HISTORICAL SOURCES IN THE CLASSROOM AND THEIR EDUCATIONAL EFFECTS

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ABSTRACT

Let History into the Mathematics Classroom is an upcoming book by the French *Commission inter-IREM Histoire et Épistémologie*. Starting from an example from this book, we will first endeavour to identify specific features of the output of this community of practice. On this basis, we will formulate a list of research questions that are not specific to the HPM community, so as to foster collaboration with mathematics education researchers working in other fields. In the second part of the paper, we will report on a recent experiment in primary school which sheds light on the extent to which meta-tasks – a notion introduced in the first part – can be successfully entrusted to young students.

1 *Let History into the Mathematics Classroom*, and reflections thereupon

Since the 1970s, in France, the *Commission inter-IREM Histoire et Épistémologie* (CiIHE) has been producing resources for teaching and teacher-training, through publications, conferences and in-service teacher-training sessions. A new book will soon be available – under the title *Let History into the Mathematics Classroom* – which reports in details on ten experiments of using historical sources in the classroom, at all levels of secondary education (Barbin, forthcoming)[1]. After presenting one of them, we will use this new book to endeavour to identify specific features of what the CiIHE[2] does, and mention some of the things it does not usually do. This will enable us to formulate a list of research questions that we feel are central for the HPM community, yet not completely specific to it, hopefully paving the way for more intense collaboration with researchers in other fields of mathematics education.

1.1 An example: When Leibniz plays dice

In the early 2000s, I came across a paper in *Historia Mathematica* on hitherto unpublished work of Leibniz bearing on games of chance (Mora-Charles, 1992). I was struck by the combination of rich conceptual content and low technical demands (hardly any prerequisites, elementary calculations). I decided to use this text for an introductory session on probability theory, for students of age 15-16.

At that time, in France, students of this age had no prior knowledge of probability theory, but were familiar with basic notions in descriptive statistics: frequencies, relative frequencies, the arithmetic mean and some of its basic properties (linearity; the fact that it can be worked out using relative frequencies only). Before the session, students were asked to

[1] Since the book is not available yet (this paper being written in April 2016), the references will not include page numbers.
[2] This endeavour is descriptive and not normative. Needless to say, many of these features are common to several groups in the HPM community.

look up a few words in a dictionary: heuristic, *a priori*, *a posterior*, empirical, "*aléatoire*" (random), and "*hasard*" (chance). For "aléatoire" and "hasard" they had to look up the etymological origin of the words (namely the word "*dé*"/die, in Latin and Arabic, respectively).

The format I chose for the session is rather unusual: 1½ hours devoted to explaining the content of a mathematical text; guidance was not provided by a worksheet; the students were given the original source, and the teacher engaged in a dialogue with the whole class about the text, along a list a carefully chosen questions and requests.

The text is taken from a rather long letter, written in French, from which I selected the following extract:

> But in order to make this matter more intelligible, I first of all say that appearance [probability] can be estimated, and even that it can be sold or bought.
>
> (…) Let us take an example. Two people are playing at dice[3]: one will win if he scores eight points again, the other if he gets five. It is a question of knowing which of the two it would be best to bet on. I say that it should be the one who needs eight points, and even that his advantage compared with the hope that the other must have, is three to two. That is to say that I could bet three écus to two for the one who needs eight points against the other without doing myself any harm. And if I bet one against one, I have a great advantage. It is true that notwithstanding the chance that I might lose; especially since the chance of losing is like two and that of winning is like three. But as time goes by, observing these rules of chance, and playing or betting often, it is constant that at the end, I will have won rather than lost.
>
> But to show that there is a greater probability for the player needing eight points, here is a demonstration. I suppose that they are playing with two dice, and that the two dice are well made, without any cheating. This being the case, it is clear that there are only two ways to reach five points; one is 1 and 4, the other 2 and 3. However there are three ways to score eight points, i.e. 2 and 6, 3 and 5 and also 4 and 4. Now each of these ways has in itself as much probability as the other as, for example, there is no reason why it cannot be said that there is more probability of getting 1 and 4 than 3 and 5. Consequently, there are as many probabilities (equal amongst themselves) as there are of ways. So if five points can only be made in two ways, but eight points can be made in three ways, it is clear that there are two chances of getting five and three chances of getting eight.
>
> (…) That being the case, it is obvious that the estimate I have just made is the one to follow. That is to say that this fundamental maxim will be the case:
>
> The chance or probability of outcome A keeps the same proportion to the chance or probability of outcome B as the number of all the ways capable of producing outcome A has proportionally to all the ways of producing outcome B, supposing all these ways are equally doable. (Chorlay, forthcoming)[4]

[3] With two dice.
[4] Many thanks to Peter Ransom for the translation.

A detailed account of the session is available in (Chorlay, forthcoming). Let us just summarize the main points:

- What does the following ratio mean: the advantage of the one who bets on a sum of eight "compared with the hope that the other must have, is three to two"?
    - If the two players bet on the results, this ratio enables them to decide whether the wages are fair or not: if you win 1 écu three times out of five (relative frequency 60%), and lose 1 écu two times out of five (relative frequency 40%), the mean win is 1 x 0.6 + (-1) x 0.4 = 0.2 écus; hence the game is structurally favourable to the player who bets on a sum of 8. Another bet would be fair for the two players: If you win 2 écus for a sum of 8 and lose 3 écus on a sum of 5, the mean win is 2 x 0.6 + (-3) x 0.4 = 0 écus.
    - More generally, this ratio enables the players to make rational decisions (here: to engage in a game of chance on terms which are not structurally detrimental to them). The ratio has no dimension, but it can be used to determine values (here in écus). Since the 17^{th} century, the rational pricing of random events has become fundamental in the banking and insurance business: Leibniz was quite right when he said that "appearance (...) can be sold or bought"!
- How does Leibniz determine the ratio 3/2 (in favour of a sum of eight), or the ratio 60% / 40% ? He explains his method in the paragraph starting with "But to show that (...)".
    - He proceeds by pure reasoning and not by observation; hence, strictly speaking, the values 60% and 40% are not relative frequencies of the kind used in descriptive statistics: descriptive statistics relies on surveys (empirical data) and works out parameters *a posteriori*. Leibniz relies on pure reasoning, the values are determined prior to any observations, that is *a priori*. In this context, these values are called probabilities and not relative frequencies; in this context, the mean is called the expected value.
    - To determine the specific values, he works out "all the ways" of getting a sum of 8 (3 ways) and a sum of 5 (2 ways). When dealing with a random experiment, we call these "ways" outcomes. Leibniz states very clearly that this reasoning makes sense only if the dice are fair, so that "each of these ways has in itself as much probability as the other"; we shall say that the outcomes are equally likely.
- Back to the first question, with a twist: "what do these values 60%, and 40% mean? Do they enable us to say what will happen at the next throw of two fair dice?" Leibniz deals with this quite clearly "It is true that notwithstanding the chance that I might lose; especially since the chance of losing is like two and that of winning is like three. But as time goes by, observing these rules of chance, and playing or betting often, it is constant that at the end, I will have won rather than lost." In this passage, Leibniz connects the probabilities (which are *a priori*, theoretical values) to relative frequencies: on a large sample of the same random experiment, the empirical relative frequencies should provide reasonable approximations of the probabilities; for these

approximations to become exact, one would have to repeat the experiment an infinite number of times (betting "often" until the "end of time"!). This statement is an informal version of an important mathematical theorem, called the law of large numbers.

- "Does the law of large number provide a means to check that the values which Leibniz determined by pure reasoning give a correct quantitative description of the random experiment?" Sure, we can use a spreadsheet or write a simulation program (taking in to accounts sums of 8 or 5 only).
    - 10 simulations of 100 repeats each show a large dispersion of the relative frequencies of sum 8. It's hard to say anything.
    - Working out a dispersion parameter (the spread, the interquartile range, or the standard deviation) with 10 simulations of 1000 repeats, then 10 of 10 000 repeats, then 10 of 100 000 repeats seems to confirm our intuitive understanding of the law of large numbers: dispersion becomes smaller as the number of repeats tends to infinity; hence the empirical relative frequencies should provide ever more accurate values of the probabilities.
    - However, something seems to be wrong: according to Leibniz, the probability of sum 8 is 0.6; but in the simulations, the estimated probability stabilizes between 0.55 and 0.56. Maybe Leibniz's reasoning is wrong, or the informal version of the law of large numbers is deceiving ...
    - There is a way out of this predicament, which is compatible with both the "fundamental maxim" (which provides a means to work out probabilities by "counting ways" when the outcomes are equally likely) and the law of large numbers. If the two dice were of different colours, or if one die was thrown twice, we would be able to distinguish between nine different and equally likely outcomes. Hence the following table is incorrect:

Case	$2+6=8$	$3+5=8$	$4+4=8$	$1+4=5$	$2+3=5$
Probability	0.2	0.2	0.2	0.2	0.2

It should be replaced by one of the two tables:

Total of 8	**Total of 5**
2 then 6	1 then 4
3 then 5	2 then 3
4 then 4	3 then 2
5 then 3	4 then 1
6 then 2	
5 ways	***4 ways***
Probability 5/9	***Probability 4/9***

Case	$2+6=8$	$3+5=8$	$4+4=8$	$1+4=5$	$2+3=5$
Probability	2/9	2/9	1/9	2/9	2/9

> The same random experiment has two models which account for the empirical relative frequencies. In the first model the 9 outcomes are equally likely, hence the probabilities can be worked out by the easy formula $\frac{number\ of\ succesful\ outcomes}{number\ of\ possible\ outcomes}$. For instance, in this random experiment, the probably of getting at least one "4" is 3/9 (three out of nine). In the second model, there are 5 unequally likely outcomes. Probabilities can be worked out using another (easy) reasoning; for instance, the probability of getting at least one "4" is 1/9 + 2/9. Both methods lead to the same values; the first one is a special case of the second one.

The teaching goals for this session are quite clear: the curriculum requires that students this age become acquainted withe the notions of "probability", "probability distribution", "expected value", "equally likely outcomes", "fair bet"; it requires they be able to determine the probability distribution on finite sample spaces, in simple cases; that they be able to work out probabilities on the basis of a probability distribution, whether the outcomes are equally likely or not. The curriculum also suggests that more epistemological aspects be tackled, such as the connection between descriptive statistics and probability theory, using an informal statement of the law of large numbers; also that, on some occasions, students come accross a multiplicy of random models for the same real-life experiment. Moreover, it is required that students use ICT in all parts of the curriculum where it can help them make sense of the maths.

From an HPM viewpoint, this session is typically on the "HM as a tool" side, as opposed to "HM as a goal". A *history* of maths course on the emergence of probability theory would probably dwell on the works of Galieo, Fermat, Pascal, Huygens; would discuss the problem of points, Pascal's wager on the expected benefits of a christian life etc. Of course it is not historically uninteresting to see that Leibniz wrote several texts on probability theory, but investigating this fact is interesting only for professional historians (of mathematics, or of philosophy). Moreover, this short passage does not say much about Leibniz's interest in probability theory; and as far as the history of probability theory is concerned, it is hardly worth a footnote. To put it in a nutshell: its potential didactical value is not directly correlated to its historical import[5].

1.2 An example of what?

On the whole, many features of this lesson plan are common to those presented in the upcoming book *Let History into the Mathematics Classroom*, which reflects the fact that the French *Commission inter-IREM Histoire et Epistémologie* (CiIHE) is not only a network but also a community of practice. Let us list some of these features:

[5] De Vittori's analysis of standard non-standard teaching sessions, namely those relying of historical sources, fully applies here: « *l'usage de l'histoire des mathématiques dans le contexte scolaire de l'exercice justifie l'extraction d'une partie épistémologique de l'histoire non dans le cadre d'une pratique historienne, mais afin d'élaborer une nouvelle forme de pédagogie* . » (de Vittori, 2015, p.19). Meaning: The use of history of mathematics in the context of a school exercise justifies the extraction of some epistemological part of history; here, the framework is not that of the practice of historians, but the elaboration of a new form of pedagogy.

