INTEGRATING HISTORY OF MATHEMATICS WITH FOUNDATIONAL CONTENTS IN THE EDUCATION OF PROSPECTIVE ELEMENTARY TEACHERS

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ABSTRACT

The proposal presented here relates to the integration of the mathematical syllabus content, including foundational issues, with selected aspects of history of mathematics in the education of prospective elementary teachers.
An analysis of the needs and cultural profile of prospective elementary teachers in Italy was the original motivation of this study. The new syllabus has been applied for several years in Rome, with good results in mathematical learning and self-confidence regarding the performance as a future mathematics teacher. Students also show an ability to use history of mathematics in the classroom with good feedback from the children.
The historical contents were used in Spain to offer a stronger mathematical background to a group of teachers and prospective teachers collaborating in an experimental workshop for children with Down´s Syndrome. Once again the result was improved appreciation of their own skills and greater autonomy and spirit of initiative.

1 The historical background: from optimism to an age of uncertainty in children's mathematical education and its impact on primary school teachers' education

Both in Italy and in Spain prospective elementary school teachers show fear and rejection in the face of mathematics, which is largely explained by a cultural atmosphere of pessimism about the difficulties of the task of teaching maths to children and also by confusion about how to carry out such a task.

A loss of certainty about methods and contents of children's mathematical education spread throughout much of the industrialized world after the Second World War. In a context in which a long, solid tradition of calculations and elementary training was thoroughly applied around the world, the pestalozzian optimism of many innovators between the second half of the 19th century and the first decades of the 20th century, most of them mathematicians (Fröbel, Mary Everest Boole, Grace Chisholm Young, Jean Macé, Charles Laisant, Maria Montessori, Rodolfo Bettazzi), was overwhelmed by a wave of pessimism.

The crisis of European science due to political evolution and the war was one main cause, because the network of international contacts, cultural centres, publishing houses and general scientific journals had suffered a severe blow. Deep contributions based on the practical experience of educators when confronted with a didactical tradition rooted in the medieval abacus schools had marked an age of confidence in the powers of the mathematical mind to be exploited in the “preparation of the child for science”. However, after many editions and translations of Laisant's *Initiation to mathematics* and Macé's *Grand-papa's arithmetic*, these booklets addressed to educators became forgotten books. In his 1938 *Mathematics in society and culture* Federico Enriques, referring to Johann Pestalozzi and Friedrich Fröbel's inspiring contributions, wrote "the formative value of mathematics can also be seen in the first grades of children's education and working-class education; because mathematical intelligence is quite precocious", referring to Johann Pestalozzi and Friedrich Fröbel's inspiring contributions. However, 1938 was the year of the race laws in Italy and Enriques was expelled from the university. Even Maria Montessori's optimism in her *Psychoarithmetics* and *Psychogeometry*, published in 1934 in pre-civil war Spain was confined to the circle of montessorian schools in the following decades.[1]

The impact of the psychological experimental approach, from Edward Thorndike to Piaget's prominent books with Alina Szeminska and Bäber Inhelder on number and concepts of space in the child can be summarized in the transition from an approach of confidence in children's intuition to a focus on the difficulties of learning mathematics – dominant through the 1950s to the 1960s. The above mentioned innovators, pedagogues and mathematicians had criticized rote learning and boring and mechanical methods of 19th century primary schools and they tried to put forward a different style of teaching specially designed for children. Alternatively, the psychological world believed it had found an obstacle to learning mathematics in the young child's mind, a kind of illogical child, not able to make logical inference similar to how Lévy-Bruhl had spoken of a primitive illogical mentality. In the 1970s and 1980s Edinburgh psychologists led by Margaret Donaldson offered a radically different vision of the strength of children's mind (Donaldson 1978, Hughes 1986). Nevertheless, in the meantime such a context of pessimism and loss of almost every deep rooted certainty on children's mathematical education led to questioning of the on-going education of young future primary schools teachers.

