THE LIGHT PREFERS THE SHORTEST

Physics and Geometry about Shortest Path Problems from Heron to Fermat

Roberto CAPONE, Immacolata D'ACUNTO, Maria Rosaria DEL SORBO, Oriana FIORE

Università di Salerno, Fisciano, Italy
rcapone@unisa.it
Università di Salerno, Fisciano, Italy
idacunto@unisa.it
IS "L. Da Vinci", Poggiomarino, Italy
marodel@gmail.com
Liceo Statale "P.E. Imbriani", Avellino, Italy
orianafio@gmail.com

ABSTRACT

Our work is a detailed report about an educational experience of a group of students of the first two years of high school. The starting point is the classical problem of the "Seven Bridges of Königsberg", but the whole historical background of our inquiry spans from Heron to Fermat, on the thread of the shortest path problem. In particular, starting from physical phenomena easily experienced in the daily life, the Fermat's least time principle is introduced. The behavior of the light is described in a geometrical optic approximation, in analogy with kinematics. The activities, based on laboratory teaching and learning, are focused on linking physics and mathematics: refraction and reflection experiments, compass-and-straightedge constructions and interactive open source geometry software, as Geogebra; recognition of isometries in problems and reasoning based on their features; working by artifacts to build knowledge and skills; translation of geometrical objects in a purely algebraic language.

1 Introduction

Educational research recommends to make the students aware about the historical and theoretic path of mathematical sciences, bringing them closer to the origins of scientific thought. It is strongly advisable to provide students with concrete references to everyday reality, in order to help them to accomplish their targets in the educational process. Most likely, one of the best ways to deal with "shortest path" problems and symmetries consists in recalling phenomena ordinarily and spontaneously occurring in nature, Arts, Physics and Biology. For an instance, Physics offers to Mathematics countless and significant opportunities to observing and experimenting about the "shortest path". In detail, geometrical optics and classical theories on the behavior of the light are two central topics: on one hand they are tightly linked to the real-world and on the other hand they are a perfect "shortest path" problem's examples.

The activities proposed in an action-research approach, such as Mathematics and Physics laboratories to learn by discovering, can be considered a part of the cultural background of a skills-based education.

In this educational process, interdisciplinarity is a useful way to facilitate the transition from knowledge to action. In fact, "(skills) cannot be reduced to a single discipline; they assume and create connections between knowledge and suggest new uses and mastery, which means 'skills beget skills'" (D'Amore, 2000).

Learning by doing is very important because "when we experience something we act upon it, we do something; then we suffer or undergo the consequences. We do something to the thing and then it does something to us in return: such is the peculiar combination. The connection of these two phases of experience measures the fruitfulness of experience…". (Dewey, 2007)

We are going to show a teaching unit starting from a historical problem, whose resolution aims to activate more skills at the same time, in a holistic approach. The ability to integrate knowledge and ways of thinking coming from many different disciplines establishes areas of expertise to produce a cognitive advancement – such as explaining a phenomenon, solving a problem or creating a product – in ways that would have been impossible or unlikely through a single discipline. Skills produce skill in creating connections between knowledge and suggest new uses and settings of known elements.

Following a constructive methodology, we suggest to start from concrete states and to prefer an inductive experimental hypothetical-deductive approach. Everything we can know is the product of an active construction of the subject; as well as constructivists, we believe that "the learning isn't the result of the development, learning is the development".

Learning by doing is our point: in addition to theoretical explanation, we essentially introduce many experimental activities about optics.

In order to overcome a fragmented knowledge "closed" within a single discipline, all the activities were structured through an interdisciplinary teaching action, which could allow students to get new points of view. Our proposal does not focus on the subject, but on methodological aspects. Students are placed at the very core of the whole process of teaching and learning. In this way, they are allowed to acquire a general attitude in asking questions, in treating problems and in mastering knowledge. Indeed, a skills-based education aims to promote "a coordinated system of knowledge and skills put in action by the party, in connection with a purpose (i.e., a task, a set of tasks or an action) generating interest and promoting good internal motivational and affective attitudes" (Pellerey 2005). Aiming to making easier the transition from knowledge to action, an interdisciplinary expertise has been proved to be the best choice. As part of an interdisciplinary design, the concept of symmetry has offered theoretical suggestions in other disciplines such as biology, chemistry, mineralogy. Pursuing skills, with this respect, imply to exploit the knowledge in an integrated way, in order to address real and concrete problems. Carrying out the project has called upon different knowledge expertise and personal resources to handle situations, while building new knowledge and skills, always with the ultimate purpose of the education of men and citizens (Tornatore, 1974).

