PANEL 1: THEORETICAL AND/OR CONCEPTUAL FRAMEWORKS FOR INTEGRATING HISTORY IN MATHEMATICS EDUCATION

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ABSTRACT

This panel considers theoretical rather than practical frameworks for the alignment of history of mathematics and mathematics education. Naturally, such a theoretical framework will have practical implications, but its main task is to set out a certain line of questioning. First of all, it must also ask, even if only implicitly, what it really means to be a theoretical framework in the first place. But, principally, it must ask the ends and the value, and, ultimately, the meaning of history of mathematics in mathematics education. Though all the authors agree that in any theoretical framework, history of mathematics, as such, must be taken as the starting point. That said, no claim is made that there must be a single theoretical framework. Nevertheless, each of the three parts of this paper emphasize similar themes. One of these is the relationship between readers and texts: what do readers bring to texts? How are they affected by the texts? How are their own mathematics selves shaped by their engagement with historical material?

1 The idea of a theoretical framework

Michael N. Fried

The task of this panel is not to set out specific practical approaches for bringing history of mathematics into the mathematics classroom: the aim is not to produce a catalog of options from which teachers can choose an approach fitting their needs, and it is not to argue for one particular ideal approach. Rather, its task is to try and delineate questions—some key questions at least—which any theoretical framework must embrace or suggest in the first place. For any theoretical framework is at bottom a set of questions giving rise to other questions.

Of these key questions, the very first, and maybe the most important question, is a reflexive one: what does it mean to be a theoretical framework in the first place? More concretely, what does it mean to be a theoretical framework, as opposed to a practical framework—what should its object be? What should it look like? Certainly, it cannot be of the form, “Do X, do Y, do Z,” or even “Try X, Y, or Z.” Such directives would imply a theoretical framework already in place. They imply that we know *why* we are to try X or Y, that we know what ends we are trying to achieve. Any theoretical educational framework ought to be cognizant of ends, of what goals one is aiming for and why they are good. It differs in this way from a theoretical framework for a natural science where the main concern is a description of how things are and why they are as they are: we would not expect a physical theory to say why it is better that like poles of a magnet repel than attract, that is, while it might say why like poles repel, it does say why they ought to repel. So a theoretical framework for history of mathematics in mathematics education should at least address the question of why we ought to learn history, what is to be gained by it, what value there is in it and what value there is in learning it in a certain way.

But it is not enough to have a reason for teaching history of mathematics. For one's end in bringing history of mathematics into mathematics teaching might have little to do with history. This is the case, surely, when teachers suggest using history of mathematics to "liven" their teaching, so that what the teacher really wants to teach, say, the quadratic formula, will be met by students more awake and more willing to learn. Let me say a little more about this motive, which, following Gulikers and Blom (2001), I have called elsewhere (Fried, 2014a), the "motivational theme."

Assume that history truly has this motivational effect on students and that, further, it is not merely the effect of novelty, that is, it persists even after it has become routine. For the motivational theme to be the basis of a theoretical framework for teaching history of mathematics in the mathematics classroom, one would have to ask what is it about history that touches students and moves them. What part of their intellectual lives is touched by history? Asking these questions is essential to the question of a theoretical framework; however, if we obtain an answer to them or even if we manage to bring out the force of them, it will be something quite different from a statement of the form, "history of mathematics increases students' motivation." The affective gain will, in this regard, be a kind of epiphenomenon only. To be sure, whether or not students enjoy mathematics or are motivated to study it with more vigor is not something to be ignored. The point is only that it itself does not provide a theoretical framework for introducing history of mathematics into mathematics teaching: in providing an end for the study of history, a theoretical framework must refer to history, not to pleasure, as important as that might be.

The purely instrumental role of history of mathematics, by pointing to ends foreign to history as such, can blind us in our search for a theoretical framework specifically for history of mathematics in mathematics education. The problem is that indeed those other ends are important goals for mathematics education. In this connection, a more difficult and subtle case in which ahistorical goals are made goals for history is that concerning students' understanding of mathematical content.

Whatever other knowledge we hope students will gain in their mathematical education—and I will return to this point shortly—one could hardly imagine excluding knowledge of concepts, propositions, and procedures central to modern mathematics, even if one can argue that this concept or this proposition or that procedure may be safely eliminated. Assuming the importance of mathematical ideas as they are understood today, history is often taken as a means of making these ideas more accessible. In the context of the motivational argument, more accessible only means more palatable; in the context of mathematical knowledge, *per se*, more accessible means more understandable or more deeply understandable, though these are not exactly the same.

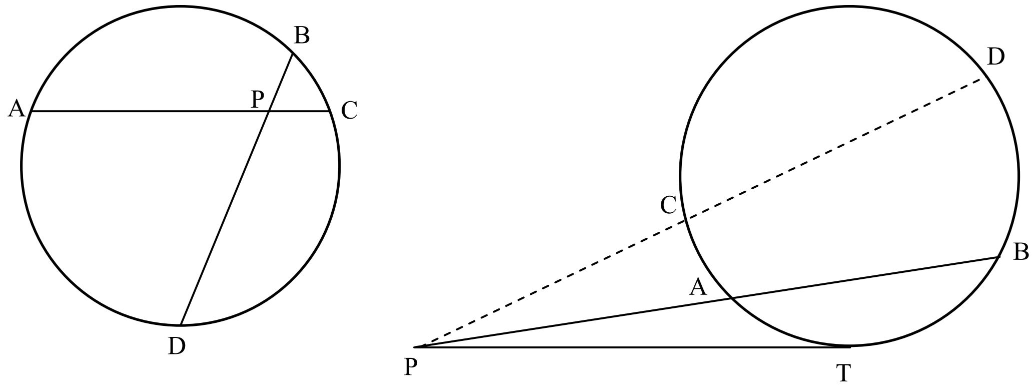
A book such as André Weil's *Number Theory An approach through history from Hammurabi to Legendre* (Weil, 2007) is good example of this use of history. The very title makes that clear: it is a book *about* number theory; its *approach* is through history. It is a good example too because Weil was a superb number theorist, and his knowledge of the relevant texts, particularly those Fermat, Euler, Lagrange, Legendre, and Gauss, is learned and thorough. He reads these texts through the eyes of a modern mathematician, through the eyes of the superb number theorist that he was. And he does so unapologetically. To take a random example, having described Fermat's judgment as to the difference between Diophantus and Vieta, Weil writes:

