FORMATIVE YEARS:
HANS FREUDENTHAL IN PREWAR AMSTERDAM

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ABSTRACT

Hans Freudenthal started his public career in mathematics education after the war. During the war, when he was expelled by the Germans from the university, he made a thorough study of the didactics of teaching arithmetic. Freudenthal himself made on several places remarks about the influences he underwent before the war, he even suggested that his "framework" on mathematics education was already formed then. Although he was in those year active as a mathematician in the first place, there can be no doubt that he was then already seriously interested in the problems of the teaching of mathematics. He had inspiring discussions with other mathematicians with unorthodox ideas about the teaching of mathematics and read publications that helped him to develop his own ideas. The seeds that were sown in those years would bear fruit long afterwards.

1 Introduction

On the morning of the 16th of November in 1930, a young man 25 years of age arrived with the night train from Berlin in Amsterdam. It was a mathematician who just had finished his PhD with a thesis on topology. He came to Amsterdam on invitation of L.E.J. Brouwer, professor in Amsterdam and a mathematician who not only was quite famous, but also highly disputed. The name of this young mathematician was Hans Freudenthal, and he came to Amsterdam as a next step in his career as a mathematician, a career that of course only just had started. It was a successful career move; since during the years that followed he produced a number of papers that established his reputation as a first class mathematician. When he came to Holland, he will not have foreseen that he would spend the rest of his long life in that country, or that he would become perhaps even more famous as a mathematics educator than as mathematician.

That career in mathematics education, what could be called a second career, started more than twenty years later, and went well under way in the sixties and seventies of the last century. In those years he was president of the ICMI, he organized the first ICME conference in 1969 in Lyon and he founded in the same year the journal *Educational Studies in Mathematics*. The ICMI honored him by creating a "Hans Freudenthal award" for "a "major cumulative program of research" in the field mathematics education, of course a clear sign of his importance.

Although Freudenthal started his public activities in mathematics education after the war, it is well known that he began to work seriously in this field during the war. Freudenthal was from a Jewish family, and although his marriage to a non-Jewish Dutch girl gave him some protection against immediate deportation, he lost his job at the University of Amsterdam. He was arrested, set free again, and later on send to a labour camp from which he escaped. On the whole he had, unplanned and no doubt unwelcome, a lot of spare time. His wife, Susanna Lutter, was highly interested in education, especially for the young children they had together, and she stimulated her husband in teaching arithmetic to their children. Freudenthal took an interest in observing how his children learned arithmetic and he began to read all he could lay his hands on about the didactics of teaching arithmetic. He filled 300 pages of a notebook with critical comments on what he had read and he composed a manuscript of 103 pages about the didactics of teaching arithmetic. The manuscript was never finished or published, but can be found in Freudenthal's personal archives that are kept in Haarlem, together with the notebook.

Figure 1. Hans Freudenthal 1905-1990.
Noord-Hollands Archief, Archief Freudenthal, inv.nr. 1914.

These activities are usually considered as his start in mathematics education. His recent biographer goes even a step further when she says about his post-war interest in mathematics education: "The seeds for this interest were, of course, already be sown during the occupation". (Bastide-van Gemert, 2015, p. 52) The use of the word "seeds" supposedly sown during the war suggests that before the war such an interest did not exist. Now that is certainly not the case. Freudenthal himself, in his autobiographical sketches, says that he was interested in educational matters already at a young age. He says: "The framework of my activities concerning mathematics education was already been formed in 1942, *or maybe even ten years before"*. (Freudenthal 1987a, p. 344, italics by the author of this paper) In a letter to Geoffrey Howson, he says that his remark in the preface of his *Mathematics as an Educational Task*, that "the first suggestions to occupy myself theoretically with education came from my wife", is most likely not correct. He wrote to Howson that "I should perhaps have said that she reinforced my interest in mathematics education". (Freudenthal, 1983). Apart from Freudenthal's own remarks, what can be said more about the formation of this framework for his interest concerning mathematics education? What are the indications that the seeds for his later didactical activities were in fact already sown in the pre-war years? Were there in those years people in the Netherlands who might have influenced him? I think there were and my aim in this presentation is to tell you something about them.