Our inspiring model was the constructivist “situated learning” (Lave, 2006) or “anchored learning”. This choice was straightforward, because the most part of learning depends on the context and on the place where the cognitive experiences were located in authentic activities, through project-based learning. The constructivism proposed the idea that learning occurs more efficiently if the learner is involved in the production of concrete objects. Our theoretical framework relies on a constructionist foundation. " The word constructionism is a mnemonic for two aspects of the theory of science education underlying this project. From constructivist theories of psychology we take a view of learning as a reconstruction rather than as a transmission of knowledge. Then we extend the idea of manipulative materials to the idea that learning is most effective when part of an activity the learner experiences as constructing a meaningful product". (Papert, 1980)

2 Educational Activities

The classical “Seven Bridges of Königsberg” problem is the starting point of our teaching activities. It is a problem inspired by a real city and a concrete situation. Konigsberg, formerly in East Prussia, today a Russian exclave on the Baltic known as Kaliningrad, is crossed by the Pregel River and its tributaries. In these rivers, there are two large islands connected with each other and with the two main areas of the city by seven bridges. Over the centuries it has been repeatedly proposed the question whether it is possible with a walk to follow a cyclic path that crosses each bridge just only once, returning towards the starting point. In 1736, Leonhard Euler faced this problem, showing that this hypothetical walk is actually impossible.

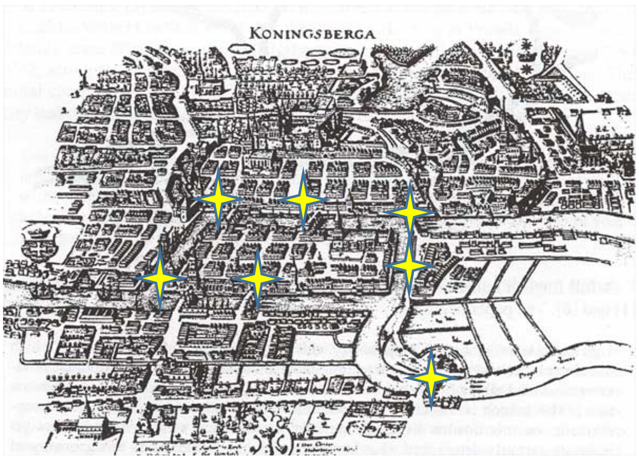

Figure 1. Königsberg’s map. The stars highlight the seven bridges’ placement.

The educational activities were broken up into steps:

Step 1 – Paper and ruler

Give students a sheet with a straight line and two points A and B, and asked them to:

1. Choose some points D, E, F, G on the straight line;
2. measure with the ruler the length of the polyline ADB, AEB, AFB and AGB;
3. sort the measures in ascending order and write the smallest measure.

Figure 2. straight line with two points, A and B

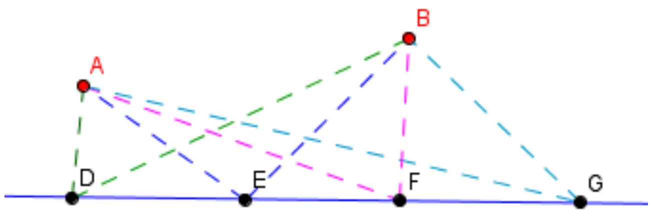

Figure 3. straight line with points and distance measures

Step 2 – Geogebra and table

Students are required to project a Geogebra file to represent the straight paths from A to B that touch the line in the point Q and then to tabulate the values found as a function of the position of the point Q on the straight line; unlike the previous phase, now, the tabulation of the points is performed by the software Geogebra, as well as the determination of the minimum.