> From our modern point of view, things look somewhat differently. Firstly, since so much of Diophantus, and even more of Viète, remains valid over arbitrary fields, we would classify this primarily as algebra; of course Viète's algebra, both in its notations and in its content, is far more advanced, and much closer to ours, than that of Diophantus. Secondly, the distinction between rational numbers and integers is not as clearcut as the above quotation from Fermat would suggest; as he knew very well, it does not apply when one is dealing with homogeneous equations. Thirdly, as will be seen presently, there is much, in Diophantus and in Viète's *Zetetica*, which in our view pertains to algebraic geometry. Moreover, modern developments have led to a better understanding of the analogies (already dimly perceived on some occasions by Leibniz and Euler) between function fields and number-fields, showing that there is sometimes little difference between solving a problem in rational numbers and solving it in a field of rational functions (Weil, 2007, p.25)

The point is not to criticize Weil's book or to praise it. It is only to make clear that what guides the exposition is Weil's own modern understanding of number theory. For him there is no problem in this. He clearly sees that the mathematical content that interests him is, perhaps in an inchoate fashion, also guiding Fermat, Euler, and the others. For him, setting out these thinkers thoughts is precisely showing the steps in the modern understanding. Is he wrong? This depends on what ultimately is the nature of mathematics, whether it is a Platonic body of ideas or a cultural production. But if mathematical content guides history itself and an historical approach for teaching mathematics, then that Platonic core content, and not history as such, is at the bottom of one's educational framework. If one is to have a truly historical framework then the historical nature of mathematics must at least be put on the table as something to question: the nature of mathematics itself must be problematized. This requires distance, or, to use the potent phrase found in Evelyn Barbin's works, *dépaysement* (e.g. Barbin, 1997), a sense of foreignness, of "displacement," regarding the mathematical thought of the past.

To take a simple example, consider proposition 35 (and 36) in Book III of Euclid's Elements: "If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other." Following typical textbook formulations, such as that of Legendre, Steiner (1826) restates the theorem and combines it with Elem.III.36 as follows:

> From an arbitrary point in the plane of a circle M, let there be straight lines PAB, PCD,.. be drawn cutting the circle: then the product (rectangle) of the segments from point to the intersection points of the circle is a constant quantity, that is: PA×PB=PC×PD=... (p. 22).



By this means, Steiner, and Louis Gaultier (1813) before him, redirects our attention from the rectangles which interest Euclid (for it is not by accident that these propositions end the book that follows upon Book II concerning equal rectangles and squares) to a point with respect to a circle and a number associated with it. For Steiner and Gaultier, *this* is the content of Elem.III.35,36. We will never think otherwise if we do not allow ourselves to be "displaced" by way Euclid himself states these propositions and how and when he chooses to prove them (see Fried, 2004, for more details). So, to reiterate, if we are to have a framework for aligning history of mathematics with mathematics education, such a framework must invite us to question what we mean by mathematical content, and not assume that it is given in advance.

But if we are to avoid a dogmatic attitude towards mathematical content so as to leave room for history. we ought also avoid a dogmatic attitude towards history itself. Our own view of mathematics of the past must be problematized. Whether or not Weil's understanding of the historical nature of mathematics is true or not, it does represent a kind relationship with the past, and there are many possible relationships (this was discussed in Fried, 2014b). In asking about a theoretical framework for history in mathematics education, we ask about history, and that means, in large part, that we ask what it means to stand facing the past: our own posture towards the past must be explored.

This kind of exploration is a kind of exploration of ourselves. It echoes to a degree with what Collingwood saw as the heart of historical knowledge. As is well-known, Collingwood held that history involves the "re-enactment of past experience," and the object of history is historical inasmuch as it can be re-enacted in one's mind. What this means is not that we become Thomas Becket, to use his example, by thinking his thoughts, but that we come to know Becket, and that we perfectly aware that this is what we are trying to do. As he puts it:

> I do not ‘simply’ become Becket, for a thinking mind is never ‘simply’ anything: it is its own activities of thought, and it is not these ‘simply’ (which, if it means anything, means ‘immediately’), for thought is not mere immediate experience but always reflection or self-knowledge, the knowledge of oneself as living in these activities. (Collingwood, 1993, p. 297)

Earlier—and repeatedly—Collingwood argues how this reflective aspect of historical knowledge makes historical knowledge simultaneously subjective and objective, and that this is the distinctive character of such knowledge:

> Historical knowledge is the knowledge of what mind has done in the past, and at the same time it is the redoing of this, the perpetuation of past acts in the present. Its object is therefore not a mere object, something outside the mind which knows it; it is an activity of thought, which can be known only insofar as the knowing mind re-enacts it and knows itself as so doing. To the historian, the activities whose history he is studying are not spectacles to be watched, but experiences to be lived through in his own mind; they are objective, or known to him, only because they are also subjective, or activities of his own. (p. 218)

If we consider that education involves not only informing students of what is known or how things are done but also forming students’ thinking, their own self-awareness as intelligent beings, and their own relationships to their world, then an educational framework for history of mathematics in mathematics would have to attend to precisely this kind of mix of the objective and subjective. And, if Collingwood is correct about his view of history, then tending to history in the history of mathematics would be one way of leading to this more general educational goal. At the same time, it shows how history of mathematics aligns mathematics education with ends of general education without sacrificing attentions to its more circumscribed goals connected to mathematical concepts, ideas, and procedures.