2 Some remarkable personalities

In the preface of his *Mathematics as an Educational Task,* Freudenthal remarks about L.E.J. Brouwer the following: "My educational interpretation of mathematics betrays the influence of L.E.J. Brouwer's view on mathematics (though not on education)". (Freudenthal, 1973) Freudenthal was Brouwer's assistant and we might suppose that his influence was exerted in the years before the war. So, our first question is: what was Brouwer's view on mathematics and how should we interpret this remark by Freudenthal?

There are others, not mentioned in the preface of his *Mathematics as an Educational Task*, but elsewhere, who certainly influenced him in the pre-war years. One of them was David van Dantzig, born in 1900, so five years older than Freudenthal. When the latter arrived in Amsterdam, Van Dantzig had already finished his mathematical studies there. But although he had a job elsewhere, he still frequently visited the Amsterdam mathematicians and he knew Freudenthal personally. Van Dantzig had highly original ideas about mathematics education and had published some articles about it. (Smid, 2000) He became after the war professor of mathematics in Amsterdam and when he died unexpectedly in 1959, Freudenthal held the memorial speech for the Dutch Mathematical Society. (Freudenthal, 1960) In this speech Freudenthal fully recognizes the influence that David van Dantzig had on him in those early years in Amsterdam. So we will have to discuss the ideas of Van Dantzig, and when we speak about Van Dantzig, we cannot but speak also about Gerrit Mannoury, a man who on his turn influenced Van Dantzig deeply and who was the colleague of Brouwer in Amsterdam.

The last person I want to speak about is usually connected with Freudenthal's activities after the war. It is Tatyana Ehrenfest-Afanassjewa, who was the heart and soul of the Dutch Mathematical Working Group, of which Freudenthal became a member after the war. But in an article by Pierre van Hiele, on occasion of Freudenthal's 70th birthday, Van Hiele recalls how enthusiastic Freudenthal in the early thirties was about one of the publications of Tatyana Ehrenfest. Van Hiele was then one of Freudenthal's students, and he describes how Freudenthal urged his students to read a small booklet, the *Uebensammlung.* (Van Hiele, 1975) Certainly in this case, it seems most appropriate to speak about a seed that was sown in Freudenthal's mind in those years, a seed that would bear fruit much later.

3 Luitzen Egbertus Jan Brouwer

L.E.J. Brouwer – Bertus for his intimi – was a mathematical genius. This reputation is based in the first place on his fundamental contributions in topology. In the years 1909-1914 he published a series of articles which, as his biographer wrote, contains not only spectacular results, such as the theory of dimension and the fix-point theorem, but also furnished new tools to breathe new life into the research of topology, which was more or less in a dead end. (Van Dalen, 2013)

Figure 2. L.E.J. Brouwer
Picture made available by the Brouwer-archives

The other element that contributed to his fame is his work on the foundations of mathematics. Brouwer was not only a mathematician, but also a philosopher with a strong mystical accent. He completely rejected the two then current theories on the foundations of mathematics; the logicism by Russell, and the formalism by Hilbert. For Brouwer, mathematics was rooted in, and created by the human mind. Crucial is the observation of the permanency of objects during consecutive points of time. That fundamental intuitive sensation of time gives, according to Brouwer, birth to the concept of counting and the series of natural numbers. All mathematics is built on this series. The continuum on the other hand, cannot it be constructed from natural numbers, nor does the continuum consist of all numbers. The continuum is in fact the separation between two different numbers, and you can construct new numbers on this continuum, but that does not mean that the continuum **is** merely a collection of numbers. As a consequence, Brouwer accepted the existence of mathematical objects only when they could be constructed in a finite number of steps. This principle gave birth to his intuitionistic mathematics, in which parts of the traditional mathematics were rejected, or at least were questioned.

