Plenary Lecture

NEW APPROACHES AND RESULTS IN THE HISTORY OF TEACHING AND LEARNING MATHEMATICS[1]

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Studies on the history of teaching and learning mathematics did not begin in recent times; rather, there were already a number of books and various types of papers published during the 19th century. The work of IMUK since 1908, the forerunner of ICMI, meant a considerable impact for historical investigations. After World War II, pertinent studies were undertaken in ever more countries. Yet, practically all the studies were undertaken within the history of some nation or some culture. They were thus bound to the respective traditions, methodologies and approaches of national educational history.

Meanwhile, the focus has changed to address comparative and international issues in this area of research. At stake is since then to unravel what are general features in the national/cultural developments and what are specific issues and what is the significance of such particular patterns. As particularly revealing have proved three issues of comparative international research:

- *the processes leading to the decisive change of mathematics from a marginal teaching subject to a major discipline, first in secondary schooling;*
- *and, related to these developments, the emergence of Mathematics for All as a program and as a major shift in socio-politics of education;*
- *the role of mathematics in the modernization of various states, in particular during the 19th century, and thus showing the social relevance of mathematics.*

The lecture presents methodological reflections, illustrative historical examples and research perspectives.

I. AN OVERVIEW ON THE DEVELOPMENT OF THE RESEARCH AREA

Research into the history of teaching and learning mathematics does not constitute an entirely new field. In fact, such research can look back to a considerable tradition. In fact, already during the 19th century, numerous pertinent studies have been published; they show the broad area opened by this research field. Best known so far are publications in Germany. One of their foci was the history of mathematics teaching at particular schools. The first such study – to my knowledge - dates of 1843 and assessed the evolution of the mathematics curriculum since the Enlightenment reforms to the Prussian Gymnasium reforms, for the Gymnasium in Arnsberg, a town in Prussian Westphalia (Fisch 1843). It systematized the periods in which mathematics teaching became steadily reinforced since the 1770s.

Another focus was on teaching methods; in 1888, Jänicke published the even today valuable study on methods to teach arithmetic (Jänicke 1888). Maybe the first monograph devoted to this area was the book, of 1887, by Siegmund Günther on the history of mathematics teaching in Germany during the Middle Ages (Günther 1887). And there were even first doctoral dissertations, one on teaching arithmetic (Stoy 1876) and one on mathematics teaching in the German state Saxony (Starke 1897).

Studies in the 19th century did not remain restricted to Germany; another study in book format was by Florian Cajori describing the history of mathematics education in the United States up to the end of the 19th century (Cajori 1890). A book by Christensen on the history of mathematics in Denmark and Norway in the 18th century dealt also with the history of mathematics teaching in these countries (Christensen 1895). In an analogous manner, a book on the history of mathematics in Finland until about 1800 studies mathematics teaching, too (Dahlin 1897).

From the beginning of the 20th century, an intense activity of publishing research studies of considerable scope and ambition reached a first peak – until World War I. Firstly remarkable is that the first doctoral theses defended in the USA in mathematics education were studies on the history of mathematics teaching (Jackson 1906; Stamper 1909). Secondly, there are the embracing studies by Germans, probably nurtured by the historicism mentality dominating there, on various key issues of the history of mathematics teaching:

- the book by Grosse on arithmetic textbooks since the 16th century inaugurated the sub-area of research on schoolbooks (Grosse 1901)
- the book by Pahl on the history of teaching mathematics and the sciences in Germany (Pahl 1913).
- even more ambitious was the approach by Timerding who gave a survey of the history of mathematics teaching, from Egypt and the Greeks to the early 20th century, with special emphasis on the teaching of mathematics in Germany during the 19th century (Timerding 1914)

This new dynamic was also partially due to the initiatives by Felix Klein to reform mathematics teaching, promoted by him as president of IMUK (*Internationale Mathematische Unterrichts-Kommission*), founded in 1908. In the series of German reports for IMUK on the state of mathematics teaching in Germany, there were several pertinent monographs: the study by Schimmack on the evolution of the reform movement (Schimmack 1911), the study by Lorey on the training of mathematics teachers (Lorey 1911) and his study on the mathematics taught at 19th century universities (Lorey 1916). Moreover, the numerous reports on the actual state of mathematics teaching in the various German states constitute today excellent sources for research on this period, in particular the well documented books by Lietzmann on the teaching of geometry and on the teaching of arithmetic (Lietzmann 1912a, 1912b). Examples from other countries are Watson on England (Watson 1909) and Heegard on Denmark (Heegard 1912).

