ANALYSIS AND GEOMETRY IN THE DEVELOPMENT OF THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER IN THE 18thAND THE 19thCENTURY

An Example of Ideas which don't appear in undergraduate Mathematics

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ABSTRACT

In an attempt to develop a new way of teaching mathematics, we can use the history to find the important ideas that articulated the now classical mathematical theories. My goal is to illustrate through an example, the Partial Differential Equations of the first order, how the different disciplines interact within a theory. This contribution aims indeed to show how the PDEs of first order constitute a point of convergence between geometry, algebra and analysis in the 19th century. Differential calculus of several variables has its origin in the study of the geometric properties of curves and some mechanical problems. Resolutions of PDEs use first analytical methods until Lagrange and Monge who provide a geometrical interpretation. The research about generalization of PDEs of first order in n variables oblige the mathematicians to use uniquely analytical methods in the first half of 19th century (Pfaff, Cauchy, Jacobi). The development of projective geometry, the birth of theory of groups, the geometrical vision of algebraic theory of invariants give the conditions for reinterpreting the general PDEs with geometrical methods (Lie, Klein). A few observations upon the teaching of Lie's Theory will follow by way of conclusion.

1 Introduction

In history many of mathematicians (including Descartes and Leibniz) stressed the difference between the *Ars inveniendi* and the *Ars expoendi* in mathematics. Concern for rigor and formalization inherited from Bourbaki has reinforced this trend to the point that the exhibition of the mathematical theories of the curriculum of graduation became dogmatic, separating the disciplines (algebra, analysis, linear geometry algebra, etc.) and deleting connections and ideas that were at the origin of their development. In an attempt to develop a new way of teaching, we can use history to find the important ideas that articulated the now classical mathematical theories. My goal is to illustrate through an example, the Partial Differential Equations (PDEs) of the first order, how the different disciplines interact within a theory.

In view of the very short time available, I will limit my talk to the development of PDEs of the first order in the 18th and 19th centuries. I pretend to show that the development of these theory is a great testimony of the close relations between geometry, algebra and analysis. For that purpose I will describe the historical conditions this theory has crossed and follow the different stages of its development which are: 1) The birth of this theory; 2) The first methods of resolution of such equations; 3) The geometric interpretation of Monge; 4) The development of analytical methods in the first half of 19th century; 5) And the great synthesis (geometric, algebraic and analytic) realized by Lie in the decade of 1870's.

2 The birth of PDEs

In accordance with (Engelsman, 1982; Grimberg, 2009), partial differentiation arose in the decades 1680-1720 from the problems enunciated by Leibniz and John Bernoulli. In this time the concept of function does not exist. Curves are described through their equation, what we would call today implicit functions. And differential calculus operates on every variable involved in the equation. Partial differentiation is then already included in the leibnizian differential algorithm and appears naturally from the problems Leibniz and Bernoulli began to treat analytically in this period, such as the envelop of parametrized curves, orthogonal trajectories, brachistochrone curves, and isoperimetrical problems. In these problems the solution involves the parameter of a family of curves and differentiation according to the parameter which was called "differentiation from curve to curve". These problems bring out also the concept of function and Euler in the 1730's reorganizes the differential calculus around the basic concept of function of one or several variables. He defines implicit functions and discusses the problem of conditions for a differential form to be complete. Independently Fontaine and Clairaut arrive to the same results (Grimberg, 2009). In this time what we call now partial differentials appear only as coefficients of a differential form. For instance, A and B in $Adx + Bdy$. The first application in mechanics appears in 1743 when Clairaut (1743) shows that a necessary equilibrium condition of a Fluid submitted to force field of components (P,Q,R) is that the force field is conservative.

In the same time the first PDE appears in the D'Alembert's 1743 *Traité de dynamique* through the study of compound pendulum. Other PDEs appear in D'Alembert's memoirs (1747, 1749a, 1749b, 1749c). The D'Alembert's methods of resolution consists in linear change of variables or what we call today Lagrange multipliers, and in the case of vibrating cords, the method of separation of variables.