Brouwer had the essence of his philosophical ideas already exposed in his PhD of 1907, but during his work on topology this part of his work remained in the shadows. It was not before the twenties that he returned with new energy to the foundations of mathematics. Brouwer was then, with Hilbert as editor in chief, one of the editors of the *Mathematische Annalen*, and so far both man respected each other. Hilbert interpreted Brouwer's new publications on intuitionism however as an almost personal attack on his ideas about the foundations of mathematics, and moreover, he feared that the future of mathematics was seriously endangered if Brouwer's ideas would prevail.

The differences between Hilbert and Brouwer resulted in a clash that deeply divided the German mathematical community. In 1927 Brouwer gave s series of lectures on intuitionism in Berlin, where he was greeted with enthusiasm. There was a certain rivalry between Berlin and Göttingen, the kingdom of Hilbert, so someone who had the guts to stand up against Hilbert received admiration in Berlin, certainly from the Berlin students. Freudenthal, being a student then in Berlin, made himself acquainted with Brouwer's ideas, followed a seminar on intuitionism, attended Brouwer's lectures and posed – in writing - some intelligent questions. (Freudenthal, 1987b p.10) The two men met, and in the following years Freudenthal sent his results in topology to Brouwer. In the summer of 1930, Brouwer invited him to come to Amsterdam to work there as his assistant. Freudenthal worked in Amsterdam on topology, and although he of course was well aware of Brouwer's intuitionism, that theory doesn't play an important role in his work. So what does Freudenthal mean by saying that "his educational interpretation of mathematics betrays the influence of L.E.J. Brouwer's view on mathematics"?

That can be explained best by a quotation of some words of Brouwer himself. In 1946, in a speech to his former colleague Gerrit Mannoury, Brouwer spoke about his early years as mathematics student in Amsterdam, when he could see mathematics only as, I quote, "a collection of truths, fascinating by their immovability, but horrifying by their lifelines". But Mannoury, said Brouwer, had shown to him that the work of a mathematician was something else then collecting such lifeless truths. The undertone of Mannoury's exposure of mathematics had been – I quote again – as follows:

> Look what I have built for you out of the structural elements of our thinking. – These are the harmonies I desired to realize. This is the scheme of construction which guided me – Behold the vision which the completed edifice suggests to us, whose realization may perhaps be attained by you or me on one day" (Van Dalen, 2013, pp. 43-44).

The tone may sound a bit swollen and exaggerated, but the meaning is clear: mathematics is made by living people; it has to be constructed by mind. Mannoury, like Brouwer himself, did not believe in eternal mathematical truths. Mathematics is not a Platonic world lying outside just waiting to be discovered, nor is it just a formal game without any bond with reality. The quotations from Brouwer can be found in a recent biography of Brouwer that has as a subtitle: *How mathematics is rooted in life*. (Van Dalen 2013) That subtitle is also a quotation, in this case from a letter of Brouwer to his PhD supervisor Korteweg, explaining to him what he wanted to do in his thesis: to show how mathematics is rooted in life. The same basic idea is summarized in a famous formulation by Hermann Weyl, in the early years one of Brouwer supporters: "Mathematics is more a way of doing then a theory".

Brouwer himself was not interested in education, only in high level mathematics and in philosophy. But Freudenthal of course really was an educator, and in this capacity one of his adages was: "mathematics as a human activity". Brouwer could have said the same, although he would not apply that to mathematics education for children, as Freudenthal did. There can be no doubt that Freudenthal was right: without being an intuitionist himself, his educational interpretation of mathematics *does* betray the influence of L.E.J. Brouwer's view on mathematics.

