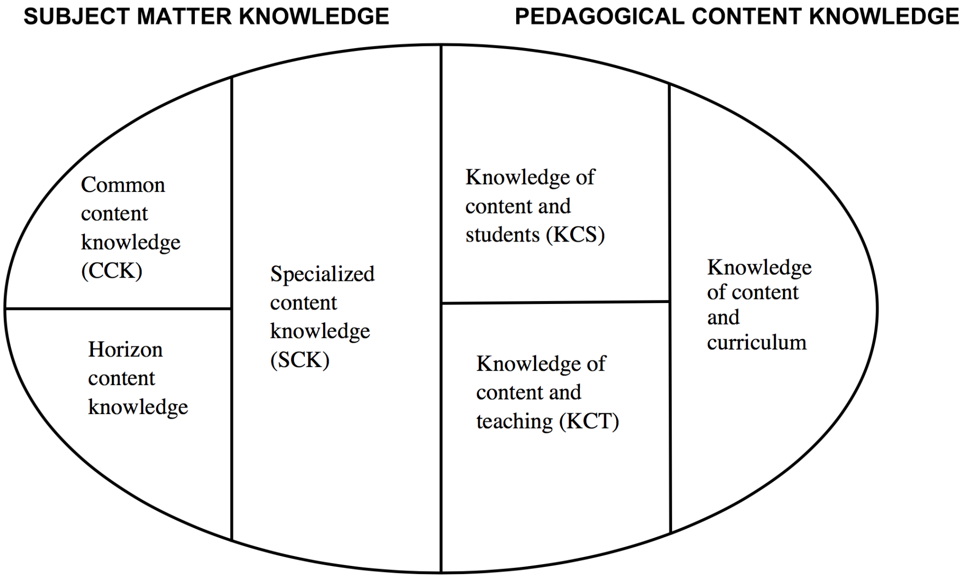

Figure 1. The common representation of MKT (Ball, Thames, & Phelps, 2008, p. 403)

This representation of categories and sub-categories of MKT might have initially served a purpose for representing one version of the forms of knowledge a teacher might employ or draw upon in teaching, but it is not necessarily considered as the core of the practice-based theory of MKT. In their discussion of the assumptions that are underlying the development of MKT at the University of Michigan, Hoover and colleagues (2014, p. 11) emphasized: (1) the role of the discipline of mathematics in and for teaching; (2) the meaning of the term "teaching" in the phrase "for teaching"; and (3) the mutual importance of both conceptual work and the validation of proposed conceptualizations in advancing early-stage research. The first assumption highlights the role of the discipline of mathematics, and this means that there must be a commitment not only to students' thinking, but also to mathematics as a discipline. The latter is important and can easily be forgotten. This emphasis on the discipline of mathematics is also what represents a significant development of Shulman's (1986) more general ideas about teachers' professional knowledge – on which the theory of MKT is based. Second, the term "teaching" is important, and Hoover and colleagues proposed that, "teaching is seen as a *plausible* conception of *professional* practice" (p. 11, original emphasis). Such conceptions – in the work of Deborah Ball and colleagues – are based on careful analyses of the work of teaching mathematics. The aim of such analyses is to identify "what is entailed mathematically in that teaching" (Hoover et al., 2014, p. 12). The focus on identifying or decomposing the work of teaching mathematics is strongly connected with the third assumption. Although many seem to associate MKT with "the egg" or multiple-choice measures as were developed as part of the Learning Mathematics for Teaching (LMT) Project, the research of Ball's group can be described as strongly conceptual and analytic. Thus, the aim of this research is to develop professionally grounded concepts that can be meaningful and usable (Ball & Bass, 2003; Hoover et al., 2014). At the core of the research efforts from Ball and her colleagues is the "focus on the mathematical tasks that teachers have to deal with in the work they do that have significant mathematical entailments" (Hoover et al., 2014, p. 13). The question about what knowledge demands are entailed in the teaching of mathematics, for these researchers, thus becomes a question of identifying recurrent mathematical tasks of teaching mathematics.