While relatively few researches were published in the Inter-War period, studies intensified after World War II and covered gradually more countries. A first book is of 1945, on geometry teaching in Finland (Nykänen 1945). Jushkevich published in 1947 and 1948 a series of papers on mathematics teaching in Russia, from the 17th to the 19th century (Jushkevich 1947-1948). A book by Prudnikov on Russian mathematics educators in the 18th and 19th centuries followed in 1956 (Prudnikov 1956).

An important impact had two volumes published in 1970 by the NCTM, the mathematics teachers association of the United States, thus giving institutional promotion to this field of study: The first volume was a reader with selections from major documents spanning the period 1831-1959 in the USA (Bidwell and Clason 1970), and the other was the NCTM Yearbook for 1970, edited by Jones and Coxford, with research studies on primary and secondary education in the USA and Canada (Jones and Coxford 1970).

From the 1980s, one remarks a rather continuous flow of publications, regarding ever more countries. Research in the history of mathematics education became now a rapidly developing area. Various trends are now visible:

- There are, on the one hand, more specialized studies for a given country; let me mention Howson's book on mathematics education in England (Howson 1982).
- There are, on the other hand, new attempts to an international history; there is the study by Schubring (1984) who researched the history from Antiquity and of various civilizations according to theoretical categories until Modern Times; Miorim's book (1998) presents the development from Antiquity; for Modern Times, it focuses on Brazil.
- The third and new trend is constituted by methodologically reflected approaches to go beyond the surface of administrative facts and decisions, with the objective to unravel the reality of teaching in school practice. These approaches rely on extensive archival research and on interdisciplinary methodology. I will comment more on these developments of methodology in the next part. A first such study is the book by Schubring (1983[1], 1991[2]), which analysed the reality of the emerging profession of mathematics teachers in Prussia. A following study was done by Siegbert Schmidt who analysed the reality of teacher training for primary schools in a specific region of Prussia (Schmidt 1991). The approach became then applied to the Netherlands where mathematics turned during the 19th century from an unwanted intruder in classical secondary schools to a major discipline in a new school type (Smid 1997).

Particularly noteworthy for this new period is the monumental work: "A History of School Mathematics", presenting studies on North America (Stanic & Kilpatrick 2003).

So far, all these activities were mainly individual initiatives. This state changed decisively with ICME 10, held in Copenhagen in 2004, when the field became internationally institutionalized the first time, as the Topic Study Group 29, on the History of Teaching and Learning Mathematics. In its preparation, a first international bibliography of relevant publications became elaborated, thanks to the cooperation of researchers from many countries. Since 2004, this international structure of a TSG was maintained and continued at each ICME. Moreover, in 2006 was founded the *International Journal for the History of Mathematics Education*, the first journal dedicated to this field of research. The journal is now in its ninth volume.

Without exaggerating one can say that a decisive climax was achieved now in 2014 when was published the first comprehensive *Handbook on the History of Mathematics Education*: it covers a wide spectrum of epochs and civilizations, countries and cultures – in 38 chapters/sections and with 40 authors. It makes not only accessible research published in the language and for readers of an individual country, but the handbook succeeded also in launching research on yet scarcely investigated regions and epochs (Karp & Schubring, eds., 2014). Clearly, the process of elaboration also proved that there are still a lot of open questions.

II. METHODOLOGICAL CHALLENGES

Research into the history of teaching and learning mathematics is confronted with a number of methodological challenges, which traditionally researchers have not been well aware of. Largely, research has had a descriptive character, focussed on data on the surface of historical phenomena and processes.

Modern research in history of education – and in particular history of school - has established, however, new patterns of methodology: it became clear that schools function as a subsystem of the respective society and that it is therefore largely sociological methods, which need to be applied. Thus, sociology of education emerged, which established methodological standards for investigating historical processes in schools. This applied to history of education in general, but research on certain school disciplines use to restrict on sociological methodology only: this is the case mainly for disciplines to which socializing functions are ascribed to, thus mother language, history and religion and where historical research therefore largely makes abstraction of the contents taught. But school disciplines like mathematics need a broader methodological approach, a more interdisciplinary one, clearly conceived of on the basis of social history – but capable to be specific for analysing the development of teaching contents.