The two sections which follow, one by Hans Niels Jahnke and one by David Guillemette, will present ways of thinking about the incorporation of history of mathematics from a theoretical point of view, as we have been speaking about it presently. We shall see that they are complementary, and in some ways both concern the relationship between students or teachers with their mathematical past and with the texts that have come down from the past.

2 The Hermeneutic Approach: Reflections and Extensions

Hans Niels Jahnke

In the following remarks, I shall present one of the more sophisticated approaches to history of mathematics in the classroom, the so-called “hermeneutic approach”. It has been elaborated in several papers, as for example Jahnke (1994) and (2012), Glaubitz (2010) and (2011). In a second step I shall extend my discussion by pointing at two papers, Kjeldsen & Blomhøj (2012) and Arcavi & Isoda (2007) which elaborate on the issue of what a student can gain by the “experiences of difference” (see Fried 2001) which history provides. The whole discussion fits very well to the catch-word „experience of the subject“ proposed by David Guillemette.

2.1 A pragmatic categorical imperative

Frequently, Otto Toeplitz’s famous saying from 1926 that going to the roots of mathematical concepts would remove the dust of time and the scratches of long use from them (Toeplitz, 2015) is interpreted in a way that we should use history when we introduce a new mathematical idea or concept in the classroom. May be, this sometimes works. But in general, a first historical appearance of an idea is clumsy and difficult to understand to a modern mind. This is no wonder since, in general, new knowledge does need a long process of clarification before we really understand it. Thus, introducing a new subject through the door of history seems to me simply unrealistic, and we should not overload the enterprise “History and pedagogy of mathematics” by demands which necessarily must lead to failure. One can call this a pragmatic categorical imperative.

Thus, the hermeneutic approach grew first of all out of the idea that you should confine history to a local experience. Students are asked to examine a source in close detail and explore its various contexts of historical, cultural and scientific nature. The hermeneutic approach will not give you an overview. Rather, it is a hope that some pupils will like history and develop a certain interest in it which might motivate them to search for further reading.

The basic guidelines of the hermeneutic procedure can be summarized in 6 principles.

(1) Students study a historical source <u>after</u> they have acquired a good understanding of the respective mathematical topic in a <u>modern</u> form and a <u>modern perspective</u>. The source is studied in a phase of teaching when the new subject-matter is applied and technical competencies are trained. Reading a source in this context is another manner of applying new concepts, quite different from usual exercises.

(2) Students gather and study information about <u>context</u> and <u>biography</u> of the author.

(3) The historical <u>peculiarity</u> of the source is kept as far as possible.

(4) Students are encouraged to produce <u>free associations</u>.

(5) The teacher insists on <u>reasoned arguments</u>, but not on accepting an interpretation which has to be shared by everybody.

(6) The historical understanding of a concept is contrasted with the modern view, that is the source should encourage processes of reflection.

2.2 What then is hermeneutics?

Simply said hermeneutics is the “art or the science of interpreting texts.” It distinguishes systematically between the author and the reader of a text and their different perspectives. This means that the strong tension between the historical perspective and the modern conception of a mathematical topic should not necessarily be smoothed out or eliminated, rather it should be embraced as essential for how a historical text may contribute to the intellectual development of a person. . The whole enterprise of reading a source rests, then, on experiences of *dépaysement*, as the French say (Barbin 1994) or *Verfremdung* ("alienation"), as the German writer Bertold Brecht would have said. Sources introduce into teaching an unwieldy element. But how comes it that such unwieldy elements do not lead to failure? This is so only when they have anchor points. The student who deals with something that he already knows but that is presented in a radically different, unfamiliar way or an unknown guise, should be able to make connections to these anchor points. In hermeneutics you would say: His horizon merges with the horizon of the past. *Horizon merging* is a term that was coined by Hans Georg Gadamer (1900 – 2002). In the horizon merging the student may begin to wonder and to reflect upon what he possibly had never thought about before. In essence he begins to develop deeper awareness. This is in fact an instance of broadening one's horizon. And it does so by utilizing a strategy of dissonance. It is well known that this kind of incompatible information ensures greater retention and ease of retrieval from memory. But to do so, there must be a familiar reference frame. It is therefore applied only to subjects that students are already familiar with.

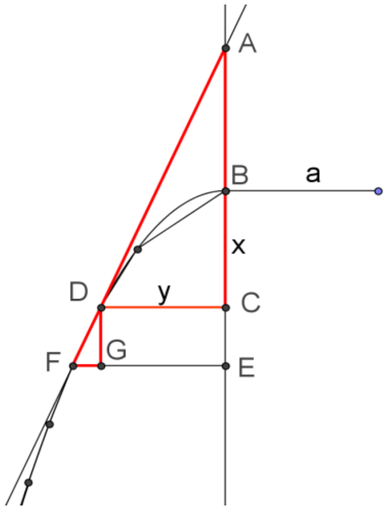
In hermeneutics the process by which the merging of horizons occurs is described by a spiral, the so-called '*hermeneutic circle*' which points to the necessity of already possessing an interpretation of a text in order to gain a new interpretation. For us as mathematics educators this appears not so strange as it might be for other people since we are used to reflect about spiral processes, the most prominent being the process of modelling.

