# BREAKING NEWS, NOVEMBER 1813 – ACCUSED ‘NUMBERS’ CLAIM GEOMETRIC ALIBI:

## A Dramatic Presentation Celebrating Joseph Gergonne

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**ABSTRACT**

Joseph-Diez Gergonne provides an early model for the importance of the role of journal editor in filtering novel and valuable ideas, encouraging correspondence, mediating disputes, influencing research priorities, shaping disciplinary discourse, and negotiating disciplinary boundaries. A paper of J.-F. Français appeared in Gergonne’s journal, *Annales de mathématiques pures et appliquées*, in 1813, on the idea of representing imaginary numbers geometrically. Gergonne placed the article and responses by himself and others in the ‘*Philosophie Mathématique*’ section of his journal. We present, in the form of a dialogue between the participants, a paraphrase of the lively interchanges in the *Annales*, including Gergonne’s copious editorial footnotes. The play is introduced, and a closing commentary given, by a historian of mathematics and a modern mathematics teacher in dialogue, providing historical and pedagogical context. Cameo appearances are made by historian Gert Schubring, John Wallis, Abraham De Moivre, Caspar Wessel, H. Valentiner, A.-M. Legendre, and the modern mathematician Barry Mazur.

## 1 Context and characters

JOHN WALLIS (1616 –1703) became Savilian Professor of Geometry at the University of Oxford in 1649. The University had been purged by Parliament of all Royalists, and Wallis may have won the Professorship as a reward for his work for the other side in the English Civil War – decrypting a Royalist letter. ABRAHAM DE MOIVRE (1667–1754) was raised a Protestant in France, in a period of religious tension. In his late teens he was imprisoned for over two years for his religious beliefs, and subsequently went to live in England. There he taught himself mathematics, published his own work, and was elected to the Royal Society when only 30. He made a living as tutor and risk consultant, solving problems of chance for financial speculators and gamblers.

Two prominent players in the early development of the geometric or graphical representation of complex numbers were not perceived as mathematicians at all. CASPAR WESSEL (1745–1818) was a Danish-Norwegian surveyer, who made his contribution in a talk to the Royal Danish Academy of the Sciences which appeared in their *Mémoires* in 1799, in Danish; a French edition appeared only in 1897, and a full English edition another century later. It has been commonly supposed until recently that the ARGAND of our story, who produced the *Essai sur un maniére de représenter les quantités imaginaires dans les constructiones géométriques* in 1806, is one Jean-Robert Argand (1768–1822), a bookseller in Paris. However a good case has been made by Gert Schubring that our Argand was a different person, and did not publish his work in any official sense, merely distributed a few copies of his manuscript to individuals. These two works, by Wessel and Argand, went almost unnoticed at first. An article of JACQUES-FRÉDERIC FRANÇAIS (1775–1833), a mathematician from Alsace, published in Gergonne's *Annales*, was instrumental in bringing Argand to communicate with Gergonne, and lay claim to having given his essay to the well-known mathematician ADRIEN-MARIE LEGENDRE (1752–1833) in 1806. Legendre, though impressed, did not pursue the ideas, but wrote about them to the older brother of Français, who found it among his deceased brother's effects; the letter has recently been discovered by Schubring. It was through Gergonne's *Annales* that Argand's work became known to many French mathematicians. Hermann Hankel's work on complex analysis in 1867 drew attention once more to Argand's contribution, provoking a French translation in 1874, and ensuring that the name of Argand would be forever attached to the complex plane. Two other relative outsiders broke the news of the geometric representation of imaginaries to the English. Adrien-Quentin Buée was a French priest who escaped to England in 1792 rather than take the oath of allegiance to the Revolutionary Constitution. He mostly wrote religious and political tracts, but published his 'Mémoire sur des quantités imaginaires' in the *Philosophical Transactions of the Royal Society* in 1806. Strangely his work was also largely unnoticed, and English mathematicians generally awoke to 'Mr Warren's planar representation of imaginaries' when John Warren published his booklet: *A Treatise on the Geometrical Representation of the Square Roots of Negative Quantities*, in 1828.

JOSEPH-DIEZ GERGONNE (1771–1859) spent some years as a Captain in the French army in the late 18th century, before taking the chair of 'transcendental mathematics' at the new École Centrale in Paris, and then moving to the University of Montpellier in 1816. He made some significant contributions to mathematics, including discovering the principle of duality in projective geometry, but he was probably most influential as the editor of the *Annales des mathématiques pures et appliquées* (*Annals of Pure and Applied Mathematics*). This journal was founded by him in 1810 when he realised how important and how hard it was to get work published, and it was sometimes called simply the *Annales de Gergonne*. Between 1810 and 1830, when he was appointed Rector of the University, he published in his journal about 200 of his own articles, as well as articles by some of the leading mathematicians of his time, including Poncelet, Servois, Bobillier, Steiner, Plücker, Chasles, Brianchon, Dupin, Lamé, and Galois.