4 Gerrit Mannoury and David van Dantzig

In his memorial speech for David van Dantzig in 1960 for the Dutch Mathematical Society, Freudenthal told that even before he came to the Netherlands, he had read two articles written by Van Dantzig about the didactics of mathematics.(Freudenthal, 1960, p. 61) That may seem a bit unlikely at first sight, since the articles are in Dutch, but maybe Freudenthal wanted in the period from August 1930, when he received the invitation by Brouwer to come to Amsterdam, until his depart in November to learn some Dutch, and he could very well have done so by reading Dutch articles. Freudenthal had a talent for learning languages, and German and Dutch are relatively cognate; so with a grammar and glossary at hand, he could certainly do such a thing. He said in his speech that he had to admit that he was very impressed by these articles. Who was Van Dantzig and what was the content of these articles?

Figure 3. David van Dantzig in 1934

David van Dantzig – not to be confused with the much more famous American mathematician George Dantzig, the so called father of linear programming – was born in 1900 in Amsterdam. He first started studying chemistry, but financial reasons made an end to this study. Gerrit Mannoury, whom we just met as a colleague of Brouwer, was then a lecturer at the university teaching mathematics to the chemistry students, and Van Dantzig, in fact much more interested in mathematics than in chemistry, wrote a letter with some questions on mathematics to Mannoury. When Van Dantzig stopped with his study in chemistry, both men stayed in contact and when van Dantzig financial circumstances improved, Mannoury persuaded him to study mathematics. Van Dantzig remained close friends with Mannoury all his life and was influenced deeply by him.

Mannoury, who as we have seen also influenced Brouwer, was a most remarkable man. He was professor of mathematics, but did not hold a university degree. He had been a schoolteacher, but in mathematics he was almost completely an autodidact. Mannoury had not only highly original ideas about mathematics, but also about the teaching of mathematics. The way mathematics was taught and the content of school mathematics was in his view worthless. There was no reason at all to bore the children with all that rubbish. Mathematics teaching was in those days always defended for its supposed "transfer" or "formative value", as a kind of gymnastic for the mind. Learning mathematics disciplines the mind and helps you to think logically. Nonsense, said Mannoury. When you learn mathematics, you just learn mathematics and nothing more, he declared, learning mathematics is like learning to play chess: you learn just that and no more. It will come to no surprise that Mannoury's ideas were not very popular or influential in the world of the Dutch mathematics teachers.

Figure 4. Gerrit Mannoury in 1917

Van Dantzig was in complete agreement with Mannoury's views and in one of the articles Freudenthal mentioned, he discusses the problem of transfer. Its title is (translated) *On the social value of teaching of mathematics.* (Van Dantzig, 1927) The traditional teaching of mathematics teaching has, according to Van Dantzig's opinion, simply no social value at all, and it should be better to teach the great majority of the children only some simple, essential techniques that they can use in daily life, and nothing more. Only for a small minority which is going to work in professions where they made use of mathematics, teaching more mathematics is useful. The teaching of mathematics could have only some *social* value if it was taught completely different: in a much more "living form", with a strong linguistic accent. Van Dantzig was not very clear how this could be done, and I think we do him no injustice if we suppose he did at that time not have a clear vision how that should be done himself.

Freudenthal remarks in his autobiographical sketches that he rejected the idea of transfer as long as he could remember. (Freudenthal, 1987, p. 359) Some pages earlier he is a bit more specific, where he says that he is convinced that he rejected the idea that mathematics was instrumental in "learning to think" already in 1932, when he gave a didactical seminar at the university in Amsterdam. (Freudenthal, 1987, p. 338) It is impossible to say if the reading of Van Dantzig article from 1927 gave him this conviction, or that its reading just corroborated an already existing, perhaps vague opinion with Freudenthal, but there can be little doubt that Van Dantzig influenced the young Freudenthal in this aspect.