This was the case in many European countries in Western Europe and in the United States, in contrast with the continuity in China or the USSR and Eastern European countries. The long term effects of this sharp contrast was aptly described by Liping Ma in her 1999 *Knowing and Teaching Elementary Mathematics: teachers understanding of fundamental mathematics in China and the United States*, which considers both the effects on children's education and on the cultural and emotive position of primary teachers.

The spread of psychological views on early maths education combined with *new math*. Meanwhile, ideas on children's first steps in mathematics had deeply changed. For the first graders, new math meant sets, trying to find intruders in sets, exercises of equipollence, and 0 as cardinal of the empty set; for pre-schoolers, down and up, inside and outside and those so called "topological notions". Geometry had never had a real relevance in primary education, but innovators had shared the feeling of the crucial role that geometry could have in transforming and improving primary mathematical education. All of this was completely

[1] Montessori wasn't a mathematician but a medical doctor, however, in Rome she had attended the discussions of mathematicians deeply concerned with basic concepts of mathematics and geometry, and her optimism was a legacy of the indefatigable Séguin, the author of the 1875 *Report on education*.

forgotten as a result of the rejection of Euclidean geometry throughout the compulsory and secondary education system.[2]

What kind of teacher should have taught new math to children? It's hardly surprising that this new trend had a deep impact on the image and profile of primary teachers. René Thom had perceptively anticipated this long-term effect in his 1972 lecture at the Exeter conference on "Developments in mathematical education" (International Commission for Mathematical Instruction, Howson 1973), emphasizing the fact that the change in mathematical programmes or contents in primary school was intended also, if not mainly, to force primary teachers to change their pedagogical methods, in order to obtain a heuristic teaching that was direct, free and constructive, that could awaken the child's interest. Twenty years later, in 1991 he remembered as a witness the dramatic impact in the 1970s France:

> Old white haired teachers, who taught elementary calculations with sticks, have been seen to be forced to come up-to-date. They were told: Sirs, what you do is ridiculous; you know nothing about set theory, and one cannot do arithmetic without understanding it. And those old teachers were forced to sit down at school desks to listen to pretentious youngsters who explained to them that they had understood nothing about numbers!" (Thom 1991).

Thom himself, wrote, that he had been a modernist *ante litteram* (he entered the Paris *École Normale Supérieure* in 1943), and was fond of the foundations of mathematics, logics, and set theory, but there have been excesses of zeal and a deep misunderstanding: that "by making the implicit mechanisms or techniques of thought conscious and explicit, one makes these techniques easier" (Thom 1973, p. 197). However, this is exactly what became the norm and set theory, Boolean algebra and topological structures were considered as the basic mathematical background of a teacher in order to work with illogical children.

The impact on primary teacher's education, life and work, and self-image as a result of the coalition of psychology with "modernism" in mathematics, as Thom put it, was severe. In many countries around the world this education lasted until very recently outside the university, in secondary education institutions, which moreover were considered of lower rank if compared with the classical language high school. In addition, his or her point of view on mathematics was that of a young student, not that of a prospective teacher. Eventually the idea of post-secondary normal schools devoted to the primary teachers education spread, or alternatively there were specific degrees in primary education in colleges of education. A modern math teacher for children aged 3 to 10 should have a psycho-pedagogical background, for some; or in other cases should learn logic and the foundations of mathematics; or a mixture of both new elements depending on the institutional set.

2 The challenge of self-confidence among prospective teachers

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[2] In fact, this did not get rid of geometry, because 3D physical didactical materials designed and commercialized in those years (Dienes, Cuisenaire) were eventually introduced in primary schools to try to solve an entangled situation, which implied appealing to children's geometrical intuition (for example comparing and adding solids).

In Italy and Spain, young students, mainly women, who enter the university to become teachers, have an experience of math courses in compulsory and secondary school lasting about twelve years. This experience often involves feelings such as annoyance or distress and a vision of mathematics reduced to pure calculation and mechanical procedures.