You start with a certain image of the text reflecting your expectations about what it might be about. Then you read the text and realize that some aspects of your image do not agree with what is said in the source. Thus, you have to modify your image, read again, modify and so on until you are satisfied with the result or simply do not like to continue. Thus we have a spiral process like

$$\text{expectation/interpretation}_i \rightarrow \text{reading}_i \rightarrow \text{modififation}_i \rightarrow \text{interpretation}_{i+1}$$

In the case of a teaching unit on Johann Bernoulli's manuscript on the differential calculus (see figure 1), students started with the expectation that Bernoulli's text might be about determining tangents to and extremal values of curves, that the idea of a limit of an infinite process might be central to the subject, that concepts like derivative and slope of a tangent (a quotient) will frequently appear and that all is sort of algebra, with new rules, but symbolic in nature. After some windings of the hermeneutic spiral they had realized that the source was in fact about tangents to and extremal values of curves, but that there was no limit concept, instead there was the difficult concept of infinitely small quantity. Also, the slope of a tangent was less important than expected. It was used in the source to get rid of the infinitely small quantities, but the target object was a second point of the tangent. Thus, the status of a slope changed to that of an auxiliary object. And so on.

On a more basic level the hermeneutic circle can be considered as a process in which a *hypothesis* is put up, tested against the source, modified, tested again and so on until the reader arrives at a satisfying result. For example, the students were asked to infer from the source what a differential is. From their knowledge of calculus some students formed the idea that a differential is something similar to a derivative. With this hypothesis they studied Bernoulli's derivation of the product rule and realized that this cannot be true, since Bernoulli did not calculate a quotient. After some further attempts they saw that a differential is in fact a difference. In a similar manner they found out what a subtangent is.



Fugure 1. Bernoulli's parabola

Can one say then that students behave like historians of mathematics when they read a source? In principle, this is the case. When they enter the source they have questions similar to those a professional historian of mathematics would ask. Roughly spoken, these questions refer to the different meanings of concepts and the different conceptual structures at the time of Bernoulli and today.

The most important difference concerns the previous knowledge a historian and a student have at their disposal. For example, consider the segment in Bernoulli's sketch of the parabola representing the parameter a. A professional historian knows of course that the segment hints at the ancient geometrical definition of the parabola. Of course, the students do not know this. For the teacher it is a difficult question whether she should tell this to her students or simply let it as it is.

2.3 What students might gain: part 1

In the hermeneutic approach, mathematics enters at least in two ways. First, there is the experience of dissonance or alienation. Students learn something about their own mathematics by experiencing and *reflecting on the contrast between modern concepts and their historical counterparts*. And the point of the „hermeneutic circle" as understood here is that the reflection is in both directions, so that the students deepen both their understanding of history and of their own set of modern conceptualizations. Second, and equally important, is the fact that in reading a source (modern) mathematics itself is *applied as a tool* (Jankvist 2009). The task to think oneself into the situation of persons living at a time long ago requires to be able to argue from the assumptions of these persons, to use their symbols and methods of calculation. This poses completely new demands on the students' abilities to argue and to prove mathematically. Thus, reading a source deepens the mathematical understanding on both levels, on that of *doing mathematics* and on that of *reflecting about mathematics*.

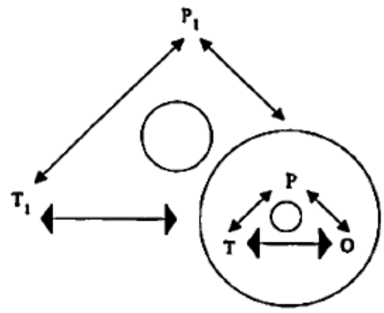


Figure 2. The ‘double circle’ (Jahnke 1994)

There is an important extra component students learn by such an hermeneutical experience. According to hermeneutics understanding a text consists in the merging of different horizons, the horizon of the reader and the horizon of the text/author. Different readers with their different backgrounds arrive at different interpretations. This situation becomes even more complicated when we take into account that the scientist herself can be considered as an ‘hermeneutician’. Her texts are the problems she is studying and, obviously, she travels through a spiral process producing ever new images of the problem at hand. This generates a primary circle symbolized in the bottom right corner of figure 2. The primary circle is itself a component of a secondary circle (the big triangle in figure 2) through which the historian/student travels in the course of her work. This might cause a feeling of participation or solidarity. The insecurity a historian experiences when she is approaching an interpretation but has not yet arrive at a satisfactory state is mirrored by the insecurity of the scientist studying a not yet solved problem.

2.4 What students might gain: part ii—meta-discursive rules

In the following I present two papers which also reflect about what students might gain by the „experience of difference“ which history provides. Kjeldsen & Blomhøj (2012) model the historical texts their university students study as „interlocutors“. Reading a text amounts to enter a dialogue. Of course, when your interlocutor simply repeats what you already have in mind you might gain a feeling of security, but you will not learn anything new. Consequently, a text will be productive and teachable only when it imparts the ‘experience of difference’. “This, of course, presupposes that the pitfalls of whig history are avoided“ (p. 331)

If this condition is satisfied something like „meta-level learning“ can happen. In order to describe what this type of learning might amount to Kjeldsen & Blomhøj take recourse to A. Sfard’s theory of thinking (Sfard 2008). There again thinking is modeled as communicating, that means the thinking of an individual is considered as a special case of “interpersonal communication“ or as a “combination of communication and cognition”. In this way, mathematics itself becomes a type of discourse and as such an autopoietic system. Communicating is a rule-regulated activity, and Sfard distinguishes between rules “concerning the content of the discourse and discursive rules about the discourse.” (p. 329)