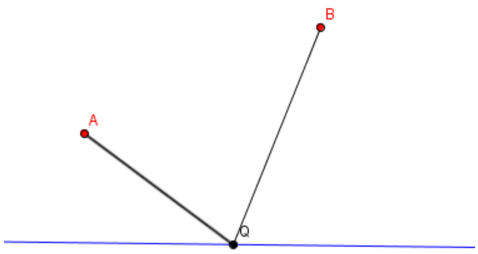

Figure 4. straight path between A, Q and B

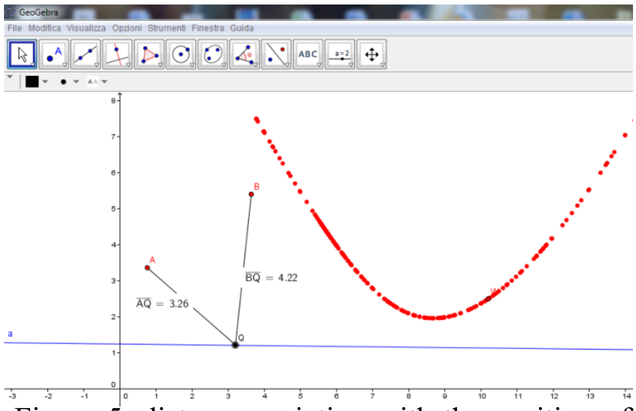

Figure 5. distance variation with the position of Q on the straight line

Step 3 – Locus of points

Students are required to design Geogebra files to graph the trend of the path length. In this step students:

1. build a locus of points, learning to deduce information by the graph;
2. conjecture that there is at least one point on the straight line Q corresponding to the shortest path.

The history of optics is closely linked to the history of geometry. Hero of Alexandria observed that the light travels in such a way that it goes to a mirror and to other points running the shortest possible *distance*.

Students are involved in a simple but concrete experience, using flat mirrors for observing the behavior of the light and the reflection of the images and guessing the geometric characterization of the shortest path.

Step 4 – Mirror and Physics laboratory

Learners are asked to analyze the image of the reflection of two points drawn on a paper sheet in front of a mirror and then perpendicular to it. The sequence of actions is:

1. simulate the mirror using two paper sheets;
2. draw the segments between the points;
3. measure the segments.

Students should consider that the segments AB’ and A’B intersect in a point P; The segments AA’ e BB’ look like they are perpendicular to the sheet’s edge placed near the mirror; point P looks like it belongs to the reflection axis.

After measuring the segments, point P comes out to be the midpoint between A and its image A’ as well as from B and its image B’ (Figure 6).

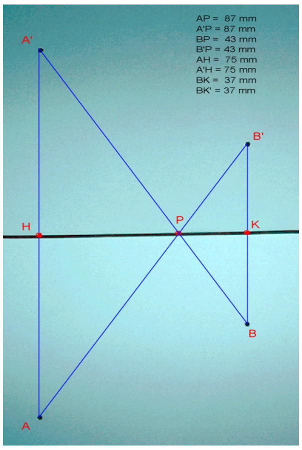

Figure 6. simulation of a mirror reflection

After that, students are asked to repeat the construction using Geogebra software to prove and verify that their measurements and observations are correct.