FRANÇOIS JOSEPH SERVOIS (1767–1847) was ordained a priest shortly after the French Revolution broke out, but before long he left the priesthood to join the army, doing mathematics in his leisure moments. He was assigned to his first academic position in 1801, recommended by Legendre, and went on to teach mathematics at a number of artillery schools. In 1804 he published a text on practical geometrical constructions for use by military officers in action, called *Little-known Solutions to Various Problems in Practical Geometry*. His most striking mathematical contribution came in 1814 when his work on the foundations of calculus came to full bloom. Following on from the work of Lagrange and of L. F. A Arbogast (1759–1803), he produced a remarkable paper 'Essai sur un nouveau mode d'exposition des principes du calcul différential', in which he defined and studied the differential in the form of what we would now call a 'linear operator'. Arbogast and Servois also initiated the study of what we now call 'functional equations', and in considering the algebraic rules needful for manipulating the symbols in the resulting 'calculus of operations', and 'calculus of functions', they made the first explicit extension of algebraic operations to entities which were neither numbers nor geometrical magnitudes. It is Servois to whom we owe the words 'distributive' and 'commutative' – standard terms in algebra today.

## 2 THE PLAY

*MARIA, a historian of mathematics, and EMMY, a modern mathematics teacher, converse frontstage, while cameo appearances are made by GERT SCHUBRING, JOHN WALLIS, ABRAHAM DE MOIVRE, BARRY MAZUR, CASPAR WESSEL, and H. VALENTINER. Then the curtain rises on a conversation taking place in November, 1813, between JOSEPH GERGONNE, JACQUES-FREDERIC FRANÇAIS, ARGAND, and FRANÇOIS JOSEPH SERVOIS, during which a cameo appearance is made by ADRIEN-MARIE LEGENDRE.*

MARIA: [*ad lib greetings to audience, in English or French*] Ladies and gentlemen, my name is Maria. Welcome to our tribute to Joseph Gergonne ... In a few minutes we will hear him in conversation with three contributors to his journal. I am your Storyteller today, to give some historical background. Please welcome Emmy, who will bring the perspective of a mathematician and mathematics teacher … [*applause*]

EMMY: Thank you, Maria!

MARIA: Emmy, will you tell us – how are complex numbers introduced to students today?

EMMY: It may be done by expecting students to take on trust the – quote [*she makes hand gestures for quote marks*] – 'imaginary' number *i*, with square taken equal to minus one—

MARIA: That's more or less what Augustin-Louis Cauchy did!

EMMY: Or it may be done by using ordered pairs of real numbers to create the algebra of complex numbers.

MARIA: As first proposed by William Rowan Hamilton in 1835!

EMMY: But, in whatever way they are introduced, the exposition is always accompanied by their 'geometric representation' in what we call 'the Argand diagram', or 'the complex plane'.

MARIA: Why is this still found so helpful? Do mathematicians not see complex numbers forming an algebra that can stand alone without the picture?

EMMY: Well, even today, few mathematicians think about real numbers without imagining them on the 'number line'. We imagine complex numbers in the *plane*. It's not only that this has been found to make the new numbers more palatable, by involving some basic human intuitions, but also because the rich geometry of the plane has proved extremely fruitful – you could say indispensable – in developing complex analysis.

MARIA: Yet, believe it or not, complex numbers were used for over two centuries before being matched with this geometrical picture.

EMMY: That's amazing! I simply cannot imagine teaching or using complex numbers without the Argand diagram. Who was Argand? Was he the first to introduce the geometrical representation?

MARIA: Argand is rather an obscure person – generally supposed to be a Parisian bookseller called Jean-Robert, who published his ideas in 1806, but scholarly doubts have been shed on these assumptions, and about his dates. The historian Gert Schubring can testify to the extraordinary difficulties of uncovering historical facts two centuries later.

[*Cameo appearance*]

SCHUBRING: Unfortunately we do not have any primary evidence as to Argand's first name. Even knowing his address in Paris in 1813 does not help, as administrative and death registers were lost in the fights of the *communards* in Paris in 1870/71. Almost all we can state – in terms recalling medieval history – is that Argand 'flourished' in 1806, 1813, and 1814.[1]

[*Exit*]

MARIA: However, the earliest statements about picturing the imaginary numbers geometrically probably came from John Wallis in Oxford, well over a century earlier in spite of his statements about the absurdity of negative numbers, even. This was in 1673, a whole century after the Italian, Rafael Bombelli, first thrust the imaginary numbers into controversial prominence on the mathematical stage.

[*Cameo appearance, while equation appears on screen*]

WALLIS:[2] I say that $\sqrt{-bc}$ may be perceived as a mean proportional between $b$ and $-c$, in that the following holds:

$-bc = \left(\sqrt{-bc}\,\right)^2$ gives $\frac{b}{\sqrt{-bc}} = \frac{\sqrt{-bc}}{-c}$

This may be exemplified in Geometry, by taking the root of the imaginary quantity as the point in the Plane distant $\sqrt{bc}$ above the line of real numbers.