The one are referred to as “object-level rules”, the other as “meta-discursive rules”. In regard to mathematics meta-discursive rules

> manifest their presence…in our ability to decide whether a given description can count as a proper mathematical definition, whether a given solution can be regarded as complete and satisfactory from a mathematical point of view, and whether the given argument can count as a final and definite confirmation of what is being claimed. (Sfard 2000, quoted in Kjeldsen & Blomhøj 2012, p. 329)

In its deep structure the process of learning mathematics amounts „to gradually modify the meta-discursive rules that govern a student’s mathematical discourse. As a consequence, an essential aspect of mathematics education is to create teaching and learning situations where meta-discursive rules are exhibited as explicit objects of reflection for students” (p. 330). There are not so many opportunities in the process of learning where students get a mirror of their own development in the development of other mathematical discourses. Consequently, to Kjeldsen & Blomhøj “history is, if not indispensable, then at least an obvious choice for learning meta-discursive rules in mathematics.” (p. 329) On a general level we find here again the idea that „experience of difference“ helps to “objectify the subjective”.

2.5 What students might gain (3): learning to listen

As a last example of reflection about what students might gain by the „experience of difference“ we present a paper by Arcavi & Isoda (2007) in which they focus on furthering the interpretative skills of students in the context of learning mathematics. One of the most important re-evaluations of teachers’ skills which flow from a constructivist approach to learning mathematics is the re-evaluation of “listening“. Arcavi & Isoda give a detailed analysis of what it should mean to teachers to listen to their students in a right way. They define “listening” as “giving careful attention to hearing what students say (and to see what they do), trying to understand it and its possible sources and entailments. ‘Listening,’ as we envision it, is not a passive undertaking,...” (p. 112) Listening in a right way benefits to the students as well as to the listener. It will lead to “a caring, receptive and empathic form ... of teacher-student conversations. If often modeled by teachers, students would feel respected and valued. Moreover, the listening modeled by the teacher may become internalized by students as a habit incorporated into their repertoire of learning techniques and interpersonal skills “ (p. 112).

On the other hand, “effective listening may influence ‘listeners,’ by making them reinspect their own knowledge, against the background of what was heard from others. Such re-inspection of the listener’s own understandings (which may have been taken for granted) may promote the re-learning of some mathematics or meta-mathematics.” (p.112) More concretely, Arcavi & Isoda identify three dimensions of listening.

(1) Listening should “unclog” automatisms which necessarily are connected with the “packaged knowledge” of expert mathematicians. Instead, a teacher should be able to set aside much of what he knows so well. “...an important part of learning to listen to students

would include, paradoxically, some kind of ‘mathematics unlearning’ on the part of the teacher” (p. 113).

(2) The right way of listening requires ‘decentering capabilities’. “...we take it to mean, the capacity to adopt the other’s perspective, to ‘wear her conceptual spectacles’ (by keeping away as much as possible our own perspectives), to test in iterative cycles our understanding of what we hear, and possibly to pursue it and apply it for a while. Such a decentering involves a deep intellectual effort to be learned and exercised.“ (3) As a consequence Arcavi & Isoda distinguish between two types of listening. “Evaluative listening” is focused on ‘wrong’ and ‘right’ and opposed to “attentive listening” or “hermeneutic listening”. “The former, which is more common, consists of listening against the background of an expected correct answer. It implies a virtual ‘measurement’ of the ‘distance’ between the student’s present state of knowledge and the desirable goal, ... However, such listening may tend to disregard the students’ thinking, the sources of their idiosyncratic ideas and their potential as a source for learning” (p. 114).

As a consequence the authors propose the thesis: “History of mathematics can provide many solution approaches (to problems) which are very different from what is common nowadays. Such solution processes may conceal the thinking behind them. Thus, one has to engage in a ‘deciphering’ exercise in order to understand what was done, what could have been the reasoning behind it and what is the mathematical substrate that makes an unusual method/approach valid and possibly general. Engaging in such an exercise bears some similarities to the process of grasping what lies behind our students’ thinking and actions. We do not claim that there may be parallels between the mathematics underlying primary sources and that of our students. What we do claim is that experiencing the process of understanding the mathematical approach of a primary historical source can be a sound preparation towards listening to students” (pp. 115-116).

Thus, it is a natural consequence that hermeneutical work with historical sources should become an important component of teacher training.

3 Concerning the lived experience of the subject, and the subject itself

David Guillemette

3.1 Introduction

For several years now that I have been involved in the training of secondary mathematics teachers. I often use historical texts reading activity and primary sources in the training classroom, partly because of my own passion and interest, but mostly because I strongly believe in its educational potential. The reactions of the students range from a total indifference to a powerful existential emotion. Indeed, some reactions are quite impressive, several students react very strongly to those activities and feel a strange vertigo from the temporal distance dividing them from the text and the author, distance which carries all the weight of history and culture. Others react dismissively, and just make fun through translation, highlighting atypical notations, reasoning or arguments. Finally some react with displeasure. These find the reading activities and the encounter with history to be an unpleasant ordeal. They feel uncomfortable, frustrated, sometime see themselves “hurt in their pride”, unable, as they have always been, to make sense with a mathematical discourse.