1. The starting point for the design of a lesson plan is usually the recognition of the fact that some local teaching need is somewhat echoed in a specific historical source. The teaching is often a syllabus requirement: modelling basic random experients in high-school, studying angles in middle-school (Guichard, forthcoming), solving problems where proportionnality holds (Morice-Singh, forthcoming) etc.
2. The background knowledge of the designer is usually historical, and the background know-how usually comes from teaching experience. Results or concepts from the didactics of mathematics are seldom mentioned.
3. Designing such a lesson plan does not require a comprehensive knowledge of the history of mathematics.
4. The chapters are explicit when it comes to background knowledge (mathematical, historical, epistemological) and motivation (including intended educational effects). They are sometimes quite explicit as to the tasks entrusted to the students. As a rule, very little is said about actual student activity, and observable educational effects.
5. The range of task is wide and has some rather specific features. Let us distinguish between two types of task:
    a. Text-reading tasks.
    b. Meta-tasks (M-tasks). I am not in a position to provide a clearcut definition for this class of tasks. To delineate it, suffice it to say that (1) they differ from directly transformative tasks (draw ..., work out ..., factorize ..., solve equation ...); transformative tasks for which the didactical system usually provides techniques and a technology – to use ATD terminology (Chevallard, 2007, 133); (2) they include: justifying, comparing, assessing, criticizing, summarizing; proving and reformulating/translating/rewriting can be listed among them, in cases when no standard background techniques are provided by the didactical system. The book has many examples of these task requests: "In each case there were issues around the understanding of the texts and checking the proofs, even of completing or adding to them." (Guyot, on inscribing a square in a triangle); "Summarize the solution and explain the method. (…) Is it true or false? (…) The translator made a mistake in the problem. What is the incorrect word and what word should replace it? (...) Transcribe the problem in French everyone can understand (take care with the wording). Represent the situation with a simple diagram" (Métin, on false position methods).
6. As far as timing is concerned, we're talking medium range: beyond 10-min exercises, yet not complete chapter designs. Several chapters of the book deal with chapter design (Guichard on angles for 11-12 year-olds), or provide series of exercises on one topic to be used as a guiding thread in a given chapter (divisions of triangles in basic plane geometry (Moyon), proportionnality (Morice-Singh), graphical methods for differential equations (Tournès, chap. 7)). Even in these cases, the activities based on history are fully compatible with curriculum requirements, and interwoven with standard texbook exercises. *Enrichment* rather than *reconstruction*; from a cognitive

viewpoint, cognitive flexibility is aimed for, with no hidden background recapitulationist model of the “ontogeny recapitulates phylogeny” type.

7. The tasks that are entrusted to the students are usually rather demanding, even difficult, which means two things: first, students say they are taxing. Second, the session designer is fully aware of this fact, and is ready to live with it (which means he/she is comfortable occasionally asking a lot from students, and running the risk of complete failure). Occasionally he/she embraces it: “I realized that the mathematical content was both accessible for 16 year-old students and sufficiently ‘hidden’ to require interpretation” (Métin). In this respect, the Leibniz session is not fully representative, at least as far as the strictly mathematical content is concerned.
8. The chapters are meant to be used as ressources for other teachers. Consequently, one has to distinguish between two intended audiences: on the one hand, the intended audience of the teaching sessions is that of secondary school students; on the other hand, the intended audience of the book, and more generally of the output of the CiIHE, is an audience of teachers and teacher-trainers.

1.3 Possible research questions

Reflecting on this list of features enables us to to elicit a range of research questions, only one of which will be touched upon in the second part of the paper.

A first series of questions concerns the first intended audience, that of students, and the (still tentative) notion of M-task:

- Can (and should) a more clearcut caracterization of this class of tasks be given ? Some features can be mentioned, to complement the first elements brought up above (5b).We chose to call them meta-tasks, since students are required to do something *with reference to*, or *about* a piece of mathematics; something which is not limited to acting *on* or *within* the mathematics involved (as is the case in “factorize (...) then solve equation (...)”). To successfully perform the task requires the identification of the relevant mathematical background knowledge, and this knowledge may involve pretty diverse and distant elements (e.g. write a computer simulation to test Leibniz’ forecast as to a game of dice); beyond background mathematical knowledge, other abilities are required, in particular when it comes to reading and writing mathematical – and, more generally, argumentative – texts. Specifying this class of tasks would help to identify:
    - The expected educational effects. In particular, is the main expected effect the passage from the “mobilizable knowledge” to “available knowledge”[6]? Another possibility was investigated by Tinne Kjeldsen, that of fostering “commognitive” conflicts (Kjeldsen, 2012).

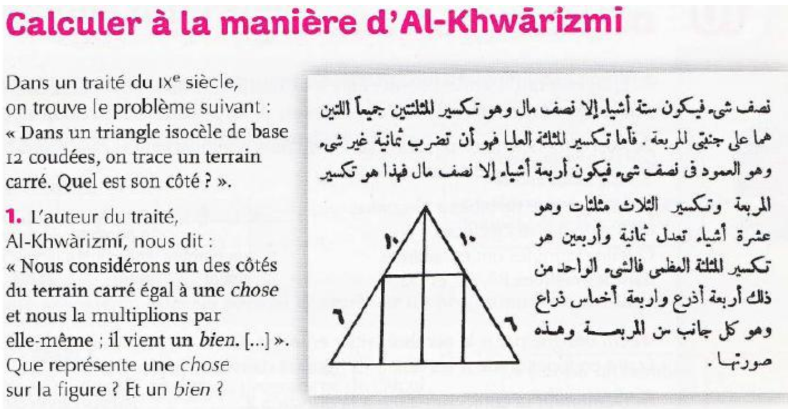
[6] This distinction has become classical in the French didactical community, on the basis of Aline Robert’s work (Robert, 1998). Some knowledge (or know-how) is mobilizable if students can apply it reasonably successfully upon request (e.g. “Use Pythagoras’ rule to work out the length of side AC”); it has become available when students are able to identify it as the relevant tool even when no indications are given.