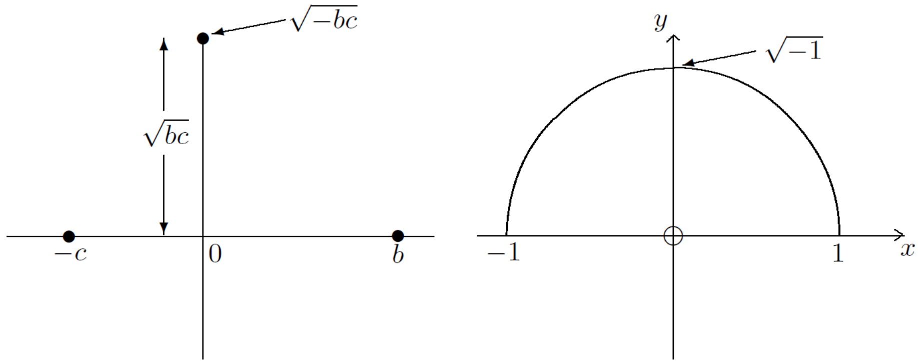
[*Exit*]

EMMY: [*diagrams and equations appear on a screen for live audience*] Here's the diagram on the left. Take $b = c = 1$, and use the modern Argand diagram on the right. Wallis says that the mean proportional between the two points *1* and minus *1* on the number line is $\sqrt{-1}$, which for us is the complex number *i*, and it corresponds to the point (*0*, *1*) in the plane, or a unit distance

[1] Paraphrased from (Schubring 2001), 132, which decribes his careful investigations, clarifying what is and is not known about Argand, as well as about Buée and some of the earlier work that influenced them. Much of what Schubring describes as 'the mystery' surrounding these obscure authors and their isolated achievements, still remains. Schubring also analyses (p. 132) how the original designation of Argand's contested biographical details arose from his 1874 editor Hoüel's unsubstantiated assumption that he originated in Geneva.

[2] Based on his *Algebra* of 1685; English translation of extract in (Smith 1959, 48).

up the perpendicular $y$-axis in our Argand diagram. It's easy to see this as a sort of half-way stage between the two points, thinking of rotations through a right-angle about the origin.




MARIA: Rotations! Wallis was onto something here, but he did not pursue it any further. And nobody of his generation was listening! What's the point, they might have asked? Tell me, Emmy, how would you convince your students that complex numbers are actually useful?

EMMY: Well, I would *love* to show them the algebraic solution of cubic equations! But, sadly, this is not in most mathematical curricula! Indeed, in many countries, high school students do not meet complex numbers at all. When I *have* discussed complex numbers with students, it has been when we reach De Moivre's theorem and its applications that I have caught some of them actually looking interested!

MARIA: Emmy, please state for us this theorem, once called Cote's theorem.

EMMY: The theorem says that, for any real number $\theta$ [theta] and any integer $n$,

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n$$

However, you know, it's when I have gone further and shown them the wonderful and unexpected connection, found by Leonhard Euler, between the real *exponential* function $e^x$ and the real trigonometric functions *cosine* and *sine*, that a few have really shown excitement. I have even known some to say 'awesome!' at that point. Here it is:

$$\cos x + i \sin x = e^x$$

MARIA: From which De Moivre's theorem follows immediately, setting $\theta = x$, then $n\theta = x$.

EMMY: That reminds me of a piece of mathematical folklore: the shortest distance between any two points in the real domain lies through the complex domain.

MARIA: Euler's inspired manipulations gave eighteenth century mathematicians – at least the Continental ones – a growing confidence in the potency of these so-called fictions – these impossible numbers.[3]

EMMY: Did De Moivre discover his theorem long before Euler discovered his?

MARIA: Euler arrived at his theorem by his usual magic around the middle of the eighteenth century. But, as early as 1707, Abraham De Moivre had caught a glimpse of the analogy

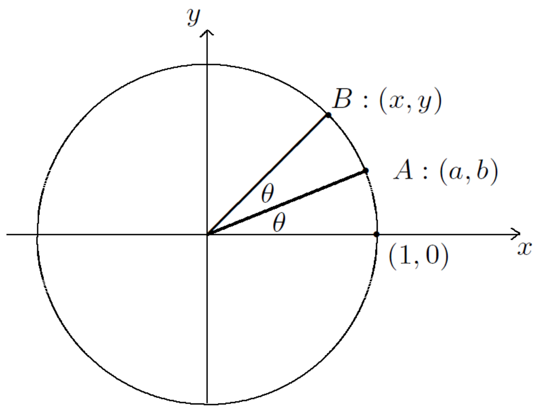
[3] (Euler 1748, 1749, 1770)

between two apparently quite different things: on the one hand, the powers of imaginary numbers, and on the other hand, the way coordinates of points in the plane change when a multiple angle is taken.

[*Cameo appearance*]

DE MOIVRE:[4] Consider a point $A$: $(a, b)$ on a circle of unit radius in the plane, whose radius vector makes angle theta with the axis. Now double the angle, and let the new point on the circle be $B$: $(x, y)$. Then it can easily be shown that the new point is given by

$$x = a^2 - b^2, \quad y = 2ab$$




But I observe that when I square the imaginary quantity $a + b\sqrt{-1}$, I obtain the imaginary quantity $a^2 + b^2 + 2ab\sqrt{-1}$ :

$$\left(a + b\sqrt{-1}\right)^2 = a^2 + b^2 + 2ab\sqrt{-1}$$

$$\left(a + b\sqrt{-1}\right)^3 = a^3 - 3ab^2 + \sqrt{-1}\,(3a^2b - b^3)$$

I observe a similar analogy when taking three times the angle, for the coordinates of the new point C on the circle are just the two expressions $a^3 - 3ab^2$, $3a^2b - b^3$ that I obtain by cubing the imaginary quantity $a + b\sqrt{-1}$. If I take the multiple n of the angle successively as 1, 2, 3, 4, 5, 6, &c., there will arise expressions of the same form as those obtained by taking the nth power of the imaginary quantity, although they are things of quite a different nature.