An easy way to face up to this situation is to give up to teach more mathematics to them, and focus on courses on mathematics education, with the hope that, combined with the maths learned in secondary school and training in psychology and pedagogy, it will be enough to tackle the mathematical education of children.

In Spain, in 1999 Miguel de Guzmán organized a European conference on the training and performance of primary teachers in mathematics at the Academy of Sciences in Madrid. A year later he spoke about the risk that the adoption of an option such as the former would entail for mathematical instruction:[3]

> our system is depriving the vast majority of people who are being trained to teach, of the possibility of a slow, calm and joyful learning of the huge wealth that the mathematical task provides for the building of thought; depriving them of the chance of appreciating maths properly through the awareness of its usefulness and its ubiquity in everyday life.
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> It is depriving them of the useful and practical knowledge of the appropriate tasks for proper learning of mathematical topics that would be required at this level; it is depriving them of the satisfaction that comes from playing and from the aesthetic pleasure that can balance, as happens with any worthwhile activity, the many humdrum and laborious efforts we have to do to reach joyful practice …
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> Is it possible for these people to approach teaching without transmitting to their pupils the fears, and the worries, conditioning their attitude to the subject?[4]

This lack of useful and practical knowledge joined with an environment of low confidence on children´s chances to understand numbers as a result of the influence of Piaget´s conceptions of the building of the idea of number (Arnau 2011).

In Italy, during the first 15 years after the introduction of a university degree in primary education,[5] a significant number of women who had already graduated in literary subjects enrolled in these studies. These women had given up mathematics for periods of up to 10 years, and this fact amplifies their unpleasant memories and fossilized vision of mathematics as simple calculus. Until 2010, the number of hours devoted to mathematics depended on the university, ranging from nothing to a course of sixty hours. Almost 30 hours were usually

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[3] In Spain, from 1970 primary teachers' education consisted of a three-years course of studies in university colleges (Escuela de Magisterio) and from 2009 it became a 5 year masters degree.

[4] Guzmán, 2000.

[5] In Italy, until 1996, the education of prospective teachers was entrusted to five-years secondary schools (Istituti Magistrali, students aged 14 to 19), where a mathematics high school programme was developed, but further education proposals focused mainly on foundations

allocated to mathematics education. The current curriculum includes more than 150 hours of mathematics and mathematics education.

Children are the main interest of prospective teachers, and they frequently think of their role in terms of caring, playing and perhaps their literary education. As a result of their own school experience, the task of providing an introduction to mathematics and science is viewed as a kind of punishment they don´t want to inflict on children. Increasing social pressure on the value of mathematics adds fear to this rejection, because learning mathematics shows clear difficulties. Mass media contribute to spread of the results of comparative research, for instance PISA o TIMSS, where countries such as Italy or Spain are in positions under the OCDE mean.

There is an internationally shared core on the basic mathematical instruction a child should received in their primary training: it focuses on numeracy, which includes measure and interpretation of tables and graphs; the differences in national curricula lie mainly in the introduction of intuitive geometry programmes.

In spite of this agreement and as a consequence of the historical evolution that we have just outlined, prospective primary teachers in countries such as Spain and Italy develop their training in a cultural environment marked by a lack of confidence about the actual possibilities of every children to learn mathematics and disorientation about the best way to reach this learning.

It is known that one's own teacher is a crucial reference model (Margolinas 2007). However, if this model is negative what can the student do?

2.1 Exploring the reasons for primary school teachers' fear and lack of selfconfidence toward learning and teaching of mathematics: an experience in Italy

Concern over mathematics as a course of study and the lack of confidence about their future role as mathematics teachers was identified by the first author of this paper in Roma Tre University as one of the possible reasons for poor performance in the basic maths course.

Between 2010-2012 four workshops aimed at pupils who show a radical rejection to the basic maths course, graduate students taking a second degree in primary education and supply teachers in primary education or kindergarten were carried out (Millán Gasca, Marrano, Spagnoletti Zeuli 2015)[6]. The goals of the workshops were to show the participants that children can enjoy mathematics and are able to learn deep mathematical concepts, and to put across a different vision of school mathematics. To achieve these goals we started from the initial interest of all of the participants in children.