The other article by Van Dantzig that Freudenthal referred to is about a problem that was heavily disputed in The Netherlands in those years: the way how mechanics should be taught on secondary education. The article had lost most of its relevance for long, but it contains some remarks that sound familiar for everyone who is acquainted with Freudenthal's ideas. Discussing the way mathematics should be taught, Van Dantzig remarks that "ready-made mathematics does not arouse anybody's interest". It is essential, he writes, that also in schoolbooks the process of *mathematization* is actively carried into effect; otherwise mathematics remains a dead object. (Van Dantzig, 1929, p. 97) Almost fifty years later Freudenthal would, in his *Mathematics as an Educational Task*, write a chapter with the title *Organization of a field by mathematizing*. In that chapter he says: "There is no mathematics without mathematizing. (…) This means teaching or even learning mathematics as mathematization". (Freudenthal, 1973, p. 134)

In his autobiographical sketches Freudenthal wrote that Van Dantzig belonged to the group of colleagues and students with whom he in the early thirties discussed the situation of Dutch mathematics teaching. It may be clear that both young men shared many ideas.

5 Van Dantzig and his ICMI-report

Van Dantzig wrote no more articles on mathematics education for more than twenty years, but in 1955 he published *The function of mathematics in modern society and its consequence for the teaching of mathematics*. It was a report for the ICMI conference of 1954 in Amsterdam. In a way, it contains the answer by Van Dantzig to the problem he posed more than two decades earlier: what is the social value of mathematics? Then, while rejecting the idea of transfer, he could not really find an answer, but now he could. (Van Dantzig, 1955) Mathematics had become an indispensable tool in modern society, and the use of mathematical models could be found everywhere. As a consequence, Van Dantzig pleads for a thorough reform of the traditional school mathematics into what he called a "consumers-mathematics", in which concepts as mathematical models, testability, and what he calls good working knowledge of graphs, algebraic computing and elementary calculus are taught. His rejection of the long outdated school mathematics is an echo of his opinion of more than twenty years earlier, but his idea for a "consumers-mathematics" is new.

Van Dantzig's report was much cited in those years in The Netherlands, but Freudenthal, while expressing his admiration for Van Dantzig didactical publications from 1927 and 1929, did not mention it at all in his memorial speech of 1960. That could be coincidental, but I think also that Van Dantzig's report contained elements that Freudenthal did not appreciate. One of these elements will have been Van Dantzig's plea to teach this consumers-mathematics as cost-effectively as possible, and to treat the student results as a mass product, to which we should apply all methods for a satisfactory quality control. That would include the extensive use of statistical methods, something Freudenthal abhorred in education. Clearly, in Van Dantzig's approach not the individual student stands in the center, but the benefit for the society as a whole. Freudenthal essentially wanted to educate individuals, and teaching mathematics was an educational task for him. For Van Dantzig, it was much more an social or economical task, to be performed for the benefit of the modern society as a whole. Van Dantzig's ideas fitted perfectly in the first decades after the war, when the reconstruction of the Western societies after the war, and the *Cold War* rivalry with the communist countries were central issues. Freudenthal's ideas on mathematics education were much more congenial with the educational climate of the seventies, when the optimal development of the individual stood in the centre.

A second reason why Freundertghal did not mention Van Dantzig's publication of 1955 might have been that Freudenthal had been very active in those years in working on a new mathematics curriculum for the Dutch schools. That new curriculum was certainly a step forward, but compared with the ideas exposed by Van Dantzig in his report, one has to admit that it was still rather old fashioned. Freudenthal however was very happy with it and saw no reason for a new curriculum at that moment. One might suspect that he did not share on that time already Van Dantzig's ideas about such a consumers-mathematics.

Some thirty years later however, such a consumers-mathematics was introduced at last. It was intended for those students who had to use mathematics as a tool, but who did not receive much instruction in mathematics in their tertiary education. Many elements described by Van Dantzig in his report of 1955 can be traced in this new program. The institute founded by Freudenthal, then still called the I.O.W.O., was the main driving force behind its introduction, and no doubt Freudenthal, then still very active, strongly supported it. But of course, there were differences. Van Dantzig focused on the content, not on the didactical approach as Freudenthal and his followers did, and Van Danztig's ideas about "cost-effectively" mathematics teaching and "quality control" of the teaching results were completely put aside.