Ball et al. (2008, p. 400) presented the following list in their attempt to conceptualize several core tasks of teaching: presenting mathematical ideas; responding to students' "why" questions; finding an example to make a specific mathematical point; recognizing what is involved in using a particular representation; linking representations to underlying ideas and to other representations; connecting a topic being taught to topics from prior or future years; explaining mathematical goals and purposes to parents; appraising and adapting the mathematical content of textbooks; modifying tasks to be either easier or harder; evaluating the plausibility of students' claims (often quickly); giving or evaluating mathematical explanations; choosing and developing useable definitions; using mathematical notation and language and critiquing its use; asking productive mathematical questions; selecting representations for particular purposes; and finally, inspecting equivalencies. This list, which was not meant to be definitive, has later been subject to further investigation, and attempts have been made to extend (e.g., Delaney, 2008) and criticize (e.g., Ng, Mosvold, & Fauskanger, 2012) it. The multiple-choice items that resulted from the LMT Project – often referred to as "the MKT items" – can be seen as attempts to operationalize these mathematical tasks of teaching. The few attempts that have already been made to investigate mathematical knowledge for teaching teachers (MKTT) build upon these foundational ideas behind MKT.

In their investigations of MKTT, both Zopf (2010) and Kim (2013) focused on investigating the tasks of teaching mathematical knowledge for teaching in teacher education. Zopf (2010) argued that the work of teaching MKT in teacher education entails a number of recurrent tasks of teaching. The following three are highlighted in particular: "selecting interpretations and representations, selecting examples, and managing the enactment of mathematical tasks for the work of teaching mathematical knowledge for teaching" (Zopf, 2010, p. 199). She suggested that there is a distinct domain of mathematical knowledge that is needed for teaching MKT in teacher education, and she referred to this as MKTT. Zopf proposed that this MKTT includes a specialized knowledge of MKT as well as a solid knowledge of the discipline of mathematics. The latter includes "knowledge about mathematical structures such as definitions, properties, theorems, and lemmas and how these are used to do mathematics; knowledge about descriptions, explanations, justifications, and proof and how these are used for mathematical work" (ibid.). Kim (2013) concurred with this in her study, and further developed a framework for teaching MKT in teacher education. This framework consists of two interrelated entities: mathematical work of teaching and knowledge about mathematics. The latter category of knowledge appears to coincide with Zopf's (2010) concept of disciplinary knowledge of mathematics, and it is particularly interesting for our study since it "is about the nature of knowledge in the discipline, such as where it comes from, how it changes, and how truth is established" (Kim, 2013, p. 12). In our reading, this points to the history of mathematics.

Mosvold, Jakobsen and Jankvist (2014) argue that history of mathematics can be useful for pre-service as well as in-service mathematics teachers, e.g. "history of mathematics can be useful for the teachers as a means to increase knowledge and awareness of possible misconceptions, obstacles and impediments related to various mathematical concepts and ideas" (ibid., p. 58). They also argue that investigations into the history of mathematics, for instance by studying historical sources, have the potential to increase the mathematical knowledge of the student teachers. As seen above, from the emerging literature on MKTT it appears evident that solid disciplinary knowledge of mathematics is required from the teacher educators (Superfine & Li, 2014; Zopf, 2010). From their review of literature on mathematical knowledge for teaching, Hoover, Mosvold, Ball and Lai (2016) argue that development of MKT in teacher education requires close connection between mathematical content and the work of teaching. Mosvold et al. (2014) suggested that history of mathematics has a lot to offer for mathematics teacher education, but the introduction of history also placed some demands on the teacher educators. In this paper we dig deeper into this phenomenon when we investigate further how history of mathematics can be introduced to future mathematics teacher educators, and how they can be prepared for using history of mathematics in their future teaching practice.

3 Educational setting and background

To become a mathematics teacher educator of primary and lower secondary teachers in Denmark, it is often favored by teacher training colleges that the educators hold a master's degree in mathematics education [2], of which the Danish School of Education at Aarhus University is the only provider in the country. To enter the master's program the student must already have a university bachelor degree, e.g. in mathematics, or a vocational bachelor degree, e.g. as a primary and lower secondary mathematics teacher. The two-year master's program consists of courses in mathematics, courses in general didactics, and a course in didactics of mathematics, several of these involving student projects, and finally a master's thesis.

The course of our interest here is *Didactics of Mathematics*, as implemented in the years of 2014 and 2015. In this course, each of the four mathematics educators within the department are given the opportunity to teach in a mathematics education topic of their own choice. One of the ideas behind this is that students in this way are also confronted with recent research, in which the mathematics educators themselves are involved. The course counts 10 ECTS (European Credit Transfer System), and each topic consists of six sessions of two to three hours of instruction and supervision each, along with group work, etc. For each of the topics, groups of students must submit a mini-project. Based on a random selection, at the end of the course the student groups are examined in one of the four mathematics education topics.