During the workshops, sixty hour projects carried out in the school apprenticeship period were discussed by the students who had designed them: a brief discussion of the theoretical framework was followed by *narration of the living experience* (Van Manen 1990), through words and images. Narration was proposed as an alternative to didactic

[6] Each workshop consisted of five sessions lasting three hours. There were 25-30 participants in each one.

models (Grégnier et al 1991). Story-telling and images were planned to achieve an effect of virtual immersion in a classroom, inspired by the Japanese *lesson study* (Isoda 2007), and to take advantage of peer coaching.

Later analysis of the experience showed that the main goal achieved was in fact increased confidence in their own actual potential as teachers or prospective teachers. This was possible thanks to a *mimetic effect*, or abandoning *feeling like the 'other'* (Scaramuzzo 2011), much more than a rational discussion on methods or pedagogical models. Because of this, the workshops dedicated progressively more time to the individual presentations of each person, sharing and analysis of school memories, discussion of fears and concerns and to reasons for hope and for assuming responsibility for children's maths education. The main focus was the formative value of mathematics that the participant had seen in the projects: participants had found a human value in mathematics.

Let us close this section with the dilemma regarding education for praxis of prospective elementary teachers in the subject of mathematics. A false dichotomy has become rooted in the mathematical education of primary teachers; between the mathematical concepts that prospective primary teachers should learn[7] and the pedagogy, psychology or didactic they have to manage. The later are sometimes understood – especially for those who will work with children – as an ability that turns them into professionals providing them with self-confidence, planning proficiency, leadership and knowing how to manage a classroom. However, these abilities alone don't work when we refer to mathematics and as a consequence mathematics is taught in primary schools with fear and this fear is transmitted from one generation to another. Hence, the origin of the need to work on the view of mathematics that prospective teachers have in order to transform them from learners to teachers. Mathematics needs to be understood as a result of human thought in its contact with the world, and history of mathematics is what can be used to show this human face of mathematics.

3 An historical approach in the training of prospective primary teachers as a source of mindfulness and interest in mathematical education

In Italy, the organization of university programmes for prospective elementary teachers has, in the past 10 years, offered the opportunity to rethink their mathematical education. Efforts have concentrated on contents, the introduction of informal language (Di Sieno, Levi, 2005, 2010), didactical methods and key issues and the connections with practice. History of mathematics had little or no role. In fact, little attention is devoted internationally to the role of history of mathematics (see for example the contributions in the volumes of the 2008 *International Handbook of Mathematics Teacher Education*), with the exception of the introduction in some textbooks of a few historical "facts" regarding for example number systems (Beckmann 2012).

[7] There has been a recent controversy in the Spanish society (March, 2013) when it became known that 83% of the applicants to be primary teachers would not be able to pass the test of knowledge and essential skills of the curriculum for twelve year old children

The first author of this paper[8] and Giorgio Israel (both historians of mathematics) have developed a syllabus with a textbook (Israel, Millán Gasca 2012) and online materials based on integration of mathematical concepts and theorems (elementary arithmetic, geometry, probability and some ideas of calculus) together with historical analysis and epistemological reflection, in order to place mathematics as part of culture as a whole[9]. The historical contents are illustrated in the classroom using images and the two hours lessons include exercises. In the written exams, students write integrated discussions of mathematical, historical and epistemological aspects.

Introducing history has a double purpose. On the one hand, to restore the contact point between mathematical concepts and reality and mankind, so that the tasks teachers design have a human sense for children and support their intuition (Donaldson 1978). On the other hand, to offer future teachers a humanistic view of mathematics which awakens the desire for sharing it with children as an integral part of the educational action, strengthening their autonomy and self-confidence in the mathematics classroom.