The low emphasis on methodology may be caused by what proves to be an illusion: the idea that research into the history of mathematics instruction presents an easy task, that this history is just a collection of facts which are observable without difficulties, and that one only needs to ‘collect’ these facts. This is in particular the view of the history of mathematics instruction as a series of administrative decisions that supposedly were transformed into practice. According to this perspective, the history basically is a history of the curriculum, of the syllabus, managed by centralist authorities. But even when the broad spectrum of historical issues is reduced to the syllabuses, the real problem is whether, and how, centralized decisions were implemented in school practice, and this opens up again the immense range of dimensions relevant to the historical development.

In fact, the history of the teaching and learning of mathematics constitutes an interdisciplinary field of study; the principal disciplines concerned are the history of mathematics and the history of education, but the general history contributes as well. Moreover, sociology is quite essential, in particular sociology of religion.

Realizing the complexity of our field of study, we might even say that it requires an even more complex methodology than the history of mathematics. Clearly, mathematics history is a part of cultural, political and social history, too, but the contents of mathematics and the evolution of its concepts occupy a far more extended domain within mathematics history than within the history of mathematics instruction. Compared with this rather dominant role of mathematical ideas and concepts, the history of teaching and learning mathematics constitutes a social reality within educational systems that needs incomparably more social categories to reveal its dimensions.

Even the entity corresponding to the structured set of mathematical concepts, namely 'school mathematics', is far from being just a derivation or a projection of the 'savoir savant' as Yves Chevallard pretended (Chevallard 1985) – well to the contrary, school mathematics develops as a product of numerous interactions, and even pressures, from and between various sectors of society. But what complicates the research in our field even more is the fact that mathematics never appears in educational systems in an independent way but always functions within structures, which are characterized by a compound of *several* school disciplines. This means that mathematics teaching and learning is always dependent on other factors that it is barely capable of influencing.

Yet, despite this fundamental and structural dependence on a concert of disciplines that in general exhibit no peaceful coexistence, the perhaps most considerable deficiency of the large majority of studies in our field is that they treat mathematics as an *isolated* teaching subject, without regarding relationships, dependencies and hierarchies in the *system* defining school learning.

It should be evident by now that marked progress in research necessitates methodological reflection and refinement. A decisive resort in doing so is presented by comparative issues – not only comparative studies of the history of various school disciplines within a given educational system, but even more importantly comparative studies on the history of mathematics instruction in different states and different cultures. It is quite natural that most research pursued or ongoing is concentrated on the history within a given nation or a given culture as the history of mathematics teaching and learning first and foremost constitutes part of the educational history of that country or culture. But in order not to end up with a collection of separate, isolated histories without interconnections, one has to establish relations between the different national histories and to reveal what is 'general' in them and what constitutes, say, cultural, or social, or political peculiarities of a specific country or culture. Practically all questions in our historical field deserve comparative studies.

The approaches and results from comparative education hence provide an essential tool for an international history of teaching and learning mathematics, in order to grasp national specificities as well as overall and global trends. Of particular methodological importance are qualitative methods, which are also applicable to the study of (historical) documents. Given the primary importance of cultural history, anthropology provides relevant methodological resources as well.

To resume: History of teaching and learning mathematics is an interdisciplinary field, interacting in particular with:

- Sociology of education
- History of education
- History of mathematics, and
- Epistemology: given the importance of the views on mathematics, which use to be specific for cultures and which determine decisively the conceptions of what should be school mathematics

III. KEY ISSUES AND NEW RESULTS

1. Origins of teaching mathematics

In popular literature, a rather idyllic picture is designed of the origins of mathematics: shepherds counting their sheep by means of producing some listings, or farmers counting the cattle. Much folklore has been designed to visualize how early mankind might have counted with fingers or even have registered results. What is reliable, however, is revealed by the famous report of Herodot on the origin of geometry in Egypt – also, there, not as the likewise idyllic history of natural phenomena by which probably unorganized settlers are confronted. Reading Herodot's report carefully, one understands that it deals with the functioning of a professional group – land surveyors – in the service of administration of a state. In fact, extensive research since the 1980s – and thanks to the use of computers then the first time applicable for evaluating an enormous number of documents – has unravelled the origin of mathematics in the somewhat parallel cultures of Mesopotamia. It is due to the innovative research methods of the team Nissen, Englund and Damerow – a historian, an archeologist and a mathematician – that the emergence of sign systems became proven as instigated by the needs of a centralized state administration, from the 4th millennium BCE on. What they were able to show was how a highly differentiated system of signs for object-bound quantities developed into a standardization yielding eventually a system of numbers, even a positional system, the sexagesimal system (Nissen et al. 1993).