4 Design and function of the six lessons

We now describe the content and purpose of the six sessions related to history in mathematics education. For each session the students were to read a collection of texts (primarily research papers), with which they were to work during a given session. Additionally, supplementary texts were provided. Students' previous experiences with history of mathematics vary from a superficial exposure to full undergraduate course work.

In session 1, students were to familiarize themselves with different arguments for and against the use of history (and epistemology) in mathematics education, as well as potential dilemmas, and of course different approaches to involving history. The assigned texts included Fried (2001), Jankvist (2009) and Niss and Højgaard (2011). The purpose of this collection of texts was to enable the students to more qualifiedly discuss concrete uses of history at different educational levels including teacher training. Session 2 focused on the role and use of theoretical frameworks in empirical studies related to a use of history in mathematics education. The students were presented with two studies (Jankvist, 2011; Kjeldsen & Blomhøj, 2012), which served as cases, and they then were to discuss the use of Sfard's (2008) framework of commognition within these two cases (the students were already somewhat familiar with this framework). As supplemental literature for this lesson, students were encouraged to examine Jankvist and Kjeldsen (2011) and the use of the Danish competency framework (Niss & Højgaard, 2011) in this.

Next, session 3 addressed the use of original sources in the teaching and learning of mathematics as well as different approaches to involve such sources (e.g. Barnett, Lodder, & Pengelley, 2014; Jankvist, 2013). Here again, one purpose was to prompt the students to argue for and against a potential use of original sources in a particular educational setting. Supplementary texts for session 3 included Glaubitz (2011) and Jankvist (2014). The topic of session 4 was that of history in mathematics teacher education and not least teachers' potential benefits of being introduced to elements of the history of mathematics. Drawing on the topics of the previous lessons, the students were to compare an older empirical study (Arcavi, Bruckheimer, & Ben-Zvi, 1982) with a newer one (Clark, 2012), and discuss the outcomes of these in the light of interpreting results by means of the MKT framework (e.g. Ball, Thames, & Phelps, 2008; students were already somewhat familiar with this), drawing also on a reading of Mosvold, Jakobsen, and Jankvist (2014).

Session 5 was designed as a workshop in which the students were to further relate the read texts to each other as well as to a constructing a concrete case of their own choice. This work eventually resulted in a submitted mini-project report (approximately 12 normal pages, plus appendices) for each student group. These reports were then presented and discussed during session 6, where each student group would also have read the report of another group in order to provide constructive feedback and to also receive feedback themselves. (In Spring 2015, the third author of this paper was present at the course in sessions 4 and 5.)

5 Students' mini-projects

In this section, we first present a list of the students' mini-projects in order to provide an overview of the topics and issues that the students themselves have chosen to address from the point of view of using history in mathematics education. Next we describe two of the mini-projects from this list in more detail; one dealing with the changing notions of the concept of function through the 18th and 19th century; and another that applies the presented theoretical constructs of the course literature in an analysis of the inclusion of history of mathematics in a secondary school mathematics textbook.

5.1 List of students' mini-projects (Spring 2014 and Spring 2015)

- The use of history in a mathematics textbook system for primary and secondary school (see illustrative example 1), asking to the use being one of history as a goal or tool, Whig history, and the use of excerpts from original sources.
- The use of original sources in relation to the introduction of the concept of function in grade 9 of secondary school (see illustrative example 2)
- Students' development of overview and judgment (Niss & Højgaard, 2011) concerning the historical development of mathematics, exemplified by the history of number systems and the number 0.
- A discussion of different mathematical discourses (cf. Sfard, 2008) in selected primary sources concerning calculation of π, and the change of teaching discourses between addressing in-issues and meta-issues (Jankvist, 2009).
- The use of Babylonian tablets for teaching 2nd-degree equations in upper secondary school as an example of using original sources in teaching (Barnett et al., 2014).
- An analysis of a HAPh-module on Boolean algebra and circuit design (Jankvist, 2013), drawing on the course literature.
- Designing a teaching module around different proofs of the Pythagorean theorem (Euclid; Liu Hui; and a modern textbook one) focusing on aspects of history as a goal in relation to proofs and proving (Jankvist, 2009).
- Using history of mathematics in mathematics teaching education (Clark, 2012), illustrated by means of Mayan mathematics (the codexes from Paris, Dresden, and Madrid).
- A discussion of historical parallelism between the coming into being of the number 0 and pupils' mathematical learning difficulties of this particular number.
- How the history of negative numbers may potentially assist pupils in their reification (Sfard, 1991) of negative numbers as mathematical objects.
- A discussion of how the history of mathematics, exemplified by the history of π, may assist pupils in developing the mathematical competencies (Niss & Højgaard, 2011) of reasoning, representation, and problem tackling.
- The history of probability theory and its potential use in the teaching of mathematics.
- Analysis of the use of history in a mathematics textbook system for primary and secondary school and the ministerial orders for the school subject of mathematics when the textbook system was published.