The role of an historical approach for achieving both objectives is a leitmotiv in Federigo Enriques' educational work (Enriques 1924-27; Enriques 1921) and more recently in the work of Miguel de Guzmán (2007). Both were a source of inspiration, along with the historical work of Cajori (1917, 1928-29). The epistemological approach of Enriques, Poincaré (1902) and Husserl (see Israel 2011) integrated the axiomatic presentation of arithmetic and geometry. This was also thanks to recent research on ancient mathematics.

3.1 Primitive concepts in arithmetic and geometry

The axiomatic approach to arithmetic and geometry has thrown an interesting light on elementary maths concepts with regard to their teaching and learning, particularly in the early stages of children's education. Dedekind and Peano explicitly referred to this in connection with their work on the concept of number (Israel, Millán Gasca 2012, Ferreirós 2008). In particular, the identification of objects and primitive relations appears helpful for identifying starting points of pathways to number and geometry. A remarkable, essential aspect is the consistency between Peano's axioms (successive, induction principle) and educational research on the central role of counting in developing number concept between 2 and 8 years and on how counting supports early additions of natural numbers (Fuson 1988).

The nature of primitive concepts in axiomatic theories is described only through the relationships among them established by the axioms. Axiomatic systems developed in the late 19th century are a culmination of the process of detachment of mathematics from reality and human mind-body experience. Therefore, it may seem paradoxical to link them to children's

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[8] A short history of mathematics aimed at prospective elementary teachers by Millán Gasca (2009) was published in the series "Quaderni a quadretti" of mathematics for primary education edited by Simonetta di Sieno.

[9] In addition, the historical evolution of children's mathematical instruction was also considered in order to put the pedagogical issues in context.

intuition: in fact, the third element is to track them in history, in Euclid's definitions and in the historical roots of these definitions.

Recent research on the ancient origin of oral and written number offers useful elements for unveiling this connection (Schmandt Besserat 1997; Nissen, Damerow, Englund 1993; on Egyptian number words see the discussion in Cartocci 2007). Moreover, Peter Damerow (2007) has considered links with neuroscience research on human cognition; Denise Schmandt Besserat has linked her thesis on the origin of symbolic thinking with anthropological research on number words and symbols in several cultures; and the historian and mathematician Enrico Giusti, has linked number, point and straight line to ancient technical experience in his essay *Hypothesis on the nature of mathematical entities* (1999). Schmandt Besserat (1999) and Giusti (2011) have even adventured to present this evidence to children in an appropriate language and with great attention to images. Cerasoli (2012) is another example of transferring results of historical research to children.

Our point of departure was that the study of the current discussion on the origin of numbers combined with a knowledge of axiomatic of Peano, including the role of recurrence (Poincaré 1902) and of repetition (Lafforgue 2010), provides prospective elementary teachers access to a deeper insight into mathematical ideas.

We want to emphasize that our proposal to discuss Peano's axioms does not mean a re-propositioning of logic and set- theoretical formal presentation of arithmetic divorced from human beings and the physical world (Kline 1977). Instead, the primitive concepts identified by the axiomatic system are studied together with historical research data on the ancient world and with epistemological reflection; and moreover other elements are brought into the framework: historical linguistics reflection on the systems number words (Greenberg 1978; see Fuson, Kwon 1992) and data from anthropological research (Cassirer 1923, Menninger 1958 and recent contributions from ethnomathematics).

This approach is more far-reaching than merely introducing a few examples of numerical systems without reference to the historical, epistemological and anthropological context or the epistemological problem. It could be considered too ambitious for an elementary school teacher. However, the fact is that when a teacher begins to work with children from 2 to 8 years old, he gets involved in the deep epistemological and anthropological problems of number, as is clear in the research of Martin Hughes (1986) on the child's contact with mathematical symbols when entering school.