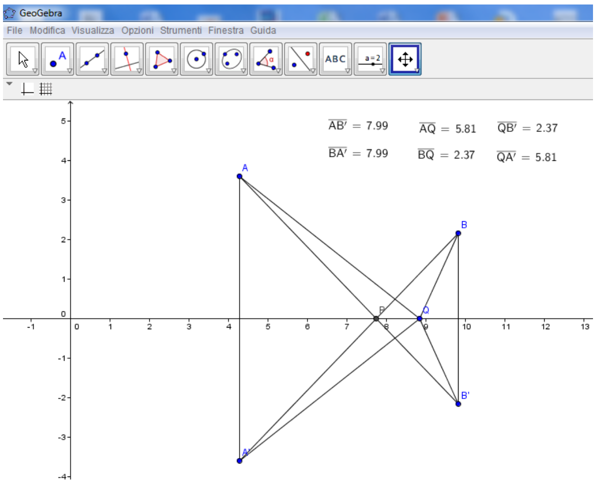

Figure 7. Geogebra simulation of reflection experiment

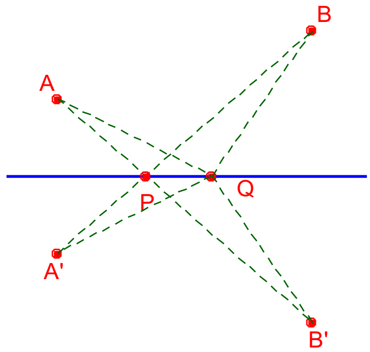

Figure 8. Drawing's details

At this point students are required to find out the characteristic properties of the point P of the straight line minimizing the length of the path APB.

Students are provided with the following directions:

1. Consider on the reflection axis a point Q different from the point P and link it to the points A, B and B';
2. Measure the polylines APB, AQB, AQB';
3. Describe your observations;

This exploration/discovery phase is supported by Geogebra software; at the end students are required to prove or disprove the conjecture: the point corresponding to the shortest path is the intersection between the segment AB' and the reflecting axis.

According to the properties of the axial symmetry, the length of the polyline AQB' is equal to the length of the polyline AQB and the length of the polyline APB' is equal to the length of the polyline APB.

Observing the triangle AB'Q we can infer that, because of the triangle inequality, the path APB' equal to APB is the shortest path, compared to any other path passing through a point P on the straight line different from P (Figure 7 and 8).

Step 5 – Angles

Students are asked to compare the magnitude of the angles between reflection axis and the segments AQ and BQ (Figures 9 and 10) and to verify the conclusions of the previous step. Students observe that, if Q overlaps P, the angles are equal.

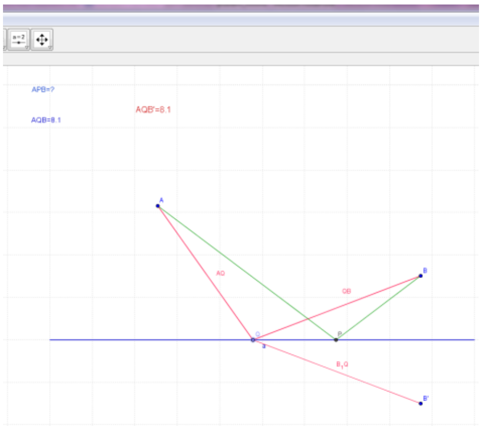

Figure 9. Angles in Geogebra simulation

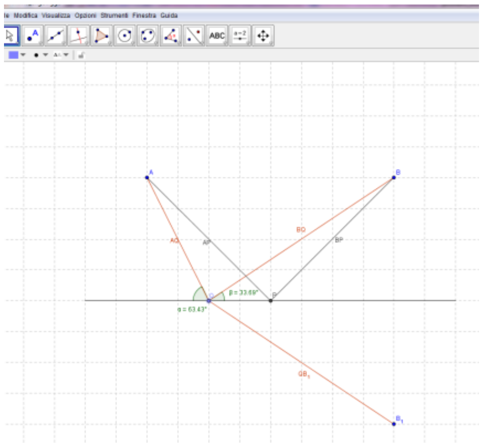

Figure 10. Details of angles

Then they make a conjecture: Angles APX and BPY are equal.

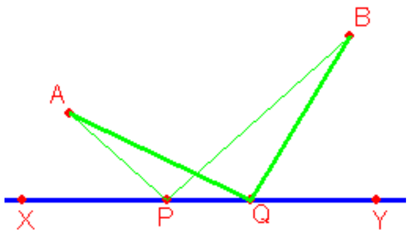

Figure 11. Further details of angles

The magnitude of the angle APX is equal to the magnitude of the angle XPA', as long as they are correspondent in a axial symmetry. The magnitude of the angle XPA' is equal to the magnitude of the angle BPY they are opposite angles. Thus, by the transitivity, the amplitude of the angle APX is equal to the amplitude of the angle BPY.