These elements, which concern specifically the lived experience of the learners, whether positive or negative, are not hazardous or inconsistent. They are not just epiphenomena, but are consubstantial of learning itself and should be, in my opinion, taken very seriously. Indeed, moving on to the lived experience of the learners, and confronting ourselves as researchers/trainers to those elements, brings us to what is inalienable concerning the encounter with history of mathematics. In a phenomenological perspective, my work has carried on the examination and the description of these experiences and such investigation has shed light “from within” the role of history in mathematics education and to reveal a fresh look and potentially new ways of thinking at its potential (Guillemette, 2015a). For my contribution to the panel, I will try to highlight various discourses concerning this lived experience of the learners in this context. These discourses, based on broad paradigms around mathematics and mathematics education, involve different perceptions, implicit or explicit, of the subject who is learning mathematics and is living the encounter of the history of the discipline. I’ll concentrate on two large tendencies or paradigm that I usually meet concerning theoretical and/or conceptual frameworks for integrating history of mathematics in mathematics education: the “humanistic” perspective and the “dialectic” perspective (in its modern Hegelian sense).

3.2 The lived experience from a “humanistic” perspective

It has been claimed that one of the main roles of the history of mathematics is to “disorient”. Indeed, history of mathematics, in a classroom or teachers training context, surprises and astonishes with the diversity the mathematical activity across cultures and the history of societies, which involves many considerations as to the form and use of mathematical objects. For many, these experiences could lead to a more cultural understanding of mathematics and invite to a historical-anthropological reflection on mathematical activity by repositioning the discipline as “human activity” (D’Enfer, Djebbar, & Radford, 2012). In other words, without excluding the development of mathematical understandings that history can play (cognitive pole), this exploration of historical and cultural dimension of the discipline, could brings the learners to take a critical look on the social aspect of mathematics, to understand the historical-cultural mechanisms of their production and to understand that, as mathematical as it is, there is no ideologically neutral knowledge, that all knowledge is part of an ethical issue for which we need to develop our sensitivity.

As Barbin put it many times, history could bring a “culture shock” in “immediately immersing the history of mathematics in history itself” (2012, p. 552, my translation). Therefore, the objective is not to read historical texts simply related to our (modern) knowledge, but rather in the context of the one who wrote them. This is when history can become a source of “epistemological astonishment” by questioning knowledge and procedures typically taken as “self-evident” (*ibid.*). For Barbin, as history invites the learners (specially here the preservice teachers) to stand out in that tone of “epistemological astonishment”, it may also invite them to investigate the following questions: “Why contemporaries did not understand such a novelty?" and "Why students do not understand?" (*ibid*.).

From a practical point of view, Barbin's claims suggest that history should allow prospective teachers to "understand the difficulties of students who are not like teachers, in a well-known country" and "better hear their questions or to better interpret their mistakes" (*id*., p. 548). More specifically, the "exotic aspects" of the history of mathematics can be a good way to "start thinking about the content taught and programs", "to sketch answers students' questions about the status of mathematical knowledge", to "avoid the fake concrete-abstract debate "and finally allow the teachers to change the way they teach, but also their educational relations" (Barbin, 1997, p. 24, my translation).

These ideas of Barbin implicitly give history a critical function in the context of learning mathematics students, a critical function that would find its point of tension in the confrontation with the history of science and objectivist philosophies that have been remained rooted in naturalism and scientific universalism. In contrast, history of mathematics, allows learners to realize their own understandings and perceptions and the particularity of their ways of dealing with mathematical objects, depending on their own contexts, cultures and experiences.

As Fried (2007; 2008) mentions, history of mathematics, in general, should be playing a central role in this quest to self-knowledge. For him it is a special contact with the history of the discipline that could make emerge in the learner some awareness of his own ideas, his individuality and his ability to confront constructively with those of others. Fried considers history, when it is taken as history and not as a means for something else, is able to contribute to the personal growth of individuals through the discovery of their own individuality. This individuality would not lead to a form of isolation, but, on the contrary, on the openness and the possibility to the exchange with the others and to understand the others. In this sense, mathematics education, through history must aim at mutual enrichment between knowledge, self-knowledge and knowledge of others.

Indeed, Fried stressed that the back-and-forth movement between the current understanding of mathematical objects and understandings from other eras brings learning to a deeper understanding of himself: "a movement towards self-knowledge, a knowledge of ourselves as a kind of creature who does mathematics, a kind of mathematical being" (Fried, 2007, p. 218). He proposes that this self-knowledge, that is to say, the knowledge of ourselves as a "mathematical-being", should be the primary objective that must give itself all forms of mathematics education based on the history of the discipline. Fried does not hesitate to emphasize the background of his thinking around these considerations by stating that: "[Education], in general, is directed towards the whole human being, and, accordingly, mathematics education, as opposed to, say, professional mathematical training, ought to contribute to students' growing into whole human beings" (p. 219).

In this perspective, the experience associated with the encounter with history of mathematics would be accompanied by an awareness and a growing movement. This would be a personal experience involving relation to ourselves (introspective element) through the history of mathematics, experience that supports the movement of growth, which is that of the learner. This "humanistic" perspective on the history of mathematics is also present, and developed, in many other speculative works in the field (e.g. Bidwell, 1993; Brown, 1996; Tang, 2007).

3.3 The lived experience from a "dialectic" perspective

Several researchers share, in large part, this view on history and its potential for teaching and learning mathematics. For example, for Radford, Furinghetti and Katz (2007), it is precisely in the highlighting and in the understanding of the links between past and current knowledge that the history of mathematics brings the most to the enrichment of the perception of discipline, and of the understanding of its genesis and its epistemology.

Above all, the authors put forward the importance of actions and events in the acquisition of knowledge, but also emphasize the crucial dimension of the possibility of introspection, confrontations and critical reflections about his own conceptions and knowledge. More specifically, history can show us how the particular meanings attributed to the mathematical objects are confined within our own experience. This limit can be exceeded only by the encounter with a truly foreign form of understanding. These theoretical elements are anchored in the dialectic tradition and specially based on the thought of philosopher Mikhaïl Bakhtin for whom "meaning only reveals its depths once it has encountered and come into contact with another, foreign meaning: they engage in a kind of dialogue, which surmounts the closeness and one-sidedness of these particular meanings" (Bakhtin, 1986, cited in Radford, Furinghetti, & Katz, 2007, p. 108).