- The conditions for such tasks to actually trigger the expected student activity.

- Many of the characteristics of M-tasks apply to “proving”, and the relation between this class and this specific task raises several questions: is the task “proving” always an M-taks, or does it depend on conditions that need to be identified? If mastering “proving” in some mathematical contexts is a major goal of mathematics education, does the inclusion of “proving” in the larger class of M-tasks help to reach this goal? Can M-tasks other than “proving” be conducive to “proving”, and in what way?
- Are M-tasks specific to the “historical sources in the classroom” context? My contention is that they are not[7], but that using a historical document as teaching (raw-)material on the one hand, and entrusting students with M-tasks on the other hand, enjoy a special connection. It would be interesting to investigate the connections between M-tasks and other protocols, such as: “True/False. Justify”, and “True/False/No way to know. Justify” questionnaires; scientific debates (Legrand, 1996); open problems for the classroom (in French: *problèmes ouverts*).
- In M-tasks, the epistemological position of students with respect to the mathematical document is not standard: students are to *investigate* an artefact (usually a mathematical text) that is a non-standard object in the maths-class and is somewhat hard to access; they are to reformulate, and assess some mathematical content on the basis of the maths they have learnt, thus acting as *experts* endowed with background knowledge. Does this imply that a specific didactical contract should be listed among the conditions for success?

A second series of questions concerns the second intended audience, that of teachers and teacher-trainers.

The most common teaching aids are textbooks, and several studies have been carried out on the way history of mathematics appears – or not – in them. On this basis, it is worth mentioning a rather unsual example. A textbook for students in the third year of middle-school featured the following exercise (fig.1), in its chapter on algebraic calculation:


Calculer à la manière d'Al-Khwārizmi

Dans un traité du IX^{e} siècle, on trouve le problème suivant : « Dans un triangle isocèle de base 12 coudées, on trace un terrain carré. Quel est son côté ? ».

1. L'auteur du traité, Al-Khwàrizmí, nous dit : « Nous considérons un des côtés du terrain carré égal à une *chose* et nous la multiplions par elle-même ; il vient un *bien*. [...] ». Que représente une *chose* sur la figure ? Et un *bien* ?

نصف شيء فيكون ستة أشياء إلا نصف مال وهو تكسير المثلثين جميعاً اللتين
هما على جنبتي المربعة . فأما تكسير المثلثة العليا فهو أن تضرب ثمانية غير شيء
وهو العمود في نصف شيء فيكون أربعة أشياء إلا نصف مال فهذا هو تكسير
المربعة وتكسير الثلاث مثلثات وهو
عشرة أشياء تعدل ثمانية وأربعين هو
تكسير المثلثة العظمى فالشيء الواحد من
ذلك أربعة أذرع واربعة أخماس ذراع
وهو كل جانب من المربعـــة وهـذه
صورتها .

[7] In this respect, we do not share de Vittori’s view (de Vittori, 2015).

2. Al-Khwàrizmí nous donne ensuite le calcul suivant :
« Quant aux deux triangles qui sont sur les flancs [...] leur aire est que tu multiplies une *chose* par six moins un demi d'une *chose*, il vient six *choses* moins la moitié d'un *bien*. »
Expliquer ce calcul.

3. Avec nos notations actuelles, si on note ℓ une *chose*, comment s'écrit un *bien* ?
Écrire le calcul précédent avec ces notations.

EXPOSÉ Qui était Al-Khwàrizmí ? Pourquoi son nom est-il important en mathématiques ?

Figure 1. Calculating in the manner of Al-Khwārizmi[8]

A few pages down, a second exercise was based on the same document; students were asked to solve the problem and "check Al-Khwārizmi's solution". This problem from the book of *al-jabr* is well-known in the CiIHE, and bears witness to the active role of Professor Ahmed Djebbar (Djebbar, 2005). For instance, an account of its use with students in vocational high-school can be found in the *Let History in the Mathematics Classroom* book (Guyot). It is however quite unusual to find such exercises in textbooks, in which parts of the original text are given, and typical text-related M-tasks are entrusted to students.

Beyond the question of tasks[9], one could investigate whether or not teachers actually use this exercise, and (if so) how? To use a metaphor from economics, is there any demand for this supply? I don't have a clue as to the "how" part of the question; however, in 2014, two in-service teachers investigated the reception of this exercise by maths-teachers, as part of a project which I supervised[10]. This project is on a much smaller scale than Professor Siu's (Siu, 2006), with a sample of 30 middle-school maths teachers; but it was focused on the reactions of the teachers to this specific exercise, on the background of their general stated view and practice of history of maths in teaching[11].

In this survey, 90% of the sample say they use the history of mathematics in their teaching: occasionally (80%) or on a regular basis (10%). In most cases (83%) this historical perspective boils down to mentioning names of famous mathematicians or well-known anecdotes. Still, 49% say they sometimes introduce new mathematical concepts or methods using history, and 41% say they occasionally ask their students to solve historical problem.

Studying their reactions to the specific exercise (fig.1) helps us to go beyond this pretty standard distribution of statements. Only 12% of the sample say they occasionnaly use original sources. 50% say they would use this exercise with their students. For those who say

[8] (Chesné, & Le Yaouanq, 2011, p.85).
[9] This exercise is commented upon in (de Vittori, 2015).
[10] Banakas P, Kerboul C. (2015) *Deux problèmes d'arpentage et de transaction commerciale en algèbre. Quelle appropriation par les enseignants ? Quels usages en classe ?* (unpublished Master project). This project was carried out in the context of the course on « history of maths and the sciences in teaching and teacher-training », in the Master of didactics of Paris-Diderot University. From a methodological point of view, this project does not meet the requirements of a scientific work; in particular, no information is given on sample selection and possible biases.
[11] Only statements were studied, using questionnaires. Studying actual practice would be a pretty challenging endeavour.

they would not, lack of sufficient historical background information is generally not the main argument – although it is mentioned by 30 % of them. They find the exercise too difficult and time consuming; it involves too much reading and writing. Of course, it could sound encouraging to see that half of the sample say they would use this exercise; in this case, they were also asked to choose a protocol (they could tick several boxes). The results were: 2 teachers answered "in the classroom, students working individually", 11 answered "in the classroom, in groupwork", and 7 answered "as a homework project" (*devoir-maison*).

