ARTICULATION OF MATHEMATICAL NOTIONS WITH QUECHUA NOTIONS ACROSS HISTORY OF MATHEMATICS AND DYNAMIC GEOMETRY

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ABSTRACT

Peru is a multilingual, multicultural country with a bilingual education system (now termed intercultural) that has developed greatly in recent years. Since 2014, the Peruvian government has offered scholarships to students from indigenous communities working to develop this education sector and cover an estimated deficit of 20,000 teachers. Prestigious universities in Lima have opened Undergraduate Programs in Intercultural Bilingual Education (IBE), which bring together students from different indigenous communities, specifically Quechua, Aymara (Andeans) and Shipibo (Amazonians). In an initial assessment, most students did not achieve a satisfactory level of performance in mathematics. This paper develops a proposal for integrating long-established Andean mathematical tools and dynamic geometry software into the learning and teaching process in order to improve attainment of mathematical competence in the intercultural education sector.

1 Research problem

Results from the 2007-2014 national census assessments indicate underperformance in mathematics by indigenous students (OMCA, 2015). The measurement curve for satisfactory performance (see Figure 1) from this time period shows a tendency toward lower performance in rural areas in comparison to the curve for urban areas. Indigenous communities are mainly located in rural areas. Over the years, the achievement gap has continued to widen. One factor that could explain this situation is that, although students study in their native language, the teaching and learning process does not consider the mathematical notions implied in the daily practices of their communities, thus divorcing school-taught mathematics from the logic of the bodies of mathematical knowledge pertaining to their cultures (González & Caraballo, 2015).

The research question in this paper is as follows: Is it possible to improve the level of numerical performance of students of Intercultural Bilingual Education using the *Yupana* for calculating basic operations, and improve their level of geometrical performance through the use of Dynamic Geometry software for designing their cultural objects?

2 Theoretical Framework

Oliveras (2015) indicated that there is a relativist paradigm in mathematics. In the same way, researchers suggest that the teaching and learning of mathematics is related to the history, language and culture of specific peoples (Barton 2008, cited by Sun & Beckmann 2015), (Ellis & Berry, 2005), (Ernest, 2005).

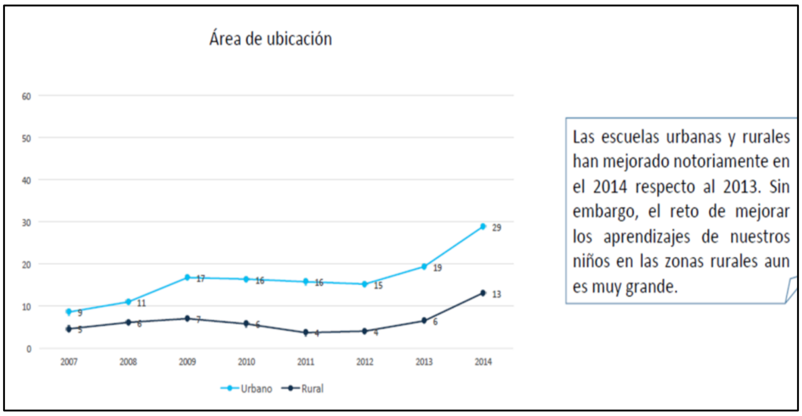

Figure 1. Percentage of students at the satisfactory level in mathematics 2007 – 2014. Rural and urban area. (OMCA, 2015)

From a cultural perspective, Cauty (2001) affirms that all cultures, create bodies of knowledge and systems for symbolically representing mathematical concepts. He believes that mathematics education should be taught in a way that conforms to the cultures of students, taking their culturally-specific mathematical conceptions into account, articulating the knowledge and skills you want students to acquire, without causing deculturation.

Over the past four decades, there has been a shift towards integrating the history of mathematics into the teaching of mathematics (Jankvist, 2009). Epistemological and didactic foundations support the introduction of the historical dimension in teaching mathematics (Barbin, 2012b), (D'Enfert, Djebbar and Radford, 2012). Similarly Barbin highlights the increasingly growing trend of teaching the history of mathematics in teacher training (2012a). Furthermore, the academic community of the HPM Group (History and Pedagogy of Mathematics) noted the need for more empirical research on integrating the history of mathematics into the teaching and learning of mathematics (Arcavi, quoted by Jankvist, Op.cit.).