In this perspective, the history of mathematics becomes itself a place where it is possible to overcome this particularity of our own understanding of mathematical objects, which is limited to our personal experiences. History is "a place to enter into a dialogue with others, and with the historical conceptual products produced by the cognitive activity of those who have preceded us in the always-changing life of cultures" (Radford, Furinghetti, & Katz, 2007, p. 109).

Through the discourse of social philosophers and psychologists, as Bakhtin (1986, 1986/2003), Ilienkov (1977) or Leontiev (1984), this perspective is marked by the Hegelian and Marxist dialectical developments. It is strongly tinged with social and historical dimension in the exploration of human life by radicalizing the very dialogic aspect of human being, of everything that a sense, a value in the world, and by placing the human being in a historical and ideological reality. Regarding the acquisition of knowledge, "action" plays a decisive role here. It is in the concrete action that the historical movement is created and that are elaborated reflections and changes. In this context, the student experience would be dominated by actions that are ways of getting in a form of "dialogue" with another culture, with different forms of understanding, in order to create the very actual social and historical tissue of the reality. In this sense, history of mathematics is a place, where it is possible to reconstruct and reinterpret the past in order to open new possibilities for learners and for future teachers.

However, as close as they seem, this “dialectic” perspective maintains some doubts about the “humanistic” one, especially with the idea that the learner must build himself free from any form of authority (understood here in both an ideological and in a pragmatic sense), a socio-culturally coded knowledge in a given socio-political context. On the one hand lies the subject conceived of itself as the master of its destiny; on the other hand, lie the regimes of truth, the discourses, and the significations of the sociocultural world in which the subject finds itself subsumed. As many other thinkers have shown, the difficult existence of the modern self unfolds against the unbearable backdrop of this dichotomy and its ensuing antinomies. The antinomies cannot be erased: they are part of the modern forms of production, the very same forms that define the modern subject (Radford, 2012).

In a way, this takes us back the still living question of Kant: “How to educate in freedom under authority?” (Kant, 1803/2004, p. 57). Indeed, the critic of the “dialectic” perspective concerns a conceptualization of emancipation that has its source in the Enlightenment. This conceptualization, built around a dualism and rationalism, is generally associated with the idea that the individual possesses somewhere in himself the conditions for personal growth, a certain intellectual potential or the possibilities of its complete socialization. In this perspective, the entourage of the learner is seen as a facilitator for the individual in his personal quest for growth.

But the “dialectic” thinkers do not attempt to resolve these contradictions and surely not propose a return to direct forms of education, but offers new avenues to rethink the very notion of emancipation. Through a sociocultural perspective, they first try to move away from a conception of a learner as “private” owners of knowledge. The learner is rather seen as an ethical subject (see Radford, 2011; 2012). In other words, they wish to avoid that education produces individuals emancipated side by side, both free and isolated. Their speech leads rather to put forward commitment, answerability and caring (Bakhtin, 1986/2003; Heidegger, 1927/1986; Levinas, 1971/2010) in the education relation, which implies a different conception of teaching-learning, as well as the classroom, and the role of the learners and the teachers.

In view of this, the history of mathematics can be seen as a place where it is possible to overcome the particularity of our own understanding of mathematics, understanding limited to our own personal experiences and sociocultural context in which we live. History of mathematics therefore offers, for Radford, and more generally for sociocultural thinkers, opportunities of meetings with ways of doing and being radically different in mathematics, distant historically and culturally. The lived experience of the students could, accordingly, be characterized by an experience of otherness in mathematics. An experience that focus not necessarily and exclusively on the relation to ourselves, but, more broadly, on a relation with the Other in mathematics, a movement toward the community.

Thus, the focus is not on an individual experiencing personal possibility of emancipation, but to the possibility for learners to discover new ways of being-in-mathematics, to open, with the others, the space of possibilities in mathematics and to respect the classroom as a politics and ethics entity, open to novelty and subversive issues (Guillemette, 2015b).

3.4 Two perspectives on the subject

Behind these two perspectives, it is possible to discern two different conceptualizations regarding the subject who is learning and who is living the encounter with the history of the discipline.

On the one hand, one can find through the “humanistic” perspective a subject that is perceived as “already given” in history. That is to say, a determined subject that is historically and culturally in search of an understanding of himself. Here, history of mathematics should contribute to the awareness of his origins, and enable the subject to understand his historical and cultural determinations. The subject, properly educated and intimately aware of its position in the world through history, will be able to deploy a certain freedom, making him responsible for his actions, improving his relation to the discipline and engaging him with lucid, rich and open activity in mathematics. A freedom that is made of independence, critical thinking, openness and curiosity, characteristics of any good scientist.

In this perspective, the learner needs to know and perceived himself in its mathematical context. This is why the experience is associated with a certain adventure, because the subject is taken, through the study of the history of mathematics, in the exploration of the intimacy of his being-in-mathematical. It is this recognition that could lead to the opening to the outside, opening to others as opportunity for mutual growth. The surprise, the doubts or the adversity, that punctuates this encounter with history, are as many experiences that characterize this awareness and this self-discovery.

This is why we can talk about a subject that is “already there”. A determined subject (culturally, historically, linguistically, socially, etc.) that is waiting to be discovered throughout history, the same history that had made him this subject hidden, inaccessible or potentially not yet noticed. History thus becomes a self-recognition tool, much as language becomes an instrument in psychoanalytic therapy.