EMMY: Students today often find this connection with the multiple angle formulas of trigonometry the most compelling justification for accepting the imaginary numbers, for no other derivation is as simple. Take a point $(a, b)$ on a unit circle $x^2 + y^2 = 1$, with polar angle θ [theta], so that $a = \cos\theta$, $b = \sin\theta$. Then double the angle of the radius vector; De Moivre observes that the coordinates of the new point will be $a^2 - b^2$, $2ab$, taking the real and imaginary parts of $\left(a + b\sqrt{-1}\right)^2$. But this point has coordinates $\cos 2\theta$, $\sin 2\theta$, so we immediately have two useful trigonometric formulas.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\cos\theta\sin\theta$$

[4] The idea is from De Moivre's paper, 'Reduction of Radicals to More Simple Terms', c. 1738, in (De Moivre 1809), but is expressed here in a form more accessible, if less intriguing, than his. He did not have unit radius, but this makes the analogy he observed clearer.

Similarly we could find the triple angle formulas by taking triple the angle, or indeed we could take any multiple $n\theta$, where $n$ is a positive integer, and take real and imaginary parts of the corresponding power of $\cos\theta + i\sin\theta$, using the binomial theorem.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

MARIA: It is interesting that De Moivre's motivation went in the opposite direction, as shown by his title: 'Reduction of Radicals to More Simple Terms'. He used a knowledge of the way the sine of a triple angle was connected to the sine of the angle, to find the sorts of cube roots of imaginary quantities that arise when Cardano's Rule is applied to the irreducible case for cubic equations.

DE MOIVRE: I use this to extract the nth root of the impossible binomial, by using a table of sines. For example, I shall show by this method that the cube roots of the impossible quantity $81 + \sqrt{-2700}$ are:

$$\frac{9}{2} + \frac{1}{2}\sqrt{-3}\,, \quad \frac{3}{2} - \frac{5}{2}\sqrt{-3}\,, \quad -3 + 2\sqrt{-3}$$

There have been several authors, and among them Dr. Wallis, who have thought that those cubic equations, which are referred to the circle, may be solved by the extraction of the cube root of an imaginary quantity, as of $81 + \sqrt{-2700}$ without any regard for the table of sines. But that is a mere fiction; and a begging of the question; for on attempting it, the result always recurs back again to the same equation as that first proposed. And the thing cannot be done directly, without the help of the table of sines, specially when the roots are irrational; as has been observed by many others.

[*Exit*]

EMMY: To me it would seem a small step from De Moivre's perception of this analogy between 'impossible numbers' and ideas from geometry, to what we now call a 'geometric representation of the algebra of complex numbers.'

MARIA: Yet it would be many decades before this unexpected connection would be clarified and published in its fully explicit form. And remember, there was not yet such a thing as 'the algebra of complex numbers'. That would take more than a century to emerge!

EMMY: So for a century no-one conceived of what we now call 'the complex plane'? But after De Moivre's paper imaginary numbers would surely have been naturally associated with the trigonometric functions?

MARIA: Yes, it's true that De Moivre's insight was used by both Euler and Gauss.[5] But neither seemed to give explicit recognition of the big geometrical picture.

EMMY: Well, when was the geometrical representation finally made explicit?

MARIA: More surprises! The great idea was arrived at and published independently by five different, relatively obscure authors, over a period of nearly three decades: Wessel in 1799,

[5] Euler's version of what we call De Moivre's theorem appeared in Chapter 8, article 132, of (Euler 1748). Gauss made use of it in his proof of the Fundamental Theorem of Algebra in 1801.

Argand in 1806 (though his work was not published until 1813), Buée in 1806, Français in 1813, and finally John Warren in 1828.

EMMY: That seems really weird to me in the twenty-first century! Did Warren not know about the earlier publications?

MARIA: It seems that almost no British mathematicians read them – one of the symptoms of the political and intellectual gulf between Britain and the Continent over that period.

EMMY: And, I suppose, the language, and the particular forum or professional organ of publication – did they appear in international journals?

MARIA: The idea of a truly international journal was still in the future. But especially interesting is that two of the earliest accounts of this fundamental connection not only appeared in very obscure places, but were by complete mathematical outsiders.

EMMY: But why did no mathematician expound the connection? What was in the minds of those mathematical giants, Euler, Lagrange, and Gauss? You suggested they used the connection freely enough!

MARIA: Yes, they did. But as late as 1801, Gauss, in his proof of the fundamental theorem, dealt with real and imaginary parts separately, never talking about the combined whole as a point in a planar picture of these numbers.

EMMY: Maybe they didn't see it as a big issue. Maybe the great Gauss never actually *saw* it that way at all! And we serve it up to our students as if obvious!

MARIA: Barry Mazur, in his book *Imagining Numbers*, expresses well the novelty and importance of the connection:

[*Cameo appearance*]

MAZUR: What a dramatic act: to find a home in our imagination for such an otherwise troublesome concept!

[*Exit*]

MARIA: But he also asks the question whether this idea was in any sense 'in the air' for those leading mathematicians.

EMMY: And what *is* the answer?