The research into the history of mathematics has made use of Inca calculation and data-recording devices, such as the yupana and the quipu, respectively, to bolster arguments in favour of integrating culturally-relevant mathematical concepts into culturally-specific mathematics pedagogy. According to this research, the yupana and quipu were used as pedagogical tools to teach mathematics to children throughout Inca history (Villavicencio, 2011).

Concerning learning and teaching geometry, Dynamic Geometry System (DGS) is a *milieu didactique,* didactic milieu. DGS was created as a specific type of digital application that allows students to build and drag mathematical objects on the screen (Arzarello, Bartolini, Leung, Mariotti & Stevenson, 2012). The multiple computational tools have rejuvenated mathematics and mathematics education.

> Any DGS figure is the result of a construction process, since it is obtained after the repeated use of tools chosen from those available in the ‘tool bar’… what makes DGS so interesting is the direct manipulation of its figures, conceived in terms of the

> embedded logic system of Euclidean geometry (Laborde and Straesser, 1990; Straesser, 2001; cited by Arzarello, et al, 2012, p. 103).

The important thing is to know how to articulate the DGS with culturally-specific Amazonian and Andean mathematical knowledge.

3 Tools of history of mathematics

This research, taking into account the cultural and historical dimension in mathematics education, applies the tools of historical Inca mathematics, the yupana and quipu, to develop numerical skills in undergraduated students in Intercultural Bilingual Education. They also studied characteristics of Inca mathematics.

Likewise, it is necessary to develop the capability of the Cabri II Plus geometry application in undergraduated students to design Amazonian and Andean cultural objects.

Both proposals were tested in IBE initial teacher training.

3.1 Andean and Amazonic number systems

The Inca number system is decimal and positional, like most Western languages. The Aymara and Shipibo cultures originally had only five and three numbers, but under the Inca empire, along with the spread of the official Quechua (Runa Simi) language, the Inca number system was adopted (Figure 2).

3.2 Yupana

Yupana comes from Quechua word, to count. The Yupana is a type of abacus for performing arithmetic operations. They were made of clay, stone or wood. The numbers were represented with corn kernels, seeds, or pebbles. A yupana can be seen in the lower left corner of a drawing of "*Primer Nueva Corónica y Buen Gobierno*", a book written by the indigenous chronicler Felipe Guaman Poma de Ayala (Figure 3). The illustration suggests that the *quipucamayoc* (Quechua word meaning, the one who owns or masters quipu) calculated with the yupana and then recorded the data on the quipu.

The yupana was known and appreciated by colonial administrators. The Spanish priest José de Acosta wrote the following about this artifact in his book *Historia Natural y Moral de las Indias*:

> To see them use another kind of quipu with maize kernels is a perfect joy. In order to effect a very difficult computation for which an able calculator would require pen and ink for the various methods of calculation these Indians make use of their kernels. They place one here, three somewhere else and eight I do not know where. They move one kernel here and three there and the fact is that they are able to complete their computation without making the smallest mistake. As a matter of fact, they are better at calculating what each one is due to pay or give than we should be with pen and ink. Whether this is not ingenious and whether these people are wild animals let those judge who will! What I consider as certain is that in what they undertake to do they are superior to us. (Acosta, 1590; quoted in Leonard & Shakiban, 2010).