6 Tatyana Ehrenfest-Afanassjewa

Freudenthal was not the only foreigner who deeply influenced Dutch mathematics teaching in the 20th century. The other one was a Russian lady, Tatyana Afanassjewa, born in Kiev 1876. At a young age she moved to St. Petersburg, where she attended secondary education, including the local gymnasium for girls. She followed courses on pedagogy and studied physics and mathematics at the St. Petersburg Woman University. She was no doubt a talented girl, and after her studies in Petersburg she went to Göttingen where she studied with Klein and Hilbert. There she met the physicist Paul Ehrenfest, with whom she married. The young couple returned to St. Petersburg where Ehrenfest tried in vain to find a job. Tatyana however could work there as a mathematics teacher and was involved in several experiments to modernize the teaching of mathematics.

In 1912, Paul Ehrenfest was at last appointed in Leiden as professor in theoretical physics, as successor of Lorentz. Tatyana followed her husband to Leiden, and although she felt never really at home in The Netherlands, she stayed there until her death in 1964.[1]

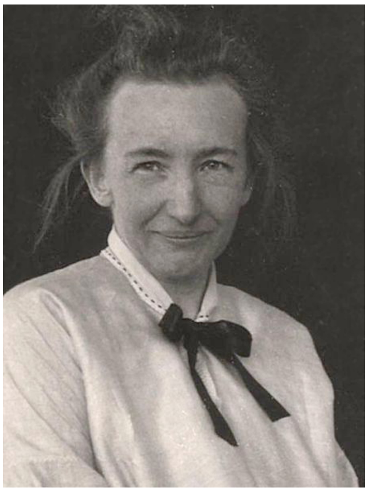

Figure 5. Tatyana Ehrenfest – Afanassjewa around 1910

Before World War II she returned several times for months during the summer to the Soviet Union to assist in the training of mathematics teachers there. In The Netherlands she taught only for one year mathematics on a school for secondary education, but her unofficial role and influence were much more important. Soon after her arrival she published a Dutch translation of one of her Russian lectures on the teaching of geometry, to "introduce herself to the Dutch mathematics teachers", as she formulated it. She succeeded in forming a discussion group of interested mathematics teachers, a group that in 1936 was transformed into the *Mathematics Working Group,* belonging to the Dutch branch of the still existing *New Education Fellowship* .[2]

In 1924, she published a brochure that would have important consequences. Its title (translated from the Dutch) was "What could and should the teaching of geometry offer to a non-mathematician". (Ehrenfest-Afanassjewa, 1924). The general opinion then in The Netherlands, but certainly not only there, was that the teaching of geometry in a deductive way, more or less in a Euclidean manner, was profitable for everybody: it made you think better and more logically. Ms. Ehrenfest had participated in 1920 and 1921 in a series of meetings at the department for pedagogy of the University of Amsterdam, in these meetings this question of transfer was discussed. (Van Dantzig, 1927) One of the other participants was Gerrit Mannoury, as we have seen a strong opponent of the idea of transfer. It seems likely that this brochure was a result of these meetings. Tatyana Ehrenfest was in favor of the idea of transfer, but certainly not in a naïve or unconditional way. The way geometry was taught in those days in The Netherlands: in a strict deductive way, right from the start, was in her opinion only detrimental for the possibility of transfer. She pleaded for a "propaedeutic"

[1] More biographical details about her life can be found in the preface by Bruno Ernst in her *Didactische Opstellen Wiskunde* (Ehrenfest-Afanassjewa 1960)
[2] Now the *World Education Fellowship*

introductory course in geometry for young children, without the proving of theorems, in which they could develop their spatial ability. Only afterwards a more systematics course, in which geometry is taught in a more traditional way, was appropriate.[3]