More recent research has revealed the origins of mathematics teaching as intimately tied to the emergence of sign systems for the same state administration. And it is highly revealing that in these origins writing and calculating constituted a unity – to the contrary of the later divergence between humanities and sciences. Researchers on the history of writing – a well known specialist is Denise Schmandt-Besserat (see

Schmandt-Besserat 1996) – and researchers on the history of mathematics agree that number and scripture originated together, in the same socio-cultural setting. Eleanor Robson, researcher on Mesopotamian mathematics, formulated the consensus of both sides recently:

> "The temple administrators of Uruk adapted token accounting to their increasingly complex needs by developing the means to record not only quantities but the objects of account as well. Thus numeracy became literate for the first time in world history" (Robson 2008, p. 28).

Writing and calculating was, thus, taught in an intertwined manner:

> "As the production of accounts entailed complex multi-base calculations, trainee scribes had to practice both writing and calculating, and they did so increasingly systematically" (Robson 2008, p. 40).

Archaeological research which had earlier on not given much attention to tokens and calculating tablets has now systematically searched for mathematical tablets and their locations; it was able to even identify buildings which had served as *edubba*, hence as schools for teaching writing and calculating, thus training scribes for the state administration (Robson 2008, p. 98). Christine Proust, in her contribution on Mesopotamia in the Handbook, was able to even reconstruct the structure of the mathematics curriculum in the *edubba*:

Level	Content	Typology	Examples
Elementary	Metrological lists (capacities, weight, surfaces, length) Metrological tables Numerical tables (reciprocals, multiplication, squares) Square and cubic roots	Types I, II and III	See Figures 3-4
Intermediate	Exercises : performing multiplications and reciprocals. Surface and volume calculations	Square-shaped tablets	See Figure 5

Figure, Proust 2014, p. 33 (in Bernard et al. 2014)

Thanks to recent research, one even knows the name of the eldest so far known person practicing mathematics: On a tablet from the palace archives, at the Syrian city of Ebla, dating from about 2350 BCE, one finds at the end: "Nammah wrote the calculation" (Robson 2008, p. 32).

A new study on China confirms this decisive function of state administration for promoting mathematics and for institutionalizing its teaching. Alexej Volkov has presented, in his chapter on China in the Handbook, in particular his research on the "School of Computation". The first such state-run School of Computations was established during the Sui dynasty (581–618) that unified China after a long period of disunity. It is known that the instruction was conducted by one or two "erudites" (*boshi*) and two “teaching assistants” (*zhujiao*) and that the number of students totalled 80. Its full development occurred from the Tang dynasty (618-907). The earliest mention of the school is dated 628; in this year, instructors were hired and students admitted. The age of the students entering the School of Computations ranged from 13 to 18. No information is available about the teaching materials used at that stage, yet one can conjecture which were the textbooks (Volkov 2014, p. 59).

After some interruption, the school opened again in 656, under the supervision of the governmental agency named “Directorate [of Education of] Sons of State” (*Guo zi jian*) – thus, a kind of ministry of education. It was from this year on that the School experienced an important functioning, in the preparation for the exams of admission to the various branches of state administration. In particular, it was in this year that the famous list of mathematical textbooks was established, which served for learning and preparing the exams at this school. This list is known in the literature as the “Ten Classics”, but as Volkov has shown, it were in fact 12 textbooks. One of them is the famous *Jiu Zhang Suan Shu* – the Nine Chapters of Mathematical Procedures, which dates back to about 300 BC and is thus a real rival to Euclid’s Elements. According to the *Tang liu dian* (Six Codes of the Tang [dynasty]) and to the *Jiu Tang shu*, the students of the school were subdivided into two groups each comprising 15 people and instructed by two “erudites” and one “teaching assistant”. The students of the first group studied treatises [1–8], and those of the second one studied treatises [9-10]. These treatises are referred to as “regular program” and “advanced program,” respectively. The study in each program usually lasted 7 years but in exceptional cases could be extended to 9 years. Treatises [11–12] were studied simultaneously with the other treatises in both programs; the time necessary for their study was not specified. The 12 textbooks of the curriculum and the extant mathematical treatises with which they are conventionally identified are shown in Figure 2. Not much is known about the procedures of instruction in the School of Computations; the only element mentioned in the extant sources is “oral explanations” provided by the instructors. There were two kinds of examinations: (1) the quizzes conducted every 10 days and (2) the examination conducted at the end of each year. A lot of details of the examinations are known (Volkov 2014, p. 63).