Número	Quechua Collao	Quechua Incahuasi Cañaris	Aimara	Shipibo Konibo
1	*huk*	*uk*	*maya*	*westiora*
2	*iskay*	*iskay*	*paya*	*rabe*
3	*kimsa*	*kimsa*	*kimsa*	*kimisha*
4	*tawa*	*ĉusku*	*pusi*	*chosko*
5	*pichqa*	*pichqa*	*qallqu*	*pichika*
6	*suqta*	*suqta*	*suxta*	*sokota*
7	*qanchis*	*qanĉis*	*paqallqu*	*kanchis*
8	*pusaq*	*pusaq*	*kimsa qallqu*	*posaka*
9	*isqun*	*isqun*	*llätunka*	*iskon*
10	*chunka*	*ĉunka*	*tunka*	*chonka*
11	*chunka hukniyuq*	*ĉunka uk*	*tunka mayani*	*chonka westiora*
12	*chunka iskayniyuq*	*ĉunka iskay*	*tunka payani*	*chonka rabe*
13	*chunka kimsayuq*	*ĉunka kimsa*	*tunka kimsani*	*chonka Kimisha*
14	*chunka tawayuq*	*ĉunka ĉusku*	*tunka pusini*	*chonka chosko*
15	*chunka pichqayuq*	*ĉunka pichqa*	*tunka qallquni*	*chonka pichika*
16	*chunka suqtayuq*	*ĉunka suqta*	*tunka suxtani*	*chonka sokota*
17	*chunka qanchisniyuq*	*ĉunka qanĉis*	*tunka paqallquni*	*chonka Kanchis*
18	*chunka pusaqniyuq*	*ĉunka pusaq*	*tunka kimsa qallquni*	*chonka posaka*
19	*chunka isqunniyuq*	*ĉunka isqun*	*tunka llätunkani*	*chonka Iskon*
20	*iskay chunka*	*iskay ĉunka*	*paya tunka*	*rabe chonka*
30	*kimsa chunka*	*kimsa ĉunka*	*kimsa tunka*	*kimisha chonka*
40	*tawa chunka*	*ĉusku ĉunka*	*pusi tunka*	*chosko chonka*
50	*pichqa chunka*	*pichqa ĉunka*	*qallqu tunka*	*pichika chonka*
60	*suqta chunka*	*suqta ĉunka*	*suxta tunka*	*sokota chonka*
70	*qanchis chunka*	*qanĉis ĉunka*	*paqallqu tunka*	*kanchis chonka*

Figure 2. Some Andean and Amazonian number systems
(Source: Ministerio de Educación de Perú, 2013)

Figure 3. Quipu and Yupana

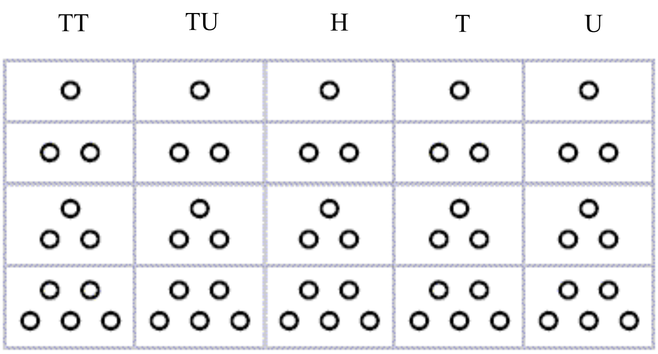

Figure 4. The yupana with a 90 degrees turn

Educators and social scientists have used the yupana in several educational research projects in Perú. Such is the case of Martha Villavicencio (2011) who worked in the Experimental Bilingual Education Project in Puno (1978-1988) and in the Andean Rural Education Program (1989-1996) in Ecuador. Likewise, the yupana was promoted as a teaching aid for the first elementary grades by the Peruvian anthropologist Daniel Chirinos (2010), in a bilingual education project in Loreto (peruvian amazon region) funded by Spanish Agency for International Development Cooperation (AECID) in Peru. ‘There are research and testimonies certifying the educational capacity of the yupana in facilitating the understanding of the positional value of numbers in the decimal system and the execution of immediate arithmetic calculations’ (Montalvo, 2012, p. 16).

The yupana (Figure 4) is composed of five rows and four columns. In the first column there are five circles in each row, in the second, three circles, in the third, two circles, and in the last column, one circle. The bottom row represents the units, the previous row is for the tens, the third from the bottom up represents the hundreds, the fourth, for the units of thousand, and the fifth, the ten-thousands (González & Caraballo, 2015). A corn kernel (seed or pebble) in a circle represents unit, ten, hundred, or thousand, according to its position in the yupana. It is used to do the arithmetic operations of addition, subtraction, multiplication and division with whole numbers in a concrete way.

Practical examples of operations with the yupana are detailed below (Mejia, 2011). Examples of addition with and without regrouping, subtraction with and without decomposing, and a multiplication with yupana are shown from Figures 5 to 8.