Although the sample is too small (and probably biased) for any conclusions to be based upon this survey, it suggests at least two avenues for research:

- If we think that this type of exercises is educationally valuable – and it seems the CiIHE does, since it is very close to what they've been supplying for years – how to make it more attractive to teachers? My interpretation of the data is that a majority of them would avoid using it, since I interpret the "as a homework-project" answer as a mere show of good-will[12]. The CiIHE books and brochures usually provide background historical information, but it may not be what teachers feel they need most. Our hypothesis is that teachers do identify M-tasks, which is why they are reluctant to use this exercise: either because they do not see the expected educational benefits; or because they would not feel comfortable/competent surpervising and assessing student work, in particular because what constitutes a "correct" answer is somewhat less easy to identify than in more standard cases.
- The fact that M-tasks are not specific to sessions based on historical sources suggests another lead. Our hypothesis is that some teachers *are* willing to occasionally engage in classroom work which *is* of an unusual nature, *does* take time, proves really *demanding* for students, and has mainly long-term expected educational effects; and some are not. To test the existence of this *venturesome-teacher profile*, one could investigate the correlation between the involvement in several teaching protocols which are M-task-rich, in particular open-ended problems (*problèmes ouverts*); maybe modelling problems as well.

In the second part of this paper, we will report on an experiment recently carried out at the primary level. The emphasis will be on a feature which is often left implicit in the publications of the CiIHE: the actual activity of students. Besides, it should contribute to establishing that, under conditions which remain to be investigated, genuine M-tasks can be entrusted to students even at the primary level.

[12] This interpretation needs grounding, of course.

2 Students' activity in an M-task-oriented teaching sequence in primary school.

2.1 Algorithmic tasks in primary school

In the fall of 2015, an experiment was carried out in four classes in the final year (or final two years) of primary school, in France (Chorlay, Masselin, & Mailloux, submitted). The original motivation for the focus on algorithm, however, lies outside of the primary school context. It lies – first – in current historical work on ancient mathematics carried out, in particular, in the SPHERE[13] research group; second, in what we feel was a fruitful introduction, that of a focus on algorithmic thinking in the current French high-school maths curriculum[14]. I must say that, as a historian and a teacher-trainer, my own view of algorithms and algorithmic thinking was significantly enriched by this unexpected partial overlap. In particular, it occurred to me that meaningful reflexive tasks can be carried out in an algorithmic context, *beyond* the two classical justificatory tasks: to prove that an algorithm is correct (relative to an *a priori* set goal); to prove that it terminates (when termination depends on some condition). In particular, two challenges can lead to M-tasks that are not directly of a justificatory nature. First, since a given algorithm has to be able to process a large class of input values, its formulation requires that some indeterminate objects be denoted and handled. Second, some pragmatic properties of an algorithm can be studied: surveyability, user-friendliness, time-and-labour cost etc.; the investigation of these properties comes naturally to mind when two algorithms performing the same function are to be compared by a user.

For primary school children we[15] selected a context which is pretty familiar to them, that of multiplication of two whole numbers; and an algorithm that is not, that of the *per gelosia* technique. Multiplication, however, was not our object of study; rather, we wanted to know to what extent reflexive tasks bearing on algorithms could be entrusted to 8-to-10-year-old students. Three 1-hour classroom sessions were designed and implemented: in the first session, the two tasks were: to perform/emulate an algorithm, and to formulate hypotheses as to its function; in the second session, the task was to write an algorithmic texts; in third session, the task was to compare the *per gelosia* technique with the one that is familiar to French students.

Before we describe the three sessions in more details, and provide elements as to what students managed to do (or did not), let us say that the whole project was of a rather exploratory nature. Since the current curriculum sets no goals as to reflexive tasks on algorithms in primary education, complete failure would have been harmless to the students – but not to our egos! Hence we felt comfortable putting them in really challenging situations,

[13] The History and Philosophy of mathematics and the Sciences research group, affiliated with the CNRS and Paris-Diderot University (UMR 7219).
[14] A third context must be mentioned, namely the in-service 3-day teacher training programme that the Paris IREM history-of-maths group designed and implemented in 2014-2015 and 2015-2016.
[15] The experiment was designed and implemented by Renaud Chorlay, Blandine Masselin and François Mailloux. In this section, the « we » denotes us three.

providing as little mathematical support as we could. The general design of the three sessions was: once the initial document was given and the task(s) spelled out, the students worked in groups of 4 for 20 to 30 minutes; the groups were eventually asked to present their work to the class under the supervision of the teacher. Our emphasis on autonomy accounts for the fact that we did not choose to work on the justification of the technique. We feel it would be very interesting at this level (and beyond), but probably with a lot of help from the teacher.

2.2 Session #1

The first session was the only one involving historical documents. The children were told that a chest had been found, which held four documents (fig. 2). The first question was: "Can you guess what these are, and why they were together?"