Table 4.1 Mathematical curriculum of the Tang School of Computations

#	Title	Duration of study	Program[a]
1	*Sunzi* 孫子 ([*Treatise of*] *Master Sun*)	1 year for two treatises together	Regular
2	*Wu cao* 五曹 (*Five Departments*)		Regular
3	*Jiu zhang* 九章 (*Nine Categories*)	3 years for twotreatises together	Regular
4	*Hai dao* 海島 (*Sea Island*)		Regular
5	*Zhang Qiujian* 張丘建 ([*Treatise of*] *Zhang Qiujian*)	1 year	Regular
6	*Xiahou Yang* 夏侯陽 ([*Treatise of*] *Xiahou Yang*)	1 year	Regular
7	*Zhou bi* 周髀 (*Gnomon of the Zhou* [*Dynasty*])	1 year for two treatises together	Regular
8	*Wu jing suan* 五經筭 (*Computations in the Five Classical Books*)		Regular
9	*Zhui shu* 綴術 (*Procedures of Mending* [=*Interpolation?*])[b]	4 years	Advanced
10	*Qi gu*[c] 緝古 (*Continuation* [*of Traditions*] *of Ancient* [*Authors*])	3 years	Advanced
11	*Ji yi* 記遺 (*Records Left Behind for Posterity*)	Not specified	Compulsory
12	*San deng shu* 三等數 (*Numbers of Three Ranks*)	Not specified	Compulsory

Figure 2: The Twelve Classics (Volkov 2014, p. 61)

2. The way to "Mathematics for All"

As one understands from the origins of mathematics teaching, it became organized by a state for its proper needs of administration and government; the origins and the first developments are therefore due to the needs of *professional* training. Mathematics as element of liberal education became instituted, in contrast, much later and not by state intervention but by initiative of higher social classes, as the first historical example - Greek city states - shows.

One uses not to be aware of a revealing contrast: while mathematics as subject of professional training functioned as a major teaching subject, it used to be taught as a marginal or auxiliary subject within liberal education.

There are therefore two pertinent research questions:

- which dynamics, which movements, which forces effected the change from a minor to a major teaching subject for mathematics within liberal education?
- What effected that the state, hitherto restricted in its actions for education to the needs of professional experts in its service, turned to act for instituting as a major subject of liberal education?

Actually, I have done a lot of research on these questions, in international comparative studies, and I am continuing this research. Basically, it entails the question: how emerged the program "Mathematics for All"? More about it will be discussed in the workshop,[2] and I am restricting here myself to two novel results:

One result concerns the conceptual framework for introducing and realizing the new function for mathematics teaching and learning and its contextualization. Traditionally, the focus in historiography for this question has been on Prussia and its neo-humanist reforms of education from 1810 on. Since Prussia was basically a Protestant country, one is lead to relate this reform conception with Max Weber's famous thesis of Protestant Ethic as the source for the rise of capitalism. Combining Weber's thesis with the Merton thesis according to which it was the Protestant context, which constituted the fundaments for the Scientific Revolution, one would think it to be quite natural that the change of the function of mathematics in secondary schools is due to Protestantism.

At a first glance, one might be confirmed that it was not due to Catholicism. In fact, Christopher Clavius (1537-1612), chief mathematician of the Jesuit order, had proposed an ambitious program, based on conceptions of the Humanism movement, for teaching mathematics in the system of Jesuit colleges becoming established in the Catholic countries in Europe and the Americas. But in the debate, on-going during the sixteenth century, about the certitude provided by mathematics, the Jesuit philosophers claimed that "mathematics does not reach the highest level of certitude, so it is not a science strictly speaking" (Paradinas 2013, p. 167). Thus, in the end, nothing became realized of Clavius's program in the *Ratio Studiorum* of 1599, and mathematics experienced a marginal role in Jesuit teaching (see Paradinas 2013).[1]

On the other hand, the change was not due to Protestantism, neither. Philipp Melanchthon (1497-1560), the principal constructor of a Protestant education system, argued intensely for teaching mathematics at the *Gymnasien* and universities. While mathematics developed there firmly at the universities, continuing what had been initiated during Humanism, mathematics at the *Gymnasien*, however, did for a long time not succeed to overcome a likewise marginal function (see Schubring 2014).