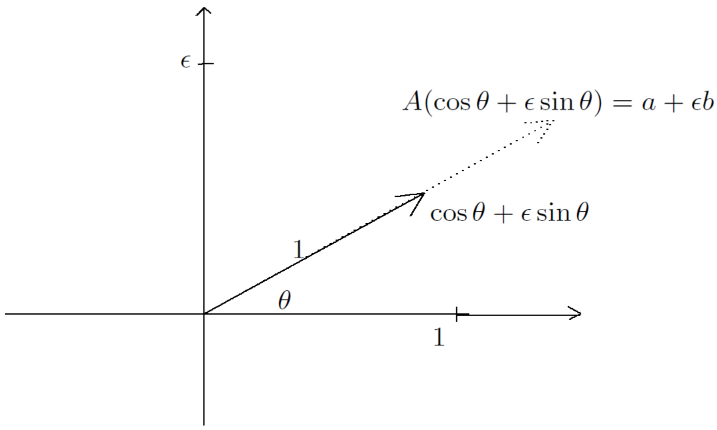
MARIA: Mazur doesn't claim to know. I don't think anybody can be sure. The fascinating fact is that the first to write clear accounts were a surveyer, and someone so obscure that there is doubt about even his first name and profession – a clock technician, or a bookseller. The earliest exposition was in a talk given to the Royal Danish Academy of the Sciences in 1797 by the Danish-Norwegian surveyer, Caspar Wessel.

[*Cameo appearance*]

WESSEL:[6] The title of my paper is 'On the Analytical Representation of Direction'. I am concerned to find a way of expressing, by means of algebra, the line segments (which I conceive as fundamental to surveying and navigation), that have both a given length and a given direction. The method I use employs the imaginary numbers.

Two straight lines are added if we unite them in such a way that the second line begins where the first one ends and then [we] pass a straight line from the first to the last point of the united lines. This line is the sum of the united lines. Now, in seeking a way of multiplying straight lines, I insist on three properties: First, the product of two lines must remain in the plane of the two lines. Second, the length of the product must be the product of the two lengths. And third, if all directions are measured as angles made with a positive unit line, which I call unity, and designate by the number 1, then the angle made by the product must be the sum of the angles made by the two lines.

Now, if I let $\epsilon$ [epsilon] denote the line with unit length perpendicular to the first line 1, and apply the three properties above, it is easy to see that $\epsilon^2 = -1$, and also $(-\epsilon)^2 = -1$. Thus a line of length 1 and direction $\theta$ [theta] can be designated as $\cos\theta + \epsilon \sin\theta$, and a line of length A and direction $\theta$ can be designated as $\mathrm{A}(\cos\theta + \epsilon\sin\theta) = a + \epsilon b$.




Now I can add and multiply these algebraic entities as follows:

$$(a + \epsilon b) + (c + \epsilon d) = (a + c) + \epsilon(b + d)$$

$$(a + \epsilon b)\times(c + \epsilon d) = (ac - bd) + \epsilon(ad + bc)$$

and I easily show that the addition and multiplication both satisfy my geometrical properties laid down earlier.

[*Exit*]

EMMY: Amazing! – so his motivations also went in the opposite direction to ours! Caspar Wessel starts with the geometrical idea of directed line segments, generalising the notion of oppositely directed positives and negatives, and sets out to represent line segments

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[6] This is a paraphrase of what he might have said in his talk, based on what he published two years later. A partial English translation is in (Smith 1959), 55–66.

algebraically. And he simultaneously provides a beautiful geometrical representation of the imaginary numbers!

MARIA: Yes. The history of mathematics is full of surprises for modern mathematicians. Unfortunately, very few mathematicians read Wessel's paper (it was written in Danish). It took a century for the French edition to come out. Here us one of the editors of that edition, Monsieur H. Valentiner:

[*Cameo appearance*]

VALENTINER:[7] It is surprising that a man can come up with a book as remarkable as the one before us, after having exceeded fifty years of age, and without producing any other scientific work, before or afterwards.

[*Exit*]

MARIA: The next major events in our story took place in 1806. In London, Adrien-Quentin Buée published a paper in the Transactions of Royal Society that he'd read to the Society the previous year.

EMMY: And did the British not sit up and take notice?

MARIA: No, it seems not. With the notable exception of George Peacock, about twenty years later. And in Paris, the same year, someone called Argand produced at least two copies of a manuscript in French: *Essai sur un maniére de représenter les quantités imaginaires dans les constructiones géométriques* (*Essay on a Way of Representing Imaginary Quantities by a Geometric Construction*).

EMMY: Was his approach different from Wessel's?

MARIA: Perhaps he was the first to clearly conceive of numbers as transformations. Like Wessel, he distinguished carefully between *rapport numérique* and *rapport de direction*, and showed how the imaginaries could be given a common construction with positives and negatives – combining and generalising the fundamental concepts of numerical or absolute value, and direction.

EMMY: His motive seems to have been slightly different, then: to bring the imaginaries into the geometric fold by extending the number line to the number plane. So the Parisian mathematicians at least must have taken note?

MARIA: Well, no. Which is not actually surprising. None of the leading mathematicians were aware of his work, except Legendre, because Argand approached him personally, and was presumably discouraged from formally submitting his manuscript.

EMMY: That's sad. So what first caught the attention of mathematicians – why is it called the Argand diagram today?

MARIA: The Continental mathematicians, at least, were alerted when a paper of J.-F. Français appeared in Joseph Gergonne's journal, *Annales de mathématiques pures et appliquées,* in

[7] Preface to (Wessel 1897).