Step 6 – Law of reflection

At this stage the students can observe the phenomenon of light reflection using, for an instance, a mirror and a laser pointer and asking them to describe this experience as a mathematical expression. A deep connection between the "minimum problem" and the light's behaviour. Students observe that the light "chooses", the shortest of many paths (heron), the shortest polyline. Moreover, they also observe that the path composed by the incident ray and the reflected ray (Figure 12) links the two points in the shortest time.

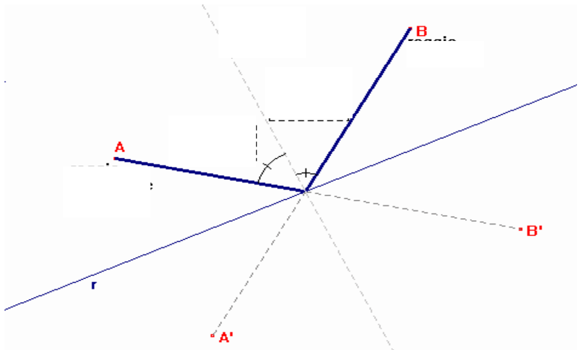

Figure 12. Law of reflection

Step 7 – Physics experiment: brachistocrone.

Students are required to guess which ball arrives first, falling down on different trajectories between two points as in the picture shown in Figure 13. It's easy to show, using an exhibit or an applet that the minimum path for a little ball moving from the point A to the point B is a curved line and not, as students can suppose in advance, a straight line. Observing this evidence, students are usually amazed.

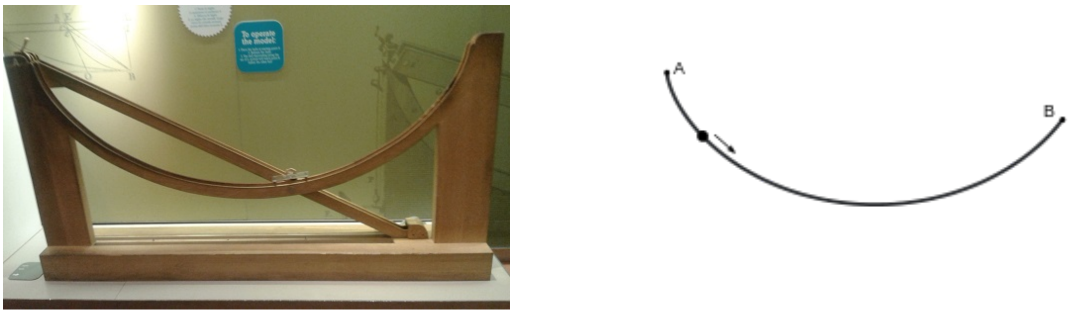

Figure 13. picture of Galileo's brachistocone and a cycloid (Galilei's museum, Firenze)

Using geometrical methods, in 1602, Galileo showed that a body subject to gravity takes less time to fall along the arc of a circumference between two points than along the corresponding segment of straight line, notwithstanding that the latter is shorter. Galileo did not realize that the "brachistochrone path" of a body leading down between two points is an arc of a cycloid, and not an arc of a circumference. The mathematical proof of brachistocronism of the cycloid was provided by Jacques Bernoulli in 1697.

Step 8 – Testing

Given the point A(0,1) and B(1,2), detect the point A' correspondent to A in an axial symmetry whose axis is the horizontal axis in a Cartesian coordinates system; then write down the equation of the straight line passing through points A' and B. Finally, find the point P as the overlap of the straight line r on the x-axis.

Chosen freely the coordinates of the points A and B, detect the coordinates of the point P as in the previous case.

Chosen freely the coordinates of the points A and B, belonging to two opposite half-planes with respect to horizontal axis (e.g. A(0,1) and B (1,-2)), detect the abscissa of the point P corresponding to the shortest path. Translate in a correct mathematical language all the choices of the previous steps.

Given two locations A and B on opposite sides of the bank of a straight river, locate where it is better to place a bridge on the river to minimize the length of the path that connects the location A to location B. (It is assumed that the banks are parallel and that the bridge is built perpendicularly to the shore).