Upon closer scrutiny, one will find, however, that it was one Catholic country where mathematics first achieved this new status. Yet, it was a quite specific Catholic country and not, let us say, a typical Catholic country: it was France. In fact, France was practicing a policy of striving for a Gallican Church – i.e. against an ultramontane obedience to the pope. The Jesuit order had been admitted to France from 1604 only in a somewhat nationalized form; it was Jansenism – an intra-Catholic reform movement in the 17th century –, which urged for a Gallican policy; consequently, it became prosecuted by the Jesuits. A key exponent of Jansenism was the philosopher and theologian Antoine Arnauld (1612-1694). In his seminal textbook *Nouveaux élémens de géométrie* (1667) he was the first to develop a theological argumentation for a primacy of mathematics in education. Jansenism thus turned out to constitute an important source for the Enlightenment movement.[2] Enlightenment in France became closely related with Rationalism, featuring thus a key importance of mathematics in general culture. It was in this context that the state created, from the middle of the 18th century, a net of military schools where mathematics constituted the leading discipline for the formation of military engineers and of officers.

While this functioning during the *Ancien Régime* still occurred within the traditional paradigm, namely the formation of military and technical experts for the needs of the state, the French Revolution effected a fundamental change: the state now assumed an overall responsibility for a public educational system and instituted in this vein mathematics as a major teaching discipline. Most characteristic is the first systematic organization of secondary education in 1802: in this first law (10 December 1802) on organizing a public education system, Latin and mathematics were declared as the two key disciplines.

> "On enseignera essentiellement dans les lycées le latin et les mathématiques" (quoted from Schubring 1984, p. 371).

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[1] The eulogies of Antonella Romano, in her book *La contre-réforme mathématique* (1999), for Clavius's program simply ignore the failure in becoming it accepted by the Jesuit order (see Schubring 2003).

[2] My research on this new kind of establishing a fundamental role of mathematics in education, by Arnauld, and on the role of Jansenism in disseminating this conception is forthcoming in the paper: "From the Few to the Many: On the Emergence of *Mathematics for All*".

3. Bipolarity of mathematics

The *second* result concerns the epistemology of school mathematics. Over extended periods, mathematics had to fight against strong resistance to achieve or to maintain acceptance in secondary schools as a legitimate teaching subject. The fight was with representatives of philology, of teaching classical languages, who claimed superior educational value for these languages. A major resource of mathematics teachers in these fights was to refer to Antiquity, to the inscription above the entrance to Plato's Academy:

ἀγεωμέτρητος μηδεὶς εἰσίτω

Although one doubts today whether such an inscription really existed there, fortunately no philologist then dared to doubt and thus it constituted an argument endowed with a certain power.

One used to think that modern secondary schools – "Realschulen" –, emerging since the 18th century and spreading rapidly during the 19th century, constituted a major backing for maintaining or achieving a strong position for mathematics in teaching. Closer scrutiny shows, however, that the propagators of "modernism", of realist oriented education, had no holistic understanding of education: they started from a principal refusal of classical oriented secondary schools, hence of traditional humanism, and strove to construct a likewise one-sided educational conception, oriented towards utility. And for legitimating this conception, mathematics was used as key argument for claiming utility as goal for the secondary schools.

Some mathematics teachers tried to avoid this embracement by agitators for *Realschulen*; for instance, Carl Friedrich Andreas Jacobi (1795-1855), the teacher who achieved to firmly establish mathematics at the *Landesschule Pforte*, a traditionally extremely humanist Saxon Gymnasium, after it had become Prussian, declared:

> "Mathematics is no modern means of education but a classical one" (Schubring 1985, p. 25).[3]

One has to consider, regarding the epistemology of school mathematics, that mathematics has a *bipolar* character: in view of its logical and foundational abilities, it belongs to the humanities; on the other hand, as correctly expressed by the term 'polytechnic', it enables enormous means of applications. It is hence remarkable that in the original neo-humanist curriculum for Prussia of 1810, the Tralles-Süvern-Plan, applications of mathematics constituted integral elements (Schubring 1991, p. 209).

It is this double-faced nature of mathematics, which constitutes the conceptual challenge for historical analyses: to be aware of this epistemological special character and to use it as a conceptual framework for concrete historical investigations.