1813.[8] Gergonne was an extremely pro-active editor, and enjoyed adding footnoted comments to articles, and challenging other mathematicians to respond. Significantly, he placed the article of Français and subsequent responses in a rather inauspicious section of his journal, called 'Philosophie Mathématique'. Let's listen in, now, to a conversation between Gergonne and three contributors.

***CURTAIN RISES***

GERGONNE: Monsieur Français, thank you for your very interesting contribution to the *Annales*. I have invited some others to join us, who have expressed views on this same subject you call *géométrie de position*. Allow me to introduce M. Argand ... and M. Servois …

[*they shake hands, saying appropriate things in early nineteenth century French style, and Gergonne ushers them to a table where they are all seated. A bottle of wine may be poured*]

FRANÇAIS:[9] [Thank you, M. Gergonne, I am honoured to meet you and colleagues. My article is merely a] sketch, very much abbreviated, of the new principles on which it is convenient, and even necessary, to base the theory of 'Geometry of Position' which I submit to the judgement of geometers. Since these principles are in formal opposition to current ideas about the nature of imaginary quantities, I expect numerous objections. But I dare to believe that a deep examination of these same principles will find them correct, and the consequences that I deduce from them, no matter how strange they may at first appear to be, will nevertheless be judged to conform to the most rigorous rules of dialectic.

GERGONNE: Far be it from me to deflate you, M. Français, but, you know, these ideas are not at all so strange as to be incapable of germinating in several heads at once.

FRANÇAIS: Indeed, M. Gergonne? May I ask who else has propounded them?

GERGONNE: Two years ago, in editing an article in the *Annales,* I was constrained to add a footnote which I thought would bring more clarity to the discussion; and in doing so I placed some imaginary numbers in a geometric configuration not dissimilar to that advocated by you.

FRANÇAIS: I do not claim that this idea is my own – I discovered it in a letter of Legendre, when going through the papers of my brother after his death.

GERGONNE: Ah – are you saying that M. Legendre is the discoverer?

FRANÇAIS: No – Legendre seemed not to regard this idea as worthy of publishing, nor was it his own idea. Indeed, he said that he obtained it as an object of pure curiosity from some other unnamed mathematician. But I believe it has more importance than he or its author gave it, and I hope that whoever it was that first produced the idea will make himself known and publish his theory in full.

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[8] (Français 1813a).

[9] This speech, except for the square bracketed part, is a direct translation, by Barry Mazur, of a passage in the article in *Annales*, p. 70. The dialogue that follows is inspired by Mazur's discussion in Chapter 11 of (Mazur 2004), including his English translation of certain phrases.

GERGONNE: [*turning to Argand*] M. Argand, you have come forward with a claim to being this unknown author!

ARGAND: Yes, indeed! I am pleased to respond to the request of M. Français for the original author of these ideas to make himself known. I can confirm that M. Legendre first heard from me of the geometric construction by which I represent the imaginary numbers, and indeed, M. Français's wish for the full exposition of the theory has been answered already some years ago, in 1806.[10]

GERGONNE: [*sharply*] I have not seen that publication!

ARGAND: Unhappily, my manuscript attracted no attention at the time. I have now had some copies printed, taking the liberty of inscribing the original date of 1806. Should any mathematicians wish for one, they may write to me. And I have one here to present to you, M. Gergonne. [*hands it to Gergonne with a bow*]

GERGONNE: I am greatly obliged... [*scrutinises the cover page*] ... and you have dedicated it to me, I see – I am honoured, Monsieur! [*sighs deeply*] I fear the state of mathematical communication is deplorable! I hear from a colleague that a man called Buée published related ideas in London – also in 1806, but I have no access to the London Transactions. And regrettably Lacroix and the other Parisian mathematicians do not take the trouble to keep me informed![11] But did you not publish your original paper seven years ago?

ARGAND: I confess I simply distributed a few copies of my *écrit*. But I did pluck up the courage to show a copy to Legendre in order to find his opinion before I could dare submit to the *Institut*. He seemed sceptical, and I went away feeling very discouraged. However I left the manuscript with him to examine further, and imagined I would contact him or hear from him in due course. Unfortunately, nothing ever came of my visit.

[*Cameo appearance, aged about 60, reminiscing directly to audience*]

LEGENDRE:[12] [*wearing a nightgown, looking distressed, carrying a lighted candle*] I cannot sleep tonight. I see a face in my dreams ... I recall that face vividly. One day an unknown person requested to see me – a young man, who wished to show me some work of his. He did not explain his motives very well, but I understood that he considered the so-called imaginary quantities to be as real as other numbers, and he represented them by lines. After he left, I brought myself to take a closer look at his manuscript, and, quite contrary to my expectations, found some quite original ideas, leading to demonstrations of trigonometric formulas and Cote's theorem. He never returned, and I realised I did not have his name or address. I wonder what happened to him. I hope I was not rude to him.

[*walks on a few steps, before turning to audience again*]

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[10] In November, Argand responded with a note (Argand 1813) in the *Annales* under the same title as his 1806 manuscript and its freshly printed copies.

[11] *Annales* 4, 1814, 367.

[12] Based loosely on an extract from a letter from Legendre to F.-J. Français, 2 November 1806, in (Schubring 2001), 129.