A similar approach has been developed in the presentation of the study of geometry. Enriques and his school devoted great attention and many essays to the continuum and the concept of straight line and point. Recent historical research on primitive geometrical concepts and relations in the text of *Elements* by Euclides (Giusti 1999); on geometrical concepts and measurement systems in ancient civilizations (see for example Hoyrup 2002) and on prehistoric archaeological material (Keller 1998) offer many elements for examining the root in human action – technique, art, architecture – and in human perception of basic geometric concepts. Epistemological reflection by Husserl (Israel 2011) and Poincaré is the third element in a triangulation useful for a better understanding of children's first steps in geometry. In order to apply these ideas to the exploration of children's naïf conceptions and to the design of geometrical activities different from recognition of three or four forms (usually triangle, circle, square and rectangle), we based our research on Thom's views on the *continuum intuition* in the emergence of consciousness in the young child. This is consistent with research on context of a child's acquisition of language (Tomasello 2003). In the case of geometry other inputs can come from history of art (Focillon 1942, Gombrich 1995) and history of techniques.[10]

Of course the main challenge is to offer an in-depth presentation including aspects of current research in history of mathematics and presenting the open research problems; and at the same time, choosing examples and issues and communicating them in a way that can meet student's expectations and enlarge their cultural horizon. As already mentioned with regard to arithmetic, the point is that teachers actually face deep issues when working with children in their first steps with number and forms, with measure, with probability. Mathematical education is not only a question of the training in skills.

This approach has already been implemented with prospective teachers in Rome from 2011 to 2015 (about 150 students each year). The result was an improvement in commitment and performance in the basic mathematics course. Moreover, the approach has also been implemented with in-service teachers in 2014-2015 with good feedback also in terms of changes in their classes; and in a training course for prospective middle-school maths and science teachers aimed at graduate students (mathematics, physics, chemistry, biology and geology). On each occasion, the positive reactions have focused on the contrast with student's school experience and on the awakening of self-confidence regarding the possibility of enjoying teaching and learning maths with children.

3.2 Other historical elements: the confluence of several ideas in the modern concept of rational number; measures in the ancient and modern worlds; probability

The approach to rational numbers aimed at elementary school teachers often uses the school vision of an entangled mixture of fractions, decimals, percentuals, time fractions, and ratios. The modern construction of Q from Z has a power of unification and it's conceptually helpful for developing a higher point of view. Again, the formal approach is combined with a consideration of several ideas – from ancient to modern times – behind the modern concept of rational numbers as an extension of integers. This means considering both symbols (for example, fractional notation in Egypt and in the Arab and Latin middle ages, sexagesimal positional notation in ancient and Greek astronomy, decimal positional position from Stevin) and concepts (geometrical ratios, arithmetical ratios and proportionality). We cannot quote historical scholarship fully, but we would

[10] "…the geometric continuum is the primordial entity. If one has any consciousness at all, it is consciousness of time and space; geometric continuity is in some way inseparably bound to conscious thought". (Thom 1971, p. 698). This vision of geometry has already inspired three educational projects with pre-scholar children (Schiopetti 2013; Colella 2014) and a PhD Thesis on mathematics with children with Down´s Syndrome (Gil Clemente 2016).

mention Cajory's still fundamental *History of mathematical notations* and Grattan-Guinness 1996.

As to probability, the axiomatic approach was presented thanks to a discussion on the ideas of chance and the origins of probability theory and the modern definition of probability (Laplace, von Mises, de Finetti). Finally, the study of measure was also integrated with recent research on metrological systems in antiquity; considering Husserl's discussion of the idea of precision as a pathway to the Greek concept of mathematics; and a discussion of the cultural Enlightenment project of the Decimal Metric System and the force against and in favour of its spread in the late modern period is presented. It's remarkable that frequently, completely ahistorical comments accompany the introduction of the decimal metric system to children in elementary schools; and of course this is one of the most hated and annoying parts of the syllabuses.

4 Learning history of mathematics for teaching mathematics to children with Down´s Syndrome: an experience in Spain

During research for a PhD Thesis on mathematics with children with Down´s Syndrome[11] (Gil Clemente, 2016) a workshop was carried out between 2014 and 2015 with eight of these children supported by a team consisting of five young teachers from 21 to 28 years old. All of them had attended sudies related to education (Therapeutic Pedagogy, Hearing and Language, Special Education ...). In these specialities, in the 1995 Spanish study plan the percentage of credits corresponding to mathematics subjects reached only 2.5% [12].