4. Function of Mathematics for Modernising society

Usually, the focus for research on the history of mathematics teaching is the major European countries: Germany, France, Italy, England. It is mainly for these countries that one investigates changes and rises in status of mathematics teaching. As a matter of fact, the structures of public education systems became established there – basically during the 19th century – and from there the structure was transmitted or imposed in one or the other variant to cultures and states in other continents. There has been recent research on how this transmission or interaction has occurred, in particular in papers published in the special Issue of the journal ZDM, dedicated to: *Turning Points in the History of Mathematics Teaching – Studies on national Policies* (vol. 44, no. 4, 2012).

As it became clear from various case studies in this special issue, establishing mathematics teaching proves to be an intentional act to modernize the respective society, to contribute to meet the demands with which the state in question is confronted. In fact, it is by actions of the respective Empire or national state, not by some individual's good will or plans of a definite social group, that mathematics becomes ascribed a function within the intended reform process. And in none of these cases, mathematics teaching became imposed from outside – it was upon proper and internally decided strategy to call for this part of knowledge.

Most telling for such ascribed modernizing functions are states suffering profound crises of their traditional modes of existing. It that ZDM issue, three non-European Empires were investigated that reveal the key functions ascribed to mathematics for re-founding the basis of state and society: China, Japan and the Ottoman Empire, all presenting Empires, which used to be regarded as unchallenged powers, but - upon being confronted with Western values or even invasions – their traditional means proved to no longer being able to maintain their status.

The case of China

As a consequence of the Second Opium War of Western powers against China and the devastating defeats (18-18), a faction rose in the government who argued that one could not continue policy in the traditional way as leading immediately into total collapse and that inner reforms were needed. As a first measure, they succeeded in getting founded a *School of Combined Learning*, at first intended as a language school to train interpreters and to thus be able to negotiate with the foreign invaders. It then developed into a college of Western learning, together with other similar colleges in more cities in China. Soon after, the new Foreign Office, also an achievement and basis of the reform faction in the government, proposed the creation of a *School of Astronomy and Mathematics*, thus adopting a reform conception proposed by wider circles in the cultural elite, calling for "self-strengthening" (Chan & Siu 2012, pp.). The opposing conservative faction tried to impede the foundation of this school, in particular since foreigners should be applied there as teachers. It argued that there was no need for teaching mathematics and that one can find experts in China for the necessary technical tasks. When the chief opponent was asked to name such domestic experts, he had to admit he knew no such one. The reformers in the government formulated the basic conviction for reforming China and being able to resist, by adopting Western science and education:

"All Western knowledge is derived from mathematics. Every Westerner of ten years of age or more studies mathematics. If we now wish to adopt Western knowledge, naturally we cannot but learn mathematics" (quoted from Chan & Siu 2012, p. 464).

The case of the Ottoman Empire

The Ottoman Empire, traditionally an expansive power, suffered but defeats during the second half of the 18th century on the Balkans. Reform-minded sultans established engineering and military schools privileging mathematics teaching, from the 1770s. Likewise similar reforms were undertaken by the partially dependent, partially independent governments of Tunisia and Egypt. In Egypt and in Tunisia, after also founding military or engineering schools, schools for general education were established, with a strong function for mathematics. The ruler of Egypt, Muhammed Ali, even sent a number of students to France to study mathematics at the *École polytechnique*. He organized to translate modern mathematics textbooks. And it is characteristic that the there still strong traditional social forces directed their destructions against the new schools, as it happened in the Ottoman Empire in 180x; in Egypt, Muhammed Ali was not strong enough to abolish the traditional system of schools run by the *ulema* and had to face the existence of two parallel systems - the modern state-run system, administered by a Ministry of Education (from 1837) and the traditional schools controlled by the *ulema* (Abdeljaouad 2012).

The case of Japan

In such situations of crisis, the state and its government concentrate on its most urgent needs for maintaining its existence and strength: on engineering and on warfare capacities; and it is thus revealing that the first functions of mathematics for the state are its rationalizing abilities: for technical and military applications. Yet, one remarks that - in a second step – the state becomes the agent for teaching mathematics as a key element of general education. Japan presents a likewise telling example for such two steps: Even before the opening of Japan to the West, by the 1867 Meji fundamental change of policy, the central government had founded a Naval Academy at Nagasaki, in 1855, "where Western mathematics was taught because of its military applications" (quoted from Ueno 2012, p. 476). As part of its profound reforms of society from 1867 on, in particular of abolishing the feudal clan system and their segregated school structures, the new government established a centralized educational system for all, with mathematics as a basic constituent. One remarks here, too, the proper intention of the state to assume responsibility for education by a public education system: practically the first measure was the creation of a Ministry of Education (Matsubara & Kusumoto 1986; Ueno 2012, pp. 477).