- The potential use of Al-Khwarizmi's *Algebra* in original source as a means for inclusion of pupils with Arab ethnicity in Danish classrooms.
- Analysis of a recent textbook system for upper secondary school, and a discussion of how this system explicitly or implicitly invites the reader (or teacher) to use history of mathematics in the teaching of the subject.

5.2 Illustrative example 1: Textbook analysis (Spring 2014)

One group focused on a Danish textbook system called *Sigma*, and in particular they looked at the books for 8th grade, which consist of one textbook for the pupils and one for the teacher. For their initial analysis of the books' use of history, the group relied on the constructs of Fried (2001) and Jankvist (2009). Firstly, the group discussed the textbook authors' purpose of using history and the degree to which they find this realized:

> In the teachers' textbook [...] we find the following statement in the chapter on Numbers and Algebra: "We believe it to be important that the pupils learn about the development of mathematics and in particular that of numbers. Although such knowledge may not have a direct yield, it assists in providing the background for a part of the world, which we live in today. Without this knowledge, mathematics [...] appears as if it has always existed in the form we are introduced to today" [Sigma 8, teachers' textbook, p. 6]. Here, knowledge of the history of mathematics is viewed as relevant in itself. Hence, generally speaking, we have to do with a goal argument [of using history]. The interesting thing then is how this is reflected in the pupils' textbook [...]. Through the entire chapter, we see a large focus on the history of mathematics. 10 out of 24 pages are dedicated entirely to history of mathematics, where the pupils are informed about the historical development of the numbers from hieroglyphs over Roman numerals to negative numbers and the number 0. Occasionally, the historical account is replaced by traditional mathematics exercises. However, there is almost no connection between the historical information and the exercises, since these can be solved without having read the historical account. Hence, the intention from the teachers' book is not clearly implemented in the pupils' book. (Færch et al., 2014, p. 3)

The group also provided another example, one on the Pythagorean Theorem, where the teachers' textbook provided an extensive account of Pythagoras, his school, and the presumed Babylonian origin of the theorem. Again, the implementation of this piece of history in the pupils' textbook is reduced to a cartoon drawing, a picture of a marble bust of Pythagoras, and several examples accompanied by modern-day notation. As pointed out by the group, the book missed an opportunity of applying excerpts from original sources here. Original sources, however, are part of the teachers' book, but as remarked by the group the book authors' intention with this remains unclear:

> ... in the teachers' book [...] six excerpts from original sources on the proof of the Pythagorean Theorem are shown [...], but no suggestions as to how the pupils may be brought to work with these sources are provided – actually, there is no mentioning of

> the sources themselves, so it is unclear why they are included in the first place. (Færch et al., 2014, p. 9)

In further relation to the discussion of purpose of using history versus approaches to using it, the group pointed out that despite it being difficult to realize 'history as a goal' through mere 'illumination approaches' (Jankvist, 2009), this appeared to be what happened in the *Sigma* system (ibid. p. 5). They continued:

> The teachers' textbook [...] contains quite a number of test exercises, but history of mathematics is not a part of any of these. In the teachers' book it is clear that history is used as a 'spice' and seen as a tool, not as a goal. In the notion of Fried (2001), what we are dealing with is a 'strategy of addition.' (Færch et al., 2014, p. 6)

With continued reference to Fried (2001), the group went on to argue that the book system had a somewhat Whig approach to using history, particularly in the pupils' textbook. Following this, the group discussed the missed opportunities of the book system in relation to fostering Sfard's commognitive conflicts, with reference to Kjeldsen and Blomhøj (2012):