Step 9 – Minimal path: Fermat and the refraction

It's a daily experience for students to see objects "broken" by refraction of light passing from one medium to another, typically from air to water. The minimum path principle of Hero of Alexandria works fine for light passing through homogenous media with the same refraction index. When light passes from one medium to another with a different refraction index, its velocity changes.

Figure 14. Refraction - Picture of a rope and its refracted image

Fermat was inspired from Heron to explain this phenomenon: due to refraction, light obviously does not choose the path of shortest *distance* but it prefers the shortest *time.* Thus the statement that the angle of incidence is equal to the angle of reflection is equivalent to the statement that the light goes to the mirror in such a way that it comes back in the *least possible time*.

Geometrical optics is just an approximation, but it is very relevant by a technical point of view and of great historical interest: the real behavior of light was discovered by Fermat about in 1650, it is called *the principle of the least time*, or *Fermat's principle*. Although we highlight to the students the well know double nature of light, our discussion is limited to the geometrical optics region, where we ignore the wavelength and the photonic character of the light. In fact, when we run reflection and refraction experiments, the wavelengths involved are very small compared to the dimensions of the equipment available for their study;

furthermore, the photon energies, using the quantum theory, are small compared with the sensitivity of the equipment.

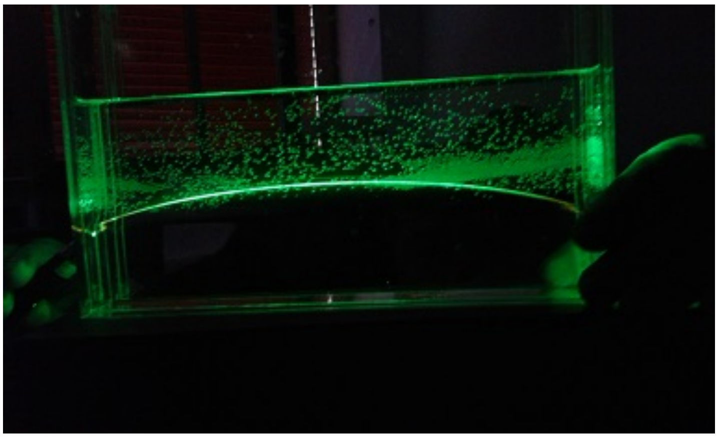

Figure 15. Experiment with a laser beam passing through water and glycerol

It is not difficult to design and build experiments in order to investigate the trajectory of the light, namely laser beam, as it goes through different media: glass, water, Plexiglas, transparent oils, glycerol, alcohol, etc. Students can see that the light propagates in a straight line. But, due to the Fermat's principle, if the light passes from one medium to another with a different refraction index, it changes direction. In the Figure 15 is shown an experiment with a laser beam passing through water and glycerol, two nonmiscible media: the beam curves!

3 Methodology

One of the most important inspiring models for our educational experiment was Inquiry Based Science Education (IBSE) model, which proposes teaching activities based on the investigation about the problems, the critic debate in groups and the research of innovative solutions, in a constructive outlook. Students deal with a problem originated from the observation of real world and teachers ask them to identify the problem making hypotheses (engage). After that, they had to plan the inquiry, exploring the variables (explore), to conduct an investigation on their own or in groups, documenting the outcomes (explain); Then, together to the teacher, who, in all the activities, operated as a scaffolder, the results were interpreted (evaluate) and communicated by formulating new problems (extend).

The students were given some assignments to be carried out using artifacts within a cycle that promotes the use of specific signs in relation to the use of special tools or artifacts, such as work in pairs or in small groups, with the artifact that promotes social exchange, together to words, plans, gestures. Students have been involved individually in different semiotic activities, especially affecting written productions. For example, after using an artifact, the students were often asked to write an individual report of their experience and their reflections, including doubts and questions arisen.

The collective discussions, finally, have been an essential part of the teaching-learning process, at the heart of the semiotic process, where the teaching-learning is based on. The whole classes were collectively engaged in a mathematical debate to answer a mathematical question, usually launched by the teacher, who explicitly formulates the topic of discussion.