The second stage is the various contributions of Euler in the same problems of mechanics, other methods of resolution, and Euler's Fluid equations. In this time the PDEs appear from the study of geometrical infinitesimal properties of the problems led by the geometric diagram. The complete analytical formulation of this type of problem will be so obtained with the *Mécanique analytique* of Lagrange where Lagrange affirms bravely in his preface that

> in this treaty you will not find diagrams, the methods I expose require neither constructions, nor geometric, nor mechanics reasoning, but only algebraic operations submitted to a regular and uniform running.

2.1 A first stage in analytical resolution of PDEs of first order.

The methods of resolution of PDEs follow the process of algebraization of the problems of mechanics. The first important stage in this process is realized by Euler in his treaty of integral calculus were he considers the partial differential equations of first order as implicit functions of 5 variables, the three variables of space were the third z is function of x and y and the two other variables are the partial differentials of z.

Lagrange elaborates a new conception of the nature of solutions. Considering an implicit function $V(x, y, z, a, b) = 0$ of three variables x, y and z, and two parameters, a and b, z being function of x and y, Lagrange shows that we can interpret such a function as what he calls a "complete integral" of a PDE of first order. The PDE $Z = 0$, originated from $V(x, y, z, a, b) = 0$, is indeed obtained resolving the system $V = 0$, $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, where a and b are eliminated (Lagrange, 1774, p. 239). From this fact, he deduces

> that all complete integral of all first order PDEs in three variables have to contain two arbitrary constants.

In this memoir, the relation between equation $V = 0$ and $Z = 0$ is based on a geometrical vision of the problem. The equation $V = 0$ representing a parametrized family of surfaces verifying the PDE, the envelop of this family verifies also the equation, but this time the particular equation $Z = 0$ without the parameters a and b. But this geometric insight established by Lagrange do not lead to the analytical method of resolution, even if we can see here the geometrical way of thinking of Lagrange.

2.2 The geometrical interpretation of Monge

The geometrical interpretation of first order PDE had to wait the works of Monge elaborated in the decade 1780 and gathered in his 1807 treaty *Application de l'Analyse à la Géométrie*. In this treaty, Monge realized a general study of surfaces and curves by analytical means and characterized developable surfaces and surfaces of revolution with PDEs, he defines the tangent plane and the normal of a surface, and characterizes the surface by the radius of curvature and as envelop of osculating circles. Monge introduces the concept of characteristic curves in the resolution of first order PDE's. In his book, the geometrical interpretation of PDEs of Lagrange, by simple analogy, turns out to be a crucial point of the theory (Monge, 1807, p. 369).

3 The analytical methods of resolution in the first half of the 19th century

The deep reason why the first further developments of the methods of resolution of PDE were analytical consists in the fact that the geometry for four and more dimensions was yet to be done while a few problems of mechanics which appeared from the works of Lagrange, Poisson, Jacobi involved more than three variables and consequently PDEs with many variables. Then a geometrical interpretation was not in this time possible. It explains the analytical way of Pfaff, Cauchy and Jacobi. A huge development of geometry, algebra and analysis will be necessary to join the conditions of the great synthesis realized by Lie.

Pfaff was then the first in a memoir dated of 1814 to begin an elaboration of methods of resolutions of PDEs involving n variables. He shows how to eliminate one by one the variables (Pfaff, 1814). And then he considers the equation $f(x_1, x_2, ..., x_n, z, p_1, ..., p_n) = 0$, the p_i being the n partial differentials of z, sets down $p_n = \varphi(x_1, \dots, x_n, z, p_1, \dots, p_{n-1})$, and integrates the equation:

$$dz - p_1 dx_1 - p_2 dx_2 - \dots - p_{n-1} dx_{n-1} - \varphi(x_1, x-2...., x-n, z, p_1, p_2, ..., p_{n-1} dx_n) = 0.$$

The demonstration was not complete and it will be achieved by Cauchy and Jacobi. Demidov observes that this equation leads to the resolution of n systems, one of which is (Demidov, 1982, p. 334):

$$\frac{dx_i}{\frac{\partial f}{dp_i}} = \frac{dz}{\sum_{x=1}^{n} p_k \frac{\partial f}{\partial p_k}} = \frac{dp_i}{-\frac{\partial f}{\partial x_j} - \frac{\partial f}{\partial z} p_j}.$$

Cauchy comes back to the general resolution of PDEs of three variables (Cauchy, 1819), and show how initial conditions lead to a solution. Cauchy constructs also solutions by means of characteristic curves even if he does not use this term. Considering an equation involving three variables x, y, z, he uses a change of variables, initial conditions, and builds solutions consisting of characteristics passing trough the curve $(x = x_0,\ z = \varphi(y))$.