In his autobiographical sketches Freudenthal discusses her article on the teaching of geometry of 1924, which he, as he says, read again for the occasion. He does not say when he read the article for the first time, but it seems unlikely that he did not read already much earlier. Freudenthal wrote to Geoffrey Howson in the letter that we have already cited before, that had met Tatyana Ehrenfest already before the war, when he attended once or twice a seminar on physics at her house (Freudenthal, 1983). Freudenthal added that he did not attend her didactical seminars on mathematics in those years. His participation in the Mathematical Working Group started only after the war, in 1947.

In the same letter to Howson, Freudenthal wrote that he partly developed his ideas "on the teaching of mathematics by opposing hers", certainly referring also to their continuing discussion about the problem of transfer. They published together in 1951 a brochure with the title "Can the teaching of mathematics contribute to educate the ability of thinking?" in which they discuss their different opinions about these problems. As Pierre van Hiele however has pointed out later, these differences are not as big as they seem at first sight, and are partly due to a vague formulation of the question. (Van Hiele, 1975).

7 The 'Uebensammlung'

While we cannot be completely sure if Freudenthal read Ms. Ehrenfest's publications of 1915 and 1924 already before the war, he certainly read her publication of 1931 soon after it had appeared. Van Hiele described Freudenthal's enthusiasm about this booklet, her *Uebensammlung zu einer geometrischen Propädeuse* . (T. Ehrenfest-Afanassjewa 1931). Pierre van Hiele was in those years one of Freudenthal's students in Amsterdam and he attended a didactical colloquium that was organized by Freudenthal. Each of the students had to present some piece of mathematics and, as van Hiele adds: "Freudenthal very cunningly choose those parts that were rather badly taught to us, so we could profit from it in two ways". The presentations of the students gave in a natural way rise to didactical discussions, and Freudenthal drew their attention to the recent published work of Ms. Ehrenfest. He lent the booklet for a few days to study at home to all the students who attended the didactical colloquium. (Van Hiele 1975).

The *Uebensammlung* is a collection of problems that should be posed within the framework of the propaedeutic geometry course that she already had proposed in 1924. She had experimented with such a propaedeutic course both in Russia before she came to Leiden as well as in The Netherlands, and she claims that when children followed this propaedeutic course, they performed better in the systematic course afterwards. The *Uebensammlung* consists of two parts: an introduction and the collection of problems. In the introduction she

[3] Her brochure elicited a firm reply from E.J. Dijksterhuis, who defended the deductive approach right from the start. Their discussion gave the impulse to the birth in 1925 of the magazine *Euclides*, now the still existing magazine of the Dutch Mathematics Teachers Association.

repeats some aspects of her 1924 brochure and explains how the collection of problems should be used. She remarks that many of the assignments should be connected with other school activities, such as drawing, making models, using toys, tools, clothes, or visiting factories and making outdoor excursions. The whole booklet breathes an atmosphere that is completely the opposite of the traditional geometry lessons of those days.

As an example the assignments 30 and 31 are shown below. (translated from the German original)

> 30. Someone walks along the edge of a quadrangular square, departing from the middle of one of the sides. Which angle did he pass through when he arrives at his starting point? The same question for a triangular, a pentagonal and a round square. The same question when he describes a shape in the form of the number 8.
>
> 31. The pupil should, on his way from home to school, on a piece of cardboard draw all angles he passes through on each crossing point he passes, by which he must determine which angle the front of his house makes with the front of the school. Let him control his result by means of a map when the streets are not straight ones, and draw his attention to possible mistakes.

E.W.A. de Moor has pointed to the connection between Ms. Ehrenfest and the Russian mathematics educator Semen Il'ich Sjochor'-Trotskij, whose courses she followed in St.