> In the teachers' textbook [...] it is stated that the pupils should become acquainted with the Roman numerals, although not to a very large extent: "The positional number system should – once more – be examined carefully with the pupils, while the Roman numerals should not be examined as much – they merely serve to illustrate the advantages of the numeral system we apply today" [Sigma 8, teachers' textbook, p. 7]. The authors' intention here is that of having one numeral system meet another in order to illustrate clearly the good idea of one of them. It is exactly in this meeting between two different discourses that the opportunity for learning arises, since the difference between the two discourses is made clear by the advantage of one of them. The intention here is for students to discover the ineffectiveness of addition in the Roman numerals as compared to our current positional system. Unfortunately, as seen before, this idea is not pursued in the pupils' textbook, which only contains little information about the Roman numerals, and not a single exercise where pupils are to work with these. (Færch et al., 2014, pp. 7-8)

5.3 Illustrative example 2: Concept of function (Spring 2014)

The case of the second group was four different definitions of the concept of function, more precisely Euler's definitions from 1748 and 1755, respectively, Dirichlet's from 1837, and a modern definition relying on the notion of sets (e.g. see Kjeldsen & Petersen, 2014). The group aimed to construct a small module to be implemented in 9th grade of secondary school, since they found that the concept of function is one that is troublesome for pupils at this grade and the beginning of upper secondary school. Hence, an assumption of the group was that such a module might assist in easing the transition phase between the two educational levels (Jankvist, 2014), meaning that they aimed at using history as a tool (Jankvist, 2009):

> We intend a half-half relationship between mathematics and history, and we use the history of mathematics as a means to teach the pupils the concept of function, i.e. our

> ‘why’ is history as a tool. We use it [history] as a motivating and cognitive tool by offering different ways to introducing the concept. (Hansen et al., 2014, p. 8)

In terms of approach, Group 2 intended a four-session module relying on the hermeneutic approach (Glaubitz, 2011):

> We find that the hermeneutic approach fits our case, because it is the contrasts between past and present that are to be examined consciously, and because it is the embracing of these tensions that provides the deeper understanding of both the mathematics itself and the history of mathematics (Barnett et al., 2014). Since we choose the hermeneutic approach we first present the pupils with the modern definition of the concept of function and afterwards the original sources. (Hansen et al., 2014, p. 9)

The idea, the group explained, is that the pupils must relate the early definitions to the modern one. In relation to Dirichlet’s definition, they said, the point of that and the modern one is actually the same, but the associated concepts have changed over time, e.g. set theory was not available at the time of Dirichlet. And, by relating the modern definition and Dirichlet’s to those of Euler, the pupils must obtain an idea of why Euler’s concept of function is insufficient for us. Following this explanation, the group addressed the potential benefits of relying on original sources:

> One of the advantages of original sources is that they promote the reader’s abilities to think like the author, and another is an understanding of the different context in which the sources are written (Barnett et al., 2014). If the pupils become aware of the historical context and try to understand what the author did, there is a chance that they also try to understand the mathematics. [...] Other advantages are, among others, to bring the pupils closer to experiencing the creation of mathematics and see the road of mathematical development, flow, errors, and success (Barnett et al., 2014). (Hansen et al., 2014, p. 9)

Finally, Group 2 touched on the discussion of having a Whig approach to history, and even though they admitted that their purpose of using history as a tool may have such consequences, the important issue is that they did so in an informed and conscious manner:

> In our case we have chosen not to use all of the original sources, because even in the Danish translations they appear too difficult. Hence, we have chosen to use only the definitions, which are what the pupils get as ‘original sources’. [...] We have found ourselves in the dilemma that either the original sources were too difficult, or we had to face that it was not possible to avoid being a ‘little’ Whig in our approach. Hence, we are aware of the fact that our approach is a little Whig, but we have found this difficult to avoid when the target group is secondary school. (Hansen et al., 2014, p. 10)