Finally Jacobi had elaborated two methods to solve a system of equations, the first in an article (Jacobi, 1837), but the second is more interesting even if it was edited after his death (Jacobi, 1862) by Clebch. This is the second method we want to describe.

Jacobi considers a system of n equations $f_i(x_1, x_2, ..., x_n, p_1, p_2, ..., p_n) = h_i$ were the parameters h_i are arbitrary constants. The p_i are functions of variables x_i with the initial equation $f_0(x_1, x_2, ..., x_n, p_1, p_2, ..., p_n) = 0$. The p_i are functions of the x_i and are partial differentials which verify the condition that $p_1 dx_1 + p_2 dx_2 + ... + p_n dx_n$ is a total differential. For this, the necessary and sufficient condition is:

$$(f_i f_k) = \sum_{l=1}^{n} \frac{\partial f_i}{\partial x_l}\frac{\partial f_k}{\partial p_l} - \frac{\partial f_i}{\partial p_l}\frac{\partial f_k}{\partial x_l} = 0, \quad i, k = 0, 1, ..., n.$$

Jacobi uses indeed the Poisson Brackets which turn possible an expression of the system which he can solve beginning from the known function f_0 and successively determining the functions f_1, f_2, etc. (Demidov, 1982, p. 337-339).

In this method Jacoby is dealing with what we call today differential operators as $A(f) = \sum_{i=1}^{n} A_i(x_1, x_2, ..., x_n)\frac{\partial f}{\partial x_i} = 0$, the condition determining solutions of the system of equation, as $A(B(f)) - B(A(f)) = 0$ which leads to the so-called Jacobi's identity, Jacobi being the first relating this property to the Poisson Brackett (Hawkins, 2000, p. 48): f, g, h, being function of $2n$ variables, x_i, p_j,

$$((f, g), h) + ((g, h), f) + ((h, f), g) = 0.$$

4 Sophus Lie

The following stage of the theory represents a great revolution which is also related to the reorganization of mathematics, especially the new vision of geometry realized by Klein in the same time of elaboration of Lie's theory. But before explaining the importance of Lie's works we have to explain the context and what was the background of Lie and Klein. In the second half of 19th century, the works of Plücker were edited in Europe offering an analytical view of projective geometry. The idea of homogeneous coordinates allows the generalization of the concept of projective space. And especially to pass from the real projective space to the complex projective space and then work in space of higher dimension than 3. In the same time the works of Grassmann, Hamilton, Cayley developed new connections between geometry, algebra and analysis. They realized the condition for a geometric investigation of n-dimensional geometry.

For instance Cayley elaborates the theory of invariants using the homogeneous coordinates defined by Plücker and uses this geometrical view to relate projective and euclidean geometry, especially in his famous 1859 Sixth Memory which became the most important source of inspiration for Klein in his deduction of non-euclidean geometry from the projective geometry and further elaboration of Erlangen's Program.

4.1 Sophus Lie and Felix Klein in Paris

The theory of groups with Silow and Jordan became also a basis of the reflexion of Klein and Lie by 1870. Lie indeed met Klein in Berlin in 1869, and visited France with him in 1870, he traveled to England and came back in Göttingen in 1872 were he was also with Klein. It's very difficult to really separate the reflexion of the two mathematicians in this period (Hawkins, 2000, p. 10-30). It was just after this period that Lie elaborated his theory of n-variable partial differential equations.

Berlin was the center of analytical research with Weierstrass, Kummer and Kronecker but the source of inspiration was more in Göttingen, with Plücker (Clebsh and Klein edited the posthumous work). Staying in Paris, Klein and Lie studied the works of Jordan and had long discussions with French mathematicians such as Darboux (Klein, 1892).