However, this perception of a subject “already there” that carries the prospect humanism is opposed to a perception of a subject which, in some way, “is not already there”. Through “dialectic” perspective, the subject is not considered as completed, determined, giving himself to himself. On the contrary, the subject is conceived as “a becoming”, constituted and constituting itself through history in a deep and fundamental ontological sense.

Consequently, history offers different experiences to this different subject. The difference can maybe be grasped through the notion of “emancipation”. Emancipation is *emancipatio*, that is to say, literally, “the fact of being released of paternal authority”, more generally to be released from an authority, a domination. In a sense, the history of mathematics holds the emancipatory elements, liberating the subject as it provides the key to formulating an answer to this quest for self-knowledge. On the other hand, the “dialectic” perspective involves a different way of understanding how one becomes free of authority. . Indeed, the class is not perceived here as a closed environment where students develop skills or a certain adaptability through negotiation process, but as the space of collaboration and cooperation so that the learners become part of the collective. The teacher therefore has the role to promote an idea of autonomy conceived as social engagement, to develop a (con)science, literally a knowing-with-others, following the idea that nobody liberates nobody, that nobody liberates himself and that that learners liberate themselves together by the means of the world (*cf.* Freire, 1974).

This perspective offers rather "ways of being and knowing as how students engage in group in their pursuit of cultural knowledge referred" (Radford, 2011, p. 15, my translation). In this context, the history of mathematics is to be the meeting place and experience where it is possible for the subject to form and to engage himself in the creation of its mathematical reality. Indeed, the acquisition of knowledge, must be taken in its etymological sense, that is to say the *adquaerere* meaning "to search". Learning could therefore be understood as a process of opening, searching for attitudes or ways of being. Learning is not a submission to a prevailing culture, much less a possession of cultural content, but rather a movement of openness to the world and to the others. With the introduction of the history of mathematics, this openness to the world is seen radicalize and takes an unusual turn. Indeed, ways of thinking and acting are multiplying around the learners in contact with the history of mathematics. These elements invite to introspection, to an awareness of our historical anchors. History puts learners in "research mode".

But is it not simply the reverse of the same coin? The subject constituting himself throughout history, but in opposite directions? On the one hand, a subject discovering himself, liberating himself and therefore growing in its powers and possibilities of mathematical development. On the other, a subject not released, but rather engaged in a creative activity embracing its historical and cultural dimension so often forgotten.

3.5 Different ways of being in research: open questions

Finally, the two positions on the subject necessarily involves different ways of conducting research. From a "humanistic" perspective, one could emphasize intimate lived experiences and the learners' new relation with the discipline. Evidence of those elements, highlighting how consciousness transforms itself in a reflexive manner can be found. On the other side, the "dialectic" perspective would try to grasp the world in common that emerges from the introduction of history into mathematics learning; it would try to explore emerging ways-of-being-in-mathematics. It would pursue this under the assumption that any movement of consciousness is itself dialogical, penetrated by and in dialogue with other movements of consciousness, and thus cannot be approached without consideration for other movements of consciousness to which it answers, and which it allows as an answer.

4 Concluding words

Michael N. Fried

Our goal in this panel, as stated in our introduction, was to set out some key questions for any theoretical framework concerning the alignment of history of mathematics with mathematics education. To this end, it was essential first to try to make clear what kind of thing a theoretical framework is, or, rather, what it is not. This is a matter that ought to remain a focus for reflection. Jankvist (2009) brought out the distinction between "history as a tool" and "history as a goal." These are not two options for a theoretical framework. History taken as something *used* rather than *studied* (Fried, 2001) subordinates history to other non-historical goals whose justifications are independent of history, even antithetical to it. Thus, "history as a tool," even if it actually serves as genuinely *useful* tool for something, cannot be the basis for a *theoretical* framework for history, for the simple reason that history taken this way is not really about history.

But one might argue that the contributions of this panel pose history as tool no less than those who would use it to motivate students to learn whatever it is the school program requires. Does not Jahnke speak of *using* history "to learn to listen"? Does not Guillemette stress the *use* of history in leading students to a "lived experience", "new ways of being-in-mathematics"? And even in the introduction, is not history conceived as something *used* as a means to "self-knowledge"? The crucial difference is that in all these cases, although one can introduce the word "use," in fact "the *uses* of history" here are expressions of the substance of history itself or of historical experience. Theoretical frameworks for history of mathematics in mathematics education, *as* theoretical frameworks, should be driven by questions centered on the historical character of mathematics, on the historical conditioning of our experience of mathematics, and, generally, the meaning of our relationship to the past.

The word "relationship" is the key one. When it comes down to it, history is not knowledge of the past if knowledge is understood as the knowing of "historical facts"—it is a fundamental historical insight that "historical facts" are always problematic. What is a fact is that we have a certain relationship to the past conditioned by texts, artifacts, and language. History is an exploration of that relationship, and it is that relationship which is the main concern of the theoretical frameworks presented in the contributions above. How those frameworks differ, how any such theoretical framework will differ from another, is in how they conceive that relationship with the past. Thus in a hermeneutic framework, which Jahnke has emphasized, one is understood to be limited in one's access to the past by one's own circumscribed experience and our unavoidable "prejudices," and, accordingly, Gadamar, the principal philosopher of modern hermeneutics, speaks of our existing within a horizon: the act of "understanding" is then one of merging horizons, that is, ours with others'. Guillemette emphasized yet another kind of relationship with the past in which one is an a dual position of looking towards "emancipation" from the authority of the past, while recognizing the creative possibilities of being bound up with the past. There are yet other kinds of relationships with the past (see Fried, 2014b). Investigation and elaboration of such relationships will continue to refine our theoretical ideas concerning history of mathematics and its place in mathematics education.

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