6 Concluding discussion

We now turn our attention to the discussion of the students’ yield in terms of ‘history in mathematics education’ as part of their future practice as teacher educators. The list of mini-projects bear witness to the way the future teacher educators attempted to connect the two primary types of knowledge pointed to for possessing MKTT: disciplinary knowledge of mathematics and mathematical work of teaching (Kim, 2013; Zopf, 2010). On the one hand, the students had the opportunity to familiarize themselves with a small piece of history of mathematics, sometimes also involving mathematics with which they were not already familiar (e.g. the group that studied Boolean algebra). On the other hand, they were expected to carefully reflect upon the way in which this history entered into a teaching and learning situation (relying on course literature), and in this process also reflect upon several of Ball and colleagues' (2008) core tasks of teaching, e.g. presenting mathematical ideas (through history) and recognizing what is involved in using a particular (historical) representation. That disciplinary knowledge and specialized knowledge of MKT make up two major components of MKTT is perhaps not so surprising. A teacher educator should possess a good portion of the six types of MKT, but they need a knowledge of MKT that is special to the work of teaching teachers. This specialized knowledge of MKT is strongly related to the deep disciplinary knowledge that is required to teach (future) mathematics teachers. Two types of MKT appear to be more in focus in relation to MKTT, namely horizon content knowledge (HCK) and knowledge of content and teaching (KCT). The two illustrative examples seem to support this.

In the first illustrative example, the students found that the textbook system authors declared that their use of history is one of history as a goal, while the students' analysis of the textbook system, exercises, tasks, etc. showed otherwise. Furthermore, the use of history in the textbook system, mainly relying on an illumination approach, often appears rather Whig, and according to the students the textbook authors missed several opportunities to include the history in a sensible manner. The group's textbook analysis did require knowledge about the curriculum in primary and secondary school (KCC) as well as the actual mathematics (CCK) and the pupils at this educational level (KCS). However, their assessment of the textbook system not adhering to its declared use of history as a goal required them to activate their HCK and KCT to a much larger extent. The observation of the textbook authors being Whig in their approach was a matter of the students' knowledge of the historical development of mathematics, i.e. related to HCK, while their observation of missed opportunities drew on HCK, KCT and SCK. In relation to MKTT, we observed an overuse of disciplinary knowledge and mathematical work of teaching. When discussing disciplinary knowledge of mathematics in MKTT, it is important to emphasize that this is a knowledge of mathematics that goes beyond common content knowledge and mathematics taught in 'regular' university courses in mathematics (cf. Hoover et al., 2016). Disciplinary knowledge of mathematics in MKTT involves knowledge of the nature of mathematics and how it has developed (Kim, 2013) – as illustrated in the discussion of this first example – and we therefore suggest that knowledge of history of mathematics and its use in teaching constitute a significant part of the knowledge that is special to the work of teaching (future) mathematics teachers.

In the second illustrative example, the students intentionally chose history as a tool to develop pupils' concept of function. They adhered to a use of original sources (the various definitions of the concept of function) through a hermeneutic approach. These students knew from their own experience that the concept of function was a difficult one for pupils (KCS), yet a central one in mathematics (CCK) and in the 9th grade curriculum (KCC). The essential practice for this group, however, is that they use their newly developed knowledge of how the concept of function has evolved (HCK) to design a teaching activity putting this into play from a cognitive perspective (KCT). If these students, in their future profession as teacher educators, were to present this activity to their student teachers, it seems plausible that it would be the aspects of HCK and KCT and the MKTT-related disciplinary knowledge of mathematics and mathematical work of teaching that would be in focus.

As argued by Smestad, Jankvist and Clark (2014), common content knowledge (CCK) and knowledge of content and curriculum (KCC) change over time with the introduction of new reforms, new curricula, etc., and such times of change teachers – and teacher educators – may rely on their HCK and KCT. As pointed to by several (e.g. Mosvold et al., 2014; Smestad et al., 2014), the history of mathematics is less prone to change, and hence offers a stable foundation for teachers in terms of HCK. A similar argument seems obvious for teacher educators, since the history of mathematics provides valuable input to developing a teacher educator's knowledge about the discipline of mathematic. In the students' mini-projects, questions about mathematical structures (definitions, properties, etc.), explanations, justifications, symbolism, formalism, proofs and proving, etc. occurred as a natural part of the process of doing these projects. And every time the students' mathematical knowledge was challenged, they also had to consider this from a pedagogical and didactical point of view relying on the course literature. Hence, we argue that the history of mathematics has an obvious role to play in the education of mathematics teacher educators as well as in illustrating the two identified components of the emerging framework of MKTT.

NOTES

1. The course described in this paper has previously been analyzed from the point of view of developing teacher competencies (Niss & Højgaard, 2011). This analysis is available in the proceedings from TWG-12, CERME-9.

2. Alternatively, the educators may hold a master's degree in mathematics. Note that in Denmark teacher educators are not required to possess a Ph.D.

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