I was so sure at first that he was just another eccentric come to trouble me ... Yet the ideas were good, and I suppose I might have helped him. However, I did write to M. Français the elder – that's Francois Joseph Français – giving him a sketch of the work, and leaving him to judge its worth. Strange – I cannot shake off this feeling of having let the young man down ... or perhaps even been guilty of unfaithfulness to mathematics.

[*shakes his head and attempts to shrug off his unease*]

But if people don't leave their address, what can one do?

[*shuffles offstage and exits*]

FRANÇAIS: I cannot speak for M. Legendre's attitude, but I do regret, Monsieur Argand, that my brother Francois seems not to have given due attention to your excellent ideas. I got the impression that neither he nor M. Legendre knew the name of the author!

GERGONNE: [*shaking his head and frowning*] Never write a manuscript without your full name, address and affiliation! You know, Monsieur Argand, your previous letters to me [*scrutinizes again the paper of Argand*] – and, indeed, also this paper you have presented to me – do not include your full name! One would imagine you have some criminal past to hide!

ARGAND: [*stiffly*] I apologise Monsieur – and assure you that my life has been exemplary and singularly uneventful! My big regret is isolation from those who might understand my work, and [*now he shows warmth*] I am overjoyed to find myself in the present illustrious company, and have given, for your *Annales*, a full résumé of my theory.

GERGONNE: [*smiling*] I trust you inserted your full name there? The printers do not care, and I may have missed that detail.

ARGAND: Ah – I must check on that ... I am pleased to say, however, that I have given an application to simplification of a well-known proof of d'Alembert.

GERGONNE: Hmmm ... [*turning to Servois*] I wonder, M. Servois, if you would be good enough to respond to both M. Français and M. Argand, regarding this curious matter of the geometrical representation of imaginary numbers. I have shown you their submissions; have you looked through them?

SERVOIS: Indeed – I have read them with interest. But, firstly, I must point out that M. Argand has made some mistaken claims, in particular, his supposed simplification of d'Alembert's inadequate proof of the fundamental theorem first satisfactorily proved by Gauss. And also his assertion that the imaginary quantity $\sqrt{-1}^{\sqrt{-1}}$ cannot be given expression in the standard form. This latter has been shown by Euler to be expressible as a real number.[13]

ARGAND: [*shocked*] Monsieur –

GERGONNE: [*interrupts, exerting editorial authority*] Let us hear what else M. Servois has to say.

[13] This and his subsequent speeches are based on a letter to the editor in November 1813 (Servois 1814).

SERVOIS: [*turning to Français*] I must also point out to you, M. Français, that you do not adequately demonstrate much of what you assert.

FRANÇAIS: [*angrily, may improvise French expostulations*] Excuse me, M. Servois, but I really do not think –

GERGONNE: [*pacifying gestures*] Gentlemen, we stand before novel ideas here, let us judge and debate with patience and respect. M. Servois, if what you claim is true, readers of this journal will be in your debt for drawing attention to these points. But I think it would be greatly appreciated if you would give your opinion as to the usefulness of this new mode of perceiving, reasoning and calculating with imaginary quantities.

SERVOIS: I myself am very doubtful that this geometrizing of the imaginary quantities, giving rise to the peculiar notation of these authors, is of great value.

FRANÇAIS & ARGAND: [*emotional gestures and French exclamations*]

GERGONNE: [*waving hands*] Monsieurs! Monsieurs! Let us hear him out!

SERVOIS: In setting up the foundations of an extraordinary doctrine, somewhat opposed to received principles, in a science such as mathematical analysis, mere analogy is hardly a sufficient mode of reasoning. I see the so-called geometrical representation as nothing more than a geometrical mask, to be worn over the analytical forms as some sort of softening disguise. But I myself regard the calculation with imaginaries, by the usual fully understood laws, as simpler and more effective than resorting to geometry.

GERGONNE: M. Servois, do you really consider it unimportant to see, at last, algebraic analysis stripped of its unintelligible and mysterious forms – stripped of the nonsense that limits it and makes it, so to speak, a cabalistic science?

SERVOIS: I see the future of algebra as pure analysis of symbols, without any need to introduce geometrical disguises and analogies. If one wishes to make the algebra more palatable by this means then, of course, one is free to think with the aid of geometry. But algebra deserves to be set free to be a science in its own right, at least when practised by initiates who do not cavil at imaginary numbers as cabalistic symbols, but accept them on an equal footing with what we call real numbers.

GERGONNE: M. Français, what is your view of this – ah – visionary description of the progress of algebra?

FRANÇAIS: [*still seething*] I have published my exposition in order to draw attention to, and give reasons for, my important affirmation, given as Corollary 3, which it seems is precisely the view of M. Servois, although he does not give cogent reasons for his view. I say: 'Imaginary numbers are just as real as positive numbers and negative numbers, and only differ in their position, which is perpendicular to the latter.' The imaginary part of $a + \sqrt{-1}\, b$ is to be taken at right angles to the line, so that the quantity, otherwise without clear meaning, can be visualised and accepted without hesitation.

ARGAND: That is quite right – I have made the same assertion –

SERVOIS: [*airily, even scornfully*] I deny that there is anything essentially 'perpendicular' about imaginary numbers – that is just the geometric mode in which you choose to see them. Analogies may help some, but they can also distract from essential commonalities and distinctions.

GERGONNE: Monsieurs Français and Argand, what is your response?