As it was clear after a discussion-group maintained with them, previous to the development of the workshop, all of them, except one, had positive memories about maths in their school years and this was the reason why they decided to voluntarily collaborate in the experience. In spite of this, they assumed that mathematics they had studied was mechanical, with a lot of arithmetic and algebraic calculations and although they had enjoyed them, the lack of real problems had made the subject useless in relation to their lives. None of them mentioned reasoning or development of thought when speaking about the contributions that the study of maths had made to their global training.

Despite their working experience with disabled children, their great vocation for education and their commitment to collaborate in the project, it was clear from the beginning that their knowledge about maths and maths education was limited. Consequently, a training course was provided for them based on the historical approach we have presents here.

The course consisted of two phases. In the first we held two sessions before the workshop and the second took place alongside the workshop.

[11] Down´s Syndrome is the most frequent cause of an intellectual disability ranging from mild to moderate. The difficulties of people with this syndrome to learn calculation-based mathematics are well known (Buckley, 2007; Faragher, 2014). These children learn slowly and need to break up the tasks into small steps. It is for this reason that an approach based on the axiomatic approach to arithmetic and overall geometry was shown to be suitable

[12] From the analysis of this curriculum, we can observe there are only 4.5 credits of mathematics over 200 credits

The first session was devoted to arithmetic. We began by introducing the teachers to the cognitive problem of counting (Gelman, Gallistel 1978) in relation to which we outlined the importance of oral counting (Fuson 1988) and how the historical origin of number can be located in this human action. Following this we tackled the axiomatic system of Peano, stressing how he reflects on his primitive concepts (one, successive) the close relationship between number and counting. Using this triple approach which is deep and fruitful we tried to give the teachers confidence in their chances as teachers.

The second session was dedicated to geometry. Initial reflexion was focused on the strong relationship that geometry has with reality and on historical and philosophical explanations for its origin. We introduced them to the idea of representative space – visual, motor and tactile – (Poincaré 1902) as the basis of the three kind of strategies we were going to use with the children to help them to explore their surroundings: sensory activities, mimetic activities that use the power of movement and symbolic activities like drawings that allow them to begin to develop their abstract thought. From this reflexion we introduced the primitive concepts and relationships in Hilbert axiomatics–point, straight line, planes, go through, to be between– and how we can build all geometry from these concepts.

These two training sessions were well accepted by the teachers. Their knowledge of geometry was even poorer than what they knew about arithmetic, therefore, they valued the ideas we presented to them on this topic. In spite of this initial lack of knowledge they were quick to realise the power of geometry for children's education, because what they learned connected with their own experience as teachers.

The next phase was developed to run alongside the ten months of the workshop. Before each session, the team met, analysed the activities and looked deeply at the mathematical content of each one. The objective was to help them to understand the goal of each activity, and to think about which attitude we should adopt as teachers to allow concepts to arise in children's minds. This kind of reflection notably improved their ability to observe mathematical processes that unfolded in each of the later session.

The main achievement of this training was the way the team members gradually began to actively collaborate in the design of their own activities. All of them had great imagination and creativity but they suffered a lack of confidence as a consequence of their poor mathematical education. The historical and epistemological approach helped to ensure that what they invented actually helped the children to learn mathematics. Knowledge of Hilbert's axiomatic was particularly fruitful in order to design geometrical activities suitable for Down´s Syndrome children.

It may seem that to work with children with an intellectual disability it isn't necessary to know a lot about maths concepts, but that it's more important to know about methodological resources adapted to their cognitive characteristics. However, one of the aspects most valued by the teachers in this training was the way in which it helped them to approach the significant concepts and make them easy for the children to understand. The historical approach and the study of axiomatics was of great help for understanding the root and origin of concepts and this, coupled with their knowledge of children, increased their conviction in the children's chances of learning and allowed them to confidently lead the children to what they wanted tem to learn.

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