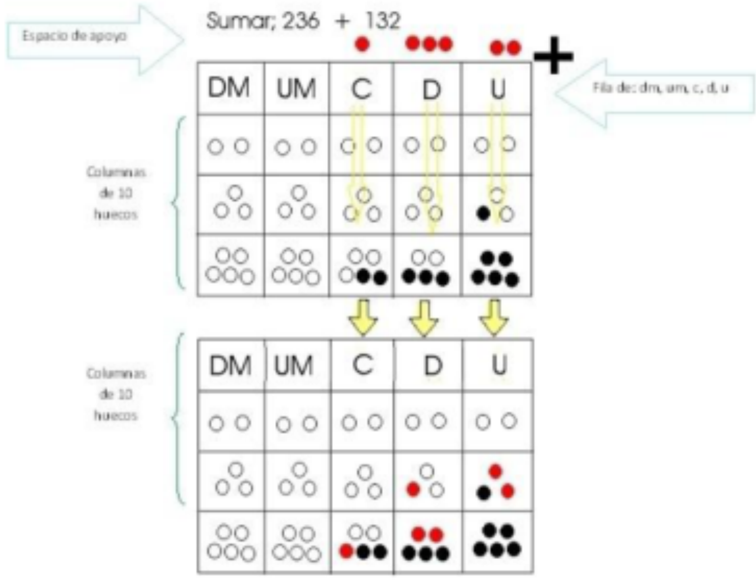

Figure 5. Sum with Yupana (Mejía, 2011)

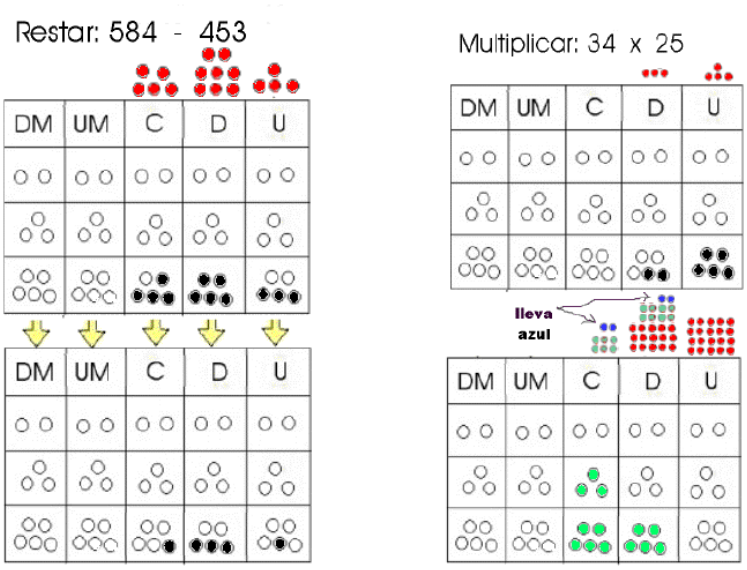

Figure 6. Subtraction and multiplication with Yupana (Mejía, 2011)

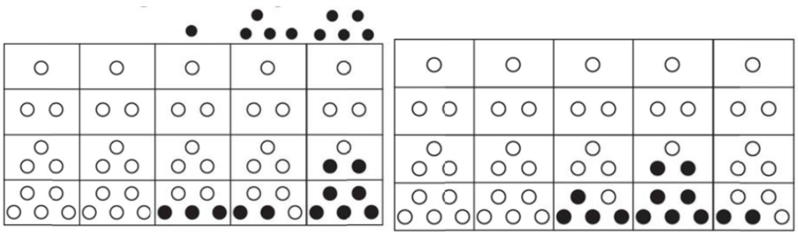

Figure 7. Sum with regrouping, 327 + 145 (González & Caraballo, 2015)

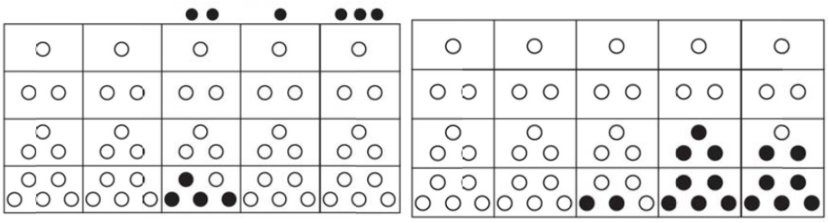

Figure 8. Subtraction with decomposing, 500 – 213 (González & Caraballo, 2015)

3.3 Quipu

Quipu is a Quechua word meaning knot. The Quipu was an Inca writing system containing statistical information, demographic data, and also a record of information of abstract topics, such as political systems, myths, etc. (Zapata, 2010). The Incas were not an exclusively oral culture. They knew how to preserve historical memory through an annotation system with knots. The quipu is not a phonetic writing system. It is a writing system using visual or tactile signals containing meaning. It is a system of three-dimensional annotations that makes sense through the combination of knots, colors, shapes and rope twisting.