The first works of Lie were about sets of lines in three dimensional projective complex space. The approach of the tetrahedron in this space is really in the spirit of the last works of Plücker. Lie's investigation, following (Hawkins, 2000, p. 2-6) consists in the consideration of a tetrahedron Δ of the complex projective space. A tetrahedron line complex Δ is determined by 4 planes. Each line meets Δ in four points. Each line intercepts in four points. Then he considers the set of lines T for which the cross ratio is the same. Lie considers then the set Θ of all projective transformations which let the vertices of the tetrahedron Δ invariant. Then for any given line he studies the orbit of this line under the projective group Θ. Another object of common research with Klein was the discovery and the study of what they called W-curves.

Two other crucial concepts in the investigation of Lie were the concept of infinitesimal transformation and contact transformation. An infinitesimal transformation is a function $x \mapsto x + dx$ which is defined by a system of linear differential equation in $\mathbb{R}^4$. These transformations form a commutative group and Lie and Klein used this tool in the study of W-curves. The concept of contact transformation is really close to Lie's method of resolution of PDEs as we will see now.

4.2 Lie's method of resolution

Lie (1872, 1873b, 1873a) interprets the equation $f(x, y, z, p, q) = 0$ as a four dimensional manifold of $\mathbb{R}^5$. The integration of equation means the determination of all manifolds $M_k, k \leq 2$ whose points satisfy

1. the equation;
2. the condition $dz - pdx - qdy = 0$.

With this interpretation, Lie can give a geometrical vision of solutions in terms of manifolds (Demidov, 1982, p. 343). In this theory the contact transformations play an important role.

Defining a contact transformation as a transformation which preserves the tangent, Lie shows that there exists a contact transformation which transforms a PDE equation in any given PDE. As Demidov observed, the demonstration of Lie was not completely rigorous. But this vision of PDEs was entirely new, and Lie's geometrical insight was crucial in this conception. The Lie conception also leads by mean of infinitesimal transformation to the identity of Jacobi, which represents indeed the beginnings of Lie' s Algebra.

In the following years, Lie will try to apply his method to second degree PDEs, and this theory will be developed by other mathematicians especially by Elie Cartan, but this is another story.

The history of PDEs of first order is a good example how a problem suffers many transformations in the time, both in the theories they required and the terms in which the problem is posed. I would wish to use the beautiful metaphor that Berger (2013) use in his book, *Geometry revealed*, that of Jacob's Ladder. To arrive to his theory, Lie had to go up in the ladder posing the problem in the new terms of differential geometry in 5-dimensional space, developing theory of contact and infinitesimal transformations. But perhaps this growth in the ladder did not turn him nearer God, because as a great mathematician of 20th century said: "we don't know if God exists, but the Book certainly does", and Lie, certainly too, wrote a few pages of the Book.

5 Conclusion

What can we deduce from this story for the elaboration of a new program of undergraduate mathematics ? A first idea is the connection between all disciplines, algebraic, analytical and geometrical methods are going together in solving problems and the exposition of graduate mathematics have to deal with this fact.

A second observation is related to the geometrical vision the student have to cultivate, even in more abstract algebra or analysis, a way Sobczyk (2013), for instance, had worked out in mathematics, and before Giaquinto (2007) tried to give an epistemological response. We have to break with the walls which separate disciplines in graduate mathematics if we want to train mathematicians and not scholastic students.

We have to emphasize finally that this path was already indicated in (Howe, 1983). In this article, as Howe enhances that Lie's Theory was taught only at graduate level, he insists too on the possibility to teach this topic in the undergraduate level (*loc. cit.* p. 601):

> While a complete discussion of Lie's Theory does require fairly elaborate preparation, a large portion of its essence is largely accessible on a much simpler level, appropriate to advanced undergraduate instruction.

More recently, Dresner (1999) wrote a text book which expounds the basics of Lie's theory of ordinary and partial differential equations. This book shows that it is possible to teach these ideas without waiting until the graduate level.

REFERENCES

Berger, M. (2013). *Géométrie vivante ou l'échelle de Jacob.* Cassini.

Cauchy, A. L. (1819). Note sur l'intégration des équations aux différences partielles du premier ordre à um nombre quelconque de variables. *Bulletin de la Société Philomatique de Paris*, *Année 1819*, 10-21.

Clairaut, A. C. (1743). *Théorie de la figure de la terre; tirée des principes de l'hydrostatique*. Paris.

D'Alembert, J. (1747). *Réflexions sur la Cause générale des vents*. Paris.