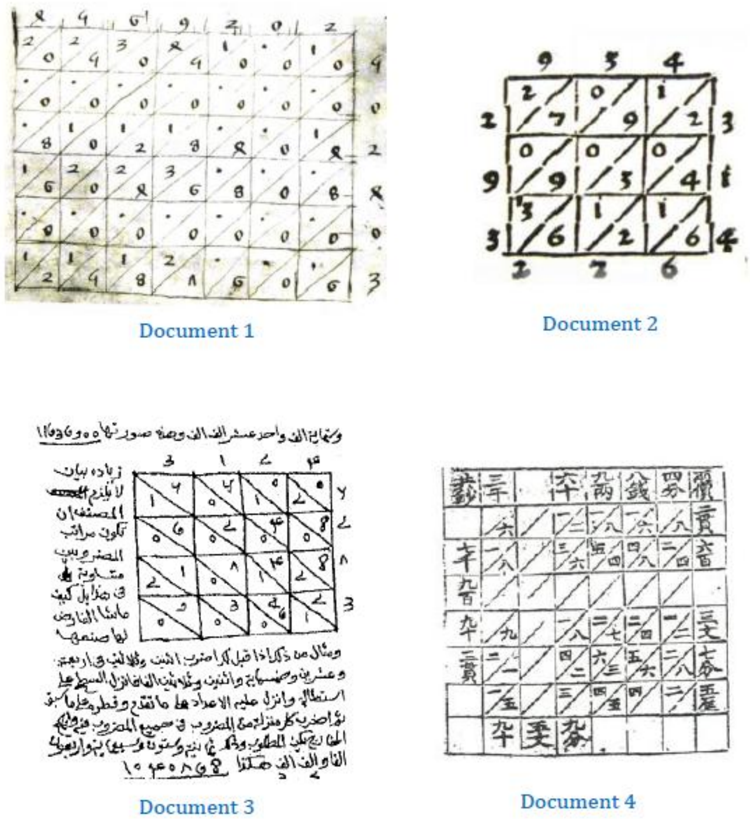


Figure 2. The documents found in the chest[16].

The students turned out to be keen observers when it came to spotting similarities and dissimilarities among the four documents. However, none surmised that they had anything to

[16] Documents 1, 2, and 4 are taken from (Chabert, 1999, p.30-32); Document 3 from (Abdeljaouad, 2005, p.61). Document 1: Latin manuscript *Tractatus de minutis philosophicis et vulgaribus* (Oxford, Bodleian Library, circa 1300). Document 2: *Treviso Arithmetic* (1478). Document 3: Fac-simile of the $Sharh\ al - talkh\bar{\imath}s$ d'$Al - Qalas\bar{a}d\bar{\imath}$ (manuscrit du $15^{\text{ème}}$ siècle, Bibliothèque de Gotha (Allemagne)). Document 4: *Jiuzhang suanfa bilei daquan*, China, 1450.

do with multiplication, or even calculation. The most common hypothesis was that these were some sort of number games; Sudoku-style. This is what we had anticipated: without the chronology of operations, and the distinction it displays between input values, intermediate steps, and output value, no calculation technique is readily identified. This is why the chest held another document – of a less historical nature however! One of us had been taped while working out the product 93x52 on a blackboard by the *gelosia* method. The movie was silent. After showing it twice (to a mesmerized audience) and freezing on the initial picture (with numbers 93 and 52 written horizontally and vertically, respectively), we asked students to reproduce on paper what they had seen; then to explain what it was all about.

The first task created a double challenge: to redraw the geometric diagram, then to recover the numbers written in it. The first challenge turned out to be really difficult, as shown in figure 3.

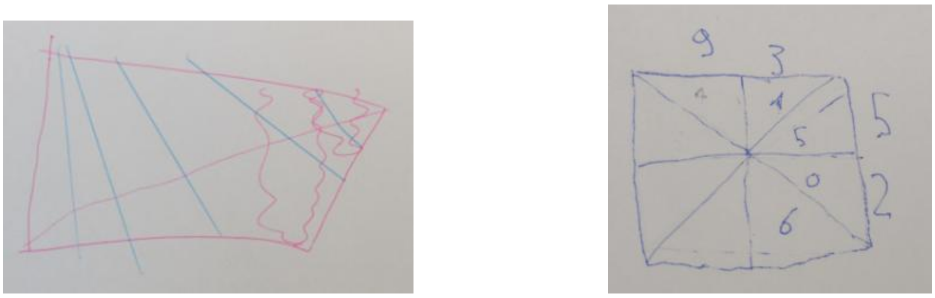


Figure 3. Incorrect diagrams.

The second challenge was quite demanding too. Some students tried to remember the numbers they had seen in the video, but it was clearly impossible to remember them all. Working in groups of four helped, since some remembered some of the numbers, and some were willing to try out ways of finding them through calculation. Eventually, most groups could complete the grid through multiplications of 1-digit numbers followed by additions. However, none spotted the structural similarity with the written technique for multiplication to which they are accustomed. To find out "what it was all about", many groups tried adding, then subtracting, then multiplying 93 and 52. The fact that 93x52 is, indeed, equal to 4 826 convinced everyone that this was a multiplication technique. Not one student asked for any kind of justification; indeed, the scenario had not been designed to trigger the need for justification.

The teacher acknowledged the fact that this is a multiplication technique, introduced the words "input values", "output value", and reminded the students of the words "factors", "product", "units digit", and "tens digit". At the end of the first session, the students were asked to carry out a few multiplications using this new technique: 23x17, 32x25, 625x8, 625x32. The last two products created a new challenge: until then, the diagrams had been 2x2 "square" grids. A new feature came into play when the initial technique had to be adapted to numbers of various lengths. It usually took students several attempts to adjust the grid, and many used a "square it up" tactic (fig. 4).

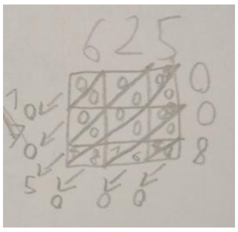


Figure 4. Square it up!

2.3 Session #2

During the next two weeks, a regular basis, students were asked to carry out multiplications either by the familiar or the by the *gelosia* technique. The second session began with a deceptively harmless request: "On paper, explain the *gelosia* method for students with no access to the video. They should be able to apply your method starting from any two whole numbers." In our analysis of the outcomes, we focused on (1) the overall correctness of the algorithm (for instance: no steps missing), (2) the semiotic means used to express the algorithm (intended for an audience with no prior knowledge of this technique), and (3) if and how these semiotic means were used to face the generality challenge ("for *any* two whole numbers"[17]).