FRANÇAIS: M. Servois, you merely *claim* equal footing for imaginaries and real numbers, but many will choose to disagree with you. By this geometric argument, it can be *shown* to be valid.

ARGAND: For my part, it was only when I saw that numbers can be regarded as transformations of points in the plane, that I found myself at peace with the imaginaries, which were previously unintelligible and mysterious to me. The number 1 leaves every point where it is; the number $-1$, which some have regarded as *reflecting* the line in its origin, can more fruitfully be seen as *rotating the plane* through $180$ degrees, or two right angles. Whereupon it becomes clear that there can be a transformation that rotates the plane through one right angle, and that this perfectly concrete and intelligible transformation then has a square equal to the transformation $-1$. Indeed, there are two such transformations – another is that which rotates the plane through three right angles, or simply a negative right angle. Thus we have two square roots of $-1$. And three cube roots, etc.

GERGONNE: Well expressed, M. Argand! Monsieurs, we shall let the arguments rest for now – I invite you all for dinner at my favourite restaurant!

[*rises to his feet and waves his arms in all-embracing gestures*]

No doubt we may await developments in both M. Servois's pure algebra and also the geometric point of view. It will be interesting to see what fruit there is from this clarification of the mysteries of imaginary quantities. If the uniting by Descartes of algebraic equations and geometric curves has been so fruitful, may we not expect comparable fertility from this new union of the algebraic with the geometric? *Viva la mathématiques!*

[*Argand, Français and Servois all stand, and follow him offstage, engaged in vigorous and emotional debate, with dramatic gesticulations*]

***CURTAIN FALLS***

MARIA: Gergonne's prophecy would be wonderfully fulfilled in the development of complex analysis, beginning with Cauchy's work just a year later.[14] Servois is, like his English contemporary, Charles Babbage, a radical formalist – he believes in symbolic expressions as the sure carriers of mathematical thought, needing no justification or geometrical representation. His views are not representative of many others of his time. EMMY: [*laughing*] Fortunately! – thank goodness for the Argand diagram! But that statement by Français that 'imaginary numbers are just as real as positive numbers and negative numbers' is now a truism of algebra.

[14] Cauchy himself attributes priority in discovery of geometric representation of imaginaries to 'un savant modeste', Henri-Dominique Truel, who mostly kept the ideas to himself from about 1786. Details are in (Schubring 2001), 136.

MARIA: Yes, indeed, but at that time it was a remarkable insight, echoing Argand's own views. There would be few affirmations like it in the literature for many decades to come! We can truly admire the foresight of Gergonne in encouraging debate on radically new ideas, bringing together, in a corner of his journal, pure algebraic formalists and pragmatic geometric conceptualists, and envisioning the power of this new union of algebra and geometry!

EMMY: I see Gergonne as an early exemplar of the immense influence that journal editors can have, and the responsibility they share, in the propagation and cross-fertilisation of ideas.

MARIA: Absolutely, and in the reception of novelty. Gergonne demonstrates what a vital role an editor can play in shaping disciplinary discourse, and negotiating disciplinary boundaries.

EMMY: Yes, I agree that editors are important agents in filtering valuable ideas, encouraging correspondence, mediating disputes, and influencing research priorities. On the negative side, however, they can sometimes be responsible for discouraging promising young researchers and killing new ideas.

MARIA: Sounds like you have had, or seen, some bad experiences, Emmy! So, you obviously think editorial influence and responsibility is just as important today in the mathematical and scientific communities, even in the age of the internet?

EMMY: Oh, yes! The business of quality control and the process of overseeing selection, and bringing to wider attention the right articles among the rising tide of submissions, is crucial for the development of mathematics. And the right articles may not emanate from the big names only!

MARIA: Right – Gergonne might have ignored the offerings of Français and Argand. Our story is a fascinating example of the way advances are made by the cooperative efforts of many lesser-known and even unknown contributors.

EMMY: And also of the vital importance of the *go-betweens* in encouraging conversation and collaboration – mentors and sponsors, journals and editors, all the activities of scientific societies.

MARIA: How significant was it that Gergonne the editor was an excellent mathematician?

EMMY: For our story, highly significant, I think. He had mathematical foresight, he enjoyed the deep respect of his contributors and could engage with them technically. Even today I think journals can benefit greatly from editors who work, or have worked, in the field. Of course, most editors now have an editorial board and a stable of trusted reviewers.

MARIA: Emmy, what corresponds today to Gergonne's innovative 'Philosophy of Mathematics' section?

EMMY: I think it's even more important, today, for journals to provide a forum for review articles, correspondence, debate and interdisciplinary discourse, as researchers are forced to focus on ever more narrow specialisations.

MARIA: Then let us celebrate Gergonne, and his editorial heirs, for their great service to mathematics! And let us also applaud the contributors to Gergonne's journal whom we saw in debate, and their forerunners in developing the fruitful ideas they were discussing –

EMMY: – which, two centuries later, are part of the basic toolkit of mathematics – the heritage of all mathematics students!

[*Maria invites the cast on stage and leads applause, then holds hands with Emmy and the rest, and they take a bow*]

## THE END


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I wish to express my indebtedness to Barry Mazur, whose book *Imagining Numbers* first gave me the inspiration for a dialogue centred around Gergonne, and to Gert Schubring, who kindly drew my attention to his investigations of Argand and pointed to the drama inherent in Argand's encounter with Legendre.


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