During the meetings, students have always been given a task to accomplish. They were asked questions about the procedure of the activities (e.g., “how are you going to do it?”, “what are you doing?” “how did you do it?”); we tried to make the student aware of the significance, function, methods and potential of their knowledge. This process of metacognition was constructed through reflection and reconstruction process of how a student learns. Experimental practice was always been followed by theoretical and group activities. In fact, in the same way, it is not possible simply teaching in an abstract, formal and theoretical way, without a context, we cannot also leave students at the very early stage of experience and “doing”.

GeoGebra software turned out to be a resource for both displaying of concepts and proof. In accordance with national indications that "the student will be able to switch from one representation register to another (numerical, graphical, functional) even using IT tools for data representation", Duval (2006), the paths have been made of minimum distance with the help of the software and compared with the paths of a light beam.

Students can see the objects handled in a dynamical geometry software in two different ways: as simple pictures, i.e. relying on the perceptive aspects of the observation, or as schemes linked to a theory, that is, relying on conceptual aspects, Fishbein (1993). In our case, rather than to speculate and formulate, students were able to take advantage of the perceptive aspects of the figures. It was not here the case to choose whether to build using ruler and compass or to exploit the potential of the dragging, because the two things are not excluding each other. In a sense, the epistemological potentialities included the verification of theoretical models and empirical data at the same time.

4 Conclusions

The purpose of the educational path shown in this work was the realization of geometric constructions with simple tools such as ruler and goniometer and open source software, to recognize, in problematic situations, isometries, to explicate them using their properties and to decode, in algebraic language, the "geometric objects."

At the end of the learning path, students learned to work in groups and socialize / share their experiences; they also learned to use knowingly the dynamical geometry software Geogebra and to deduce, infer and understand aspects of the real world through the mathematical formalism and through abstract models; besides, they learned to highlight possible links and / or analogies setting them in a unitary context.

The ruler and compass constructions and the Geogebra software as semiotic mediation tools encouraged communication, stimulated discussions and facilitated the sharing of knowledge. This methodology, as a final outcome, simplified and accelerated the understanding of the properties of geometric transformations. It has also led to a shift from a colloquial language register to an advanced language, suitable to formalize definitions and properties.

REFERENCES

Bartolini Bussi, M. G. & Mariotti, M. A. (2008), Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English (ed.), *Handbook of International Research in Mathematics Education* (second edition). Routledge.

Bern, M. W., & Graham, R. L. (1989). The shortest-network problem. *Scientific American*, *260*(1), 84-89.

D'Amore, B. (2000). La Didattica della Matematica alla svolta del millennio: Radici, collegamenti e interessi. *La matematica e la sua didattica*, *3*, 407-422.

Dewey, J. (2007). *Democracy and Education*. Teddington*:* Echo Library.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*, 103-131.

Feynman, R. D. (2006). *QED, The strange theory of light and matter*. Princeton science library.

Fishbein, E. (1993) "The theory of figural concepts." *Educational studies in mathematics*.

Kourant, R., & Robbins, H. (1941). *What is Mathematics*. Oxford University Press.

Lave, J., & Wenger, E. (2006). *L'apprendimento situato. Dall'osservazione alla partecipazione attiva nei contesti sociali*. Edizioni Erickson.

Mushkarov, O. (2010). "A*lla ricerca dei cammini minimi*". Retrieved from http://www.math2earth.v3d.sk/publications/08_In%20search%20of%20shortest%20paths.pdf

Papert, S. (1990). *Constructionist learning. A 5th Anniversary Collection of Papers, Reflecting Report Research, Projects in Progress, and Essays by the Epistomeology.* Cambridge: Learning group MIT Media Laboratory, 3.

Papert, S. (1980). *Mindstorms: Children, Computers, and Powerful Ideas*. New York: Basic Books.

Pellerey, M. (2005). Verso una nuova metodologia di ricerca educativa: La Ricerca basata su progetti (Design-Based Research). *Orientamenti pedagogici*, *52*(5), 721-737.

Radford, L. (1997). On psychology, historical epistemology, and the teaching of mathematics: Towards a socio-cultural history of mathematics. *For the learning of mathematics*, *17*(1), 26-33.

Tornatore, L. (1974). *Educazione e conoscenza*. Loescher.