We will not go into (1) in any details: we expected very satisfactory results, and very satisfactory they were. As to (2), the most common form was that of the "comic strip" displaying the successive states of a grid while multiplication was performed on an example assuming a generic role; in addition, arrows associated relevant parts of the successive diagrams with short textual instructions. This form has many advantages: it does capture the chronological aspect of the algorithm; it expresses the diagrammatic steps with a diagram, and the numerical operations through a written explanation of a general instruction ("multiply (...)") applied to a generic example ("(...) this by that"). For the students, it is probably reminiscent of what they are regularly exposed to, either in textbooks or when the teachers show a new calculation technique. However, relying on a single – if generic – example fails to meet all the requirements, since it gives no clue as to how to adjust the grid to the length of the input numbers. At the other end of the spectrum, two groups provided a purely textual explanation, using no diagrams and no generic examples. These are fully general description of the goal and the main steps, but the objects to which the instructions apply are not captured specifically enough for anyone to be able to perform the algorithm on two given numbers using only these guidelines.

Several groups faced the generality challenge, more or less directly, and with a variety of semiotic means. In order to denote the fact that their explanation had to apply to any numbers, some drew specific grids (displaying the variability of size) but left them blank, or wrote question marks to denote the location of indeterminate input digits (fig.5).

[17] Although this paper is no place to dwell on this topic, it is worth mentioning that the role of generality as an epistemological value, and the ways of expressing/capturing the general has been a topic of historical research in the SPHere team for over a decade. A long-term project recently led to the publication of (Chemla, Chorlay, & Rabouin, 2016).

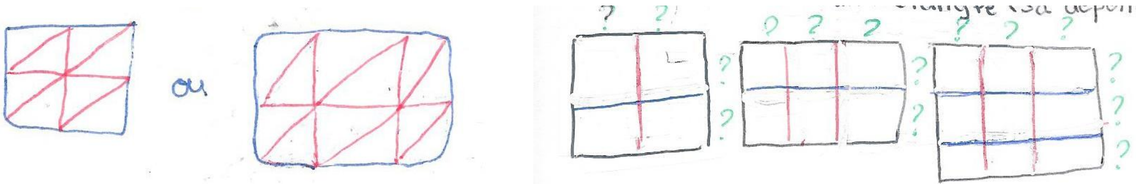


Figure 5. Grids suitable for "any" numbers

One group (fig.6) opted for a textual approach (as opposed to the comic-strip approach) but inserted two (non-generic) examples to illustrate the fact that "one draws a table which depends on the numbers".

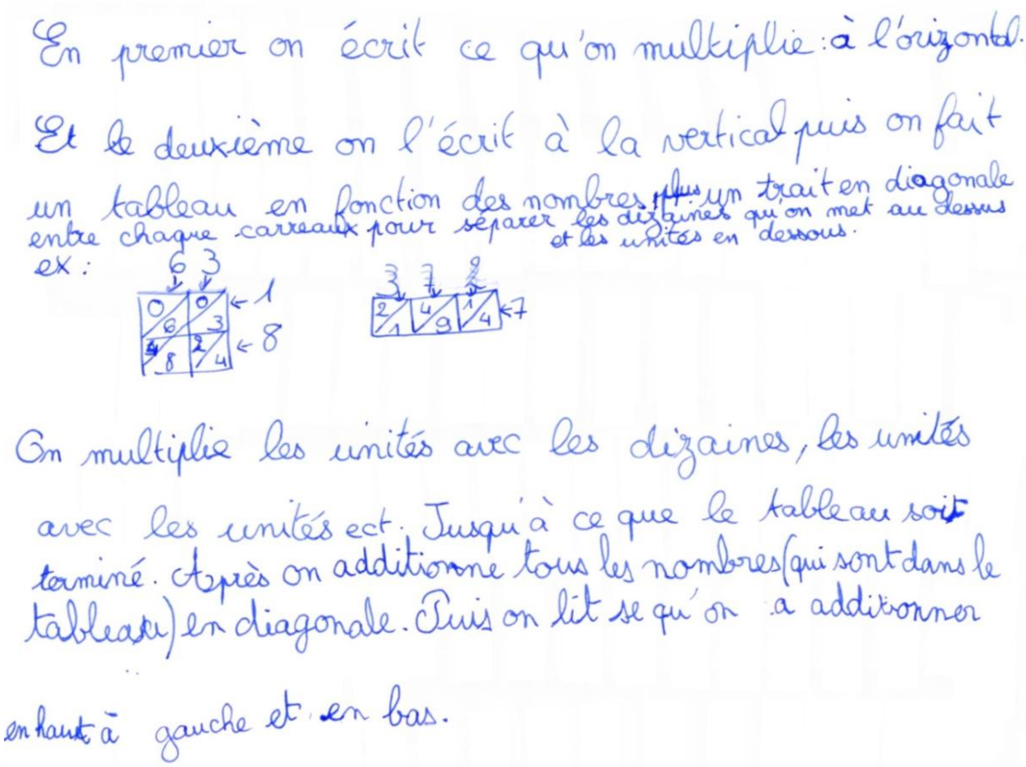


Figure 6. Moving away from generic examples.

In other groups, the step where the dimension of the grid is chosen is either hinted at ("it depends on the factors") or expressed as a semi-formal rule ("the more numbers, the more squares in the grid"). During the final part of session 2, the teacher asked more specifically for a general rule to choose the dimension of the grid. In one class, one student mentioned "the number of digits"; everyone seemed to agree.

2.4 Session #3

The third session was devoted to the comparison between the *gelosia* method and the one that has been familiar to the students for about two years: "If you were to recommend one of the two techniques for another class, which one would you choose, and why?" Again, we chose to entrust students with a set of quite complex tasks; in particular, we chose not to provide or hint at criteria for comparison, so students had to identify criteria themselves before engaging in comparison.