D'Alembert, J. L. R. (1743). *Traité de dynamique*. Paris.

D'Alembert, J. L. R. (1749a). Recherches sur la courbe que forme une corde tendue, mise en vibration. *Hist. de l'Ac. des Sc. de Berlin*, *3*, 214-219.

D'Alembert, J. L. R. (1749b). Suite des recherches sur la courbe que forme une corde tendue, mise en vibration. *Hist. de l'Ac. des Sc. de Berlin*, *3*, 220-229.

D'Alembert, J. L. R. (1749c). *Theoria resistentiae quam patitur corpus in fluidomotum, ex principiis omninò novis et simplissimis deducta, habitâ ratione tum velocitatis, figurae, et massae corporis moti, tum densitatis et compressionis partium fluidi, publisher=Manuscript in Berlin-Brandenburgische Akademie der Wissenschafte Akademie-archiv-I-M478*.

Demidov, S. S. (1982). The Study of Partial Differential Equations of the First Order in the 18th and 19th century. *Archives for History of Exact sciences*, *26*(4), 325-350.

Dresner, L. (1999). *Applications of Lie's Theory of Ordinary and Partial Differential Equations*. Institute of Physics Publishing, London.

Engelsman, S. B. (1982). *Families of Curves and the Origins of partial diferenciation*. North Holland.

Giaquinto, M. (2007). *Visual Thinking in Mathmatics, An epistemologica Study*. Oxford University Press.

Grimberg, G. E. (2009). Elaboração do cálculo diferencial leibniziano, A Gênese da teoria das funcões de várias variáveis no século XVIII. In D. Flament & W. Barroso (Eds.), *Dualidade Álgebra Geometria* (p. 59-137). Casas das Musas Brasilia.

Hawkins, T. (2000). *Emergence of the Theory of Lie Groups*. Springer.

Howe, R. (1983). Very Basic Lie Theory. *The American Mathematical Monthly*, *90*(9), 600-623.

Jacobi, C. G. (1837). Über die Reduction der integration der partiellen Differentialgleichungen Erster Ordnung zwischen irgend einer Zahl Variabeln auf die Integration eines einzigen Systems gewöhnlicher Differentialgleichungen. *J. reine und angew. Math.*, *17*, 1-189.

Jacobi, C. G. (1862). Über die Reduction der integration der partiellen Differentialgleichungen Erster Ordnung zwischen irgend einer Zahl Variabeln auf die Integration eines einzigen Systems gewöhnlicher Differentialgleichungen. *J. reine und angew. Math.*, *60*, 1-181.

Klein, F. (1892). *Über Lie's und meine Arbeiten aus den Jahren 1870-72 Von F. Klein und S. Lie. Über Lie's und meine ältere geometrische Arbeiten, von F. Klein (Erster Borkumer, Entwurf von August 1892)*. Manuscript, NSUB Göttingen, Cod. Ms f. kLEIN 22 GII, 24 p.

Lagrange, J. L. (1774). Sur les integrales particulières des équations différentielles. *Nouveaux Mémoires de l'Académie royale des Sciences et Belles-Lettre de Berlin*, *5*, 197-275.

Lie, S. (1872). Zur Theorie partieller Differentialgleichungen erster Ordnung, inbesondere iiber eine Klassifikation derselben. *Gottinger Nachrichten*, *1872*, 473-489.

Lie, S. (1873a). Partielle Differentialgleichungen erster Ordnung, in denen die unbekannte Funktion explizite vorkommt. *Forhandlinger Christiania*, *1874*, 52-85.

Lie, S. (1873b). Uber partielle Differentialgleichungen erster Ordnung. *Forhandlinger Christiania*, *1874*, 1-51.

Monge, G. (1807). *Applications de l'analyse à la géométrie*. Paris.

Pfaff, J. F. (1814). Methodus generalis, aequationes differentiarum particularum, necnon aequationes differentiales vulgares, utrasque primi ordinis inter quotcumque variabiles, complete integrandi. *Königlichen Akademie der Wissenschaften zu Berlin*, *1814-1815*, 76-136.

Sobczyk, G. (2013). *New Foundations in Mathematics, The Geometric Concept of Number*. Birkhaüser.