Petersburg (De Moor 1999, pp. 271-274). There is no doubt some affinity between the ideas of Sjochor'-Trotskij and Ms. Ehrenfest, but De Moor also underlines that the problems in the *Uebensammlung* are not only much more realistic, but also much more creative and original that those found with Sjocher'-Trotskij.

Freudenthal calls the *Uebensammlung* a "masterpiece", on condition that you do not use it as merely a propaedeutic step on the way to a systematics course. (Freudenthal 1987) He adds that he only slowly began to see the extra-value of these problems; it was in fact when he began to see the outlines of his *realistic mathematics education.* In the seventies, the former I.O.W.O., now the Freudenthal Institute developed a new curriculum for teaching geometry in primary schools. This curriculum had remarkably much in common with Ms. Ehrenfest's view on the intuitive introduction to geometry and its content had also a striking resemblance to several activities described in her *Uebensammlung.* (De Moor, 1999, p. 686).

In his letter to Howson of 1983, Freudenthal writes that he regrets that he mentioned Tatyana Ehrenfest as late as on page 405 of his *Mathematics as an Educational Task,* where he tells about her *Uebensammlung.* He should have mentioned her, he wrote, already in the preface, among those who influenced him. Now this is a curious mistake by Freudenthal, since he mentions her not as late as on page 405, but already on the pages 119-120. What he remarks there, makes clear that her influence was even more fundamental. One of the elements that Freudenthal underlines again and again is that the teaching of mathematics should be done by the method of what he called *guided re-invention.* In *Mathematics as an Educational Task* a complete chapter is devoted to that method. In page 120 Freudenthal writes the following illuminating phrase: "It [that is the method of re-invention] dawned upon me when I was studying the work of T. Ehrenfest and her disciples, both in their classrooms and in discussions with them". Visiting their classrooms and having discussions with them will have taken place after the war, when Freudenthal participated in the Mathematics Working Group. But "studying the work of T. Ehrenfest" refers certainly also to the reading of her *Uebensammlung*, since he mentioned that work on the page before as a "beautiful older example" of an analysis of active geometry . When Freudenthal read the *Uebensammlung* in the early thirties, he certainly did not fully realize its importance, as he writes himself. But, to paraphrase his biography, the seeds were sown.

8 Conclusion

When Freudenthal came from Berlin to Amsterdam, he experienced what we would call now a culture shock. The mathematics department in Amsterdam was small and teaching was old fashioned. Brouwer was a man not easy to work with, as "almost as approachable as a minefield", as somebody wrote. The mathematical scientific climate was not in any way comparable with that in Germany. According to Freudenthal, it was only his cooperation with Brouwer's other assistant, Witold Hurewicz, that helped him to survive scientifically. (Freudenthal 1987a, p. 117)

But it is also possible to draw a more positive picture. The teaching of mathematics in the schools and universities of the Netherlands in those years might have been rather backward, things started to change. Brouwer's ideas about the foundations of mathematics helped Freudenthal to form his own educational interpretation of mathematics. He met people like Gerrit Mannoury and David van Dantzig, men with unorthodox ideas about the teaching of mathematics with whom he could have inspiring discussions. He became acquainted with the work of Ms. Ehrenfest, who had not only unorthodox ideas, but also showed him a way how these ideas could be put into practice. He could organize didactical colloquia, that were attended by students as Pierre van Hiele and Dina Geldof, who married later on and became after the war his first PhD students in mathematics education. Pierre van Hieles theoretical work, known as the Van Hiele level theory, and Dina Geldofs practical work influenced Freudenthal deeply. Their contacts, already laid before the war proved to be very fruitful after the war.

The thirties may have been difficult years for the young Freudenthal, but they were certainly not barren or fruitless. Concerning mathematics, he established his reputation, concerning mathematics education; he could build slowly on his framework. The seeds sown in those years would bear fruit later, in his second career as a mathematics educator.

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