All groups of students came up with objective and explicit criteria – i.e. went beyond comments such as "we recommend ... because we like it better than the other". As a matter of fact, all the students but one (in four classes of about 25 students each) preferred the *gelosia* method; the fact that they had a point to make was probably a necessary condition for a real involvement in the task. Autonomous work led to the following arguments: (1) the *gelosia* method is more surveyable (once it's been performed you can easily check all the steps in the grid); (2) in the *gelosia* method, carries may appear only in the additive phase, whereas in the usual method carries may appear in the multiplicative phase, which is a standard cause for mistakes; (3) in the usual method, when the second factor is not a 1-digit number, one has to shift calculation lines to the left (usually by writing extra zeroes), which is a common cause of mistake when the shift is forgotten or faulty. It is striking that all these criteria had long ago been identified by researchers in the didactics of mathematics, in particular in the comparative study that Guy Brousseau carried out since the 1970s (Brousseau, 2007).

All these criteria are of a pragmatic (or ergonomic, as Brousseau put it) nature; they all dealt with the comparative user-friendliness of the techniques. We also wanted to see if students would come up with criteria that would be reminiscent of algorithmic complexity; that is, if the number of steps (of a similar nature) would be worth comparing, for the same two input values. No groups tackled this issue in the written sheets. However, some hinted orally at the fact that they found one of the two methods faster than the other. For these groups, the teacher prompted further investigation into this aspect: "Can you think of ways to measure which is the faster?" In various groups, two different strategies were selected. Some used watches to actually measure the time it took to work out the same product by either method. Others began counting the number of intermediate steps, which turns out to be both tricky (the steps are of different types) and tedious. However, the written outcomes sometimes display an unexpected semiotic feature (fig.7)

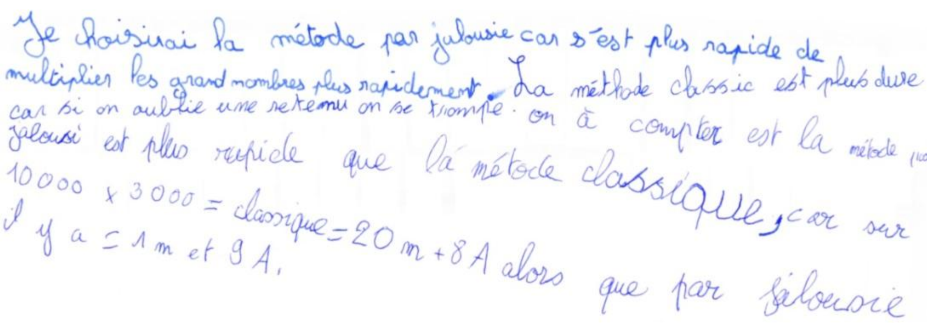

Je choisirai la métode par jalousie car s'est plus rapide de
multiplier les grand nombres plus rapidement. La méthode classic est plus dure
car si on oublie une retenu on se trompe. on à compter est la métode
jalousi est plus rapide que la métode classique, car sur
10 000 x 3 000 = classique = 20 m + 8 A alors que par jalousie
il y a = 1 m et 9 A.

Figure 7. Comparing the two techniques in terms of number of steps.

Here the students summarized their finding using a letter coding: "10 000 x 3 000 = classical = 20m + 8A" *vs* "by gelosia there are = 1m + 9A". Of course this is not symbolic algebra: the expressions are not calculated upon; numbers are not to be substituted for letters. It would be well worth investigating where this shorthand comes from[18].

Conclusion

It goes without saying that HPM – as a field of practice and research – *has* specific features, in particular when we take into account the connection with history of mathematics as a field of knowledge (Fried, 2001) (Chorlay & Hosson, 2016). However, in this paper we chose to focus on what we feel is *not* specific to the use of historical sources in the classroom, in spite of the fact that all the lesson plans or teaching sequences we presented make explicit use of such sources. Rather, we focused on tasks, intended student activity, and actual student activity. In the first part of the paper, among other things, we attempted to delineate the class of Meta-tasks (M-tasks), which is central to the HPM approach – at least as far as it is practiced in the CiIHE – yet not specific to it.

In the second part of the paper, we reported on an experiment carried out recently on a medium scale (four classes), in the final two years of primary education. Although multiplication of whole numbers was the underlying mathematical content, this is not what was at stake for us in the experiment. Rather, we wanted to investigate the extent to which typical M-tasks such as "formulating a *general* method" and "comparing two methods" could be entrusted to young students, in conditions where very little support was provided by the teachers. It turned out that the session which proved the most difficult for students was generally the first one, in which the tasks were more standard (to draw a complex diagram from memory, and identify/recover numerical operations on an example). When given sufficient time to think things through, students did remarkably well in sessions #2 and #3. A more thorough report should appear soon in (Chorlay, Masselin, & Mailloux, submitted).

We would like to conclude assuming the position of the devil's advocate, in order to point to open questions and suggest further comparative work. The experiment in primary school raises at least three interconnected questions: What was our teaching goal? Why did the 3-session-sequence work? What were its actual educational effects? To the first question, we could argue that we had research questions and not teaching goals. But that wouldn't say it all, because we probably also believe that entrusting M-tasks to students does have beneficial educational effects; work has to be done in order to identify these precisely. As to "why it worked" (if it did), the issue can be investigated by comparing with other sessions which did not work so well. In this respect, we are looking forward to the completion of Charlotte de Varent's doctoral work[19] on the use of Ancient mathematics as a means to make rather advanced students (age 15-16) question seemingly elementary mathematics (namely: the formula for the area of a rectangle, and the role of units of lengths and areas). As to the study of the educational effects, we feel we could benefit from a comparison with recent work in the

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[18] Two usual suspects would be: (1) expressing magnitudes with several units (e.g. a length of 1km 300 m, a weight of 1kg 850 g), (2) expressing whole numbers in terms of units, tens, hundreds etc. (e.g. to carry out an addition: 7t + 3u + 5t + 3u = 12t + 6u = 10t + 2t + 6u = 1h + 2t + 6u = 126).

[19] *Ancient mathematics in secondary schools: issues, current practices, and perspectives*. Dissertation supervised by Christine Proust (SPHere. UMR 7219-Paris Diderot) and Nicolas Decamp (LDAR. Paris Diderot).

didactics of physics, where the effect of teaching sequences based on history – and occasionally of original sources – is studied in terms of conceptual change (Merle, 2002), (Décamp & Hosson, 2011).

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