The societies that spoke languages with same root, as root of Indo-European languages, developed a common path towards phonetic writing, despite being mutually unintelligible. In those nearby territories, where languages from various roots interacted, was developed ideographic writing, because the signs did not represent sounds, but concepts- such as the knots in the case of quipu - and it allowed people from different languages to read without translation. Hence, the information recorded in the quipu could be interpreted by different quipucamayocs, despite they spoke different languages.

The Quipu is formed by a horizontal main rope. Several vertical hanging ropes are tied to the main rope. In the hanging ropes are the knots that contain information. In the mathematical quipu, numerical information was coded using the Inca number system. The numbers were represented according to the turns of the knots and according to the position of the knot in the hanging rope. The knots farther away from the main cord are units, as they rise, represent the tens, the hundreds, the thousands and the ten-thousands.

3.4 Application of tools of history of mathematics in the pre-service training

As has been previously discussed, several projects have been attempted since the advent of the IBE in 1978 to incorporate culturally-specific mathematical concepts and ancient tools into the primary education. The novelty of the present proposal is that these methods have been used by undergraduates of IBE to achieve basic learning of numbers and operations. Undergraduate students in the IBE Program of the Faculty of Education at the Cayetano Heredia Peruvian University, in 2015 school year, in the course Mathematics 1 and Ethnomathematics, learned to add, subtract and multiply with Yupana. They also studied the Quipu and its relationship to the Yupana.

The efficacy of these tools can be seen in videos where students can add, subtract and multiply, both in Spanish and in their native languages, Quechua, Aymara and Shipibo. The videos are posted on YouTube and can be seen in the following link. https://www.youtube.com/channel/UCgPeZpHDhIEpnIWgNM5-Csw/playlists The videos were uploaded in March 2015.

4 Tools of Dynamic Geometry

The focus of our research in dynamic geometry was designing a didactic proposal that incorporates the geometric designs in Amazonian and Andean culture as teaching tools.

In learning geometry, the research question posed was: How do we get that indigenous students use the fundamental concepts of geometry to build their cultural designs? For solve the problem posed is more advisable that geometric concepts appear in constructed designs, in a dynamic and interactive learning environment.

The Quechua, Aymara and Shipibo communities make designs using lines, segments, polygons, applying geometric transformations, all without a theoretical knowledge, having learned these techniques as part of the handed-down cultural knowledge passed from one generation to the next. The students gained a theoretical understanding of the geometrical concepts underlying their cultural art works through using Cabri II Plus software. Students built their Amazonian (Figure 9) and Andean (Figure 10) geometric designs using polygons, lines, segments, geometric transformations. This activity reinforced their geometrical knowledge in a meaningful way that allows it to take root in their minds as a practical tool and not just an intellectual abstraction. The designs can be seen on the following link https://www.flickr.com/photos/63485986@N02/sets/72157649003841112/

The process that was used to develop these patterns with Cabri can be found in the short course presented at the XIV Inter-American Conference on Mathematical Education (Bonilla, 2015).

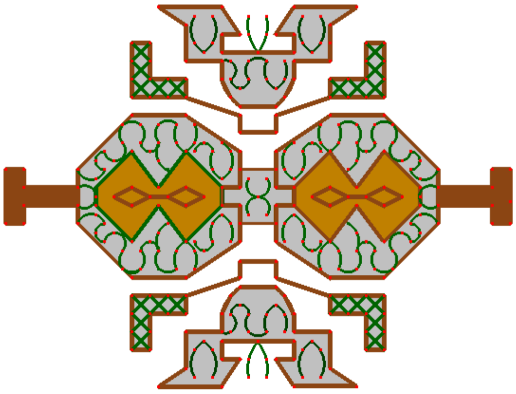

Figure 9. Shipibo design (Amazonian)

Figure 10. Quechua design (Andean)

5 Conclusions

At the end of the activities, tests were applied to measure numeracy skills in the development of basic operations, and to identify the knowledge and construction of the fundamentals of geometry, figures and geometric transformations. Students were able to increase their level of performance in the numeric domain and in the geometric domain with the proposed activities. The most important achievements are located in the affective and emotional field. The study of yupana and quipu, making the designs of their cultures, reaffirmed their identity, revaluing their cultural objects. The affirmation and approval of their work raises their self-esteem, injecting enthusiasm to develop new projects in their academic training, becoming aware of its potential.

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