5. Connections between cultural values in a society and dominant epistemology of mathematics

The case of teaching mathematics in Italy during the 19th century, and in particular from the national unity after 1860, used to raise bewilderment among non-Italians, while Italian historians used to praise the mathematician Luigi Cremona, the main responsible for the curricular and organizational decisions of 1867:

- Euclid's Elements were introduced as textbook – and not in an adapted version like in England, but in the (translated) original;
- from 1878 on, the status of mathematics teaching in secondary schools weakened ever more – as only such case in a European country.

Research over the last 20 years has clarified already some points to understand this specific development, due to an epistemology extolling pure Greek geometry and aligning this to the dominant classicist Italian culture, but only recently a key structure was revealed by Italian researchers for the international public. While one had always spoken of the *liceo* – and that was understood as *the* secondary school in Italy – it became now clear that secondary school in Italy was divided in two sections – the *ginnasio* and the *liceo* – and that in the *ginnasio* mathematics was a minor subject while it should have been a major one in the following *liceo*. Even so, the meaning of the division is still not clear since – contrary to other countries where such a division means that the first structure is destined also for students not intending to continue to university, thus covering also teaching for non-academic professions – the *ginnasio-liceo* in Italy was conceived of to prepare only for university studies. A look on the timetables for the teaching hours for mathematics already provides first insights and then questions:

School type/grade	1860	1862	1865
Ginnasio I	1		1
Ginnasio II	1		1
Ginnasio III	1		2
Ginnasio IV	3		2
Ginnasio V	3		2
Liceo I	8	6	6
Liceo II	0	2	3
Liceo III	3	3	3

Figure 3: number of weekly teaching hours for mathematics in unified Italy
(no change in the *ginnasio* in 1862)

We look primarily at the time tables for *ginnasio* and for *liceo* from 1860, the beginning of political unity of Italy, until shortly before 1867 and remark at first that in few years the time-tables were changed several times. Actually, this presents a characteristic of the Italian school policy – there was quite few stability in this system. Secondly, we remark a quite low rank of mathematics in the *liceo*. But let us now look at the time-table of 1867, elaborated by a committee of mathematicians, presided by Luigi Cremona:

Gin I	Gin II	Gin III	Gin IV	Gin V	Liceo I	Liceo II	Liceo III
0	0	0	0	5	6	7,5	0

Figure 4: weekly hours for mathematics teaching 1867 (Giacardi & Scoth 2014, p. 211)

This meant a self-decided exclusion of mathematics from the first four years of the secondary school and a concentration in the upper middle part – even in the last grade, important for the final exam, no mathematics should be taught. Looking now at the list of contents ascribed to the teaching in these few grades, one already begins to understand the conception of school mathematics and its epistemology:

G I	G II	G III	G IV	GIN V	LIC I	LIC II	L III
-	-	-	-	Arithm + geom: Euclid 1	Aritm + Alg + Geom: Euclid 2, 3	Algebra, Trigon., Geometry: Euclid 4, 5, 6, 11, 12 and circle, cylinder, cone, sphere accord to Archimedes	-

Figure 5: contents of mathematics teaching in the programme 1867 (ibid., p. 212)

The conception of Cremona had been that only good mathematics can constitute meaningful school mathematics and good mathematics for him was rigorous mathematics, organized deductively. Moreover, the best such mathematics was for him Euclid's geometry. Thus, Euclid's Elements were prescribed as textbook, and since students of the lower grades were not thought of to be mature enough for such a geometry, the teaching of mathematics should begin at only a late stage. In that first grade with mathematics, the last grade of the *ginnasio*, there should be taught also a bit of arithmetic: but only as "rational arithmetic" - a very Italian concept for school mathematics: also arithmetic should be taught in an axiomatic and deductive manner. And remark the overloading in the second grade of the *liceo*: too many weekly hours, too many mathematical topics to be taught!

Cremona believed that the role of the *ginnasio*-*liceo* was not to give students a mass of knowledge, but rather to provide a method for dealing effectively with problems. In particular, for geometry he suggested following the Euclidean method because 'this is the most appropriate for creating in young minds the habit of inflexible rigour in reasoning'; he exhorted teachers not to contaminate 'the purity of ancient geometry, transforming geometric theorems into algebraic formulas, that is, substituting concrete magnitudes ... for their measurements' (Giacardi & Scoth 2014, p. 212). Likewise, arithmetic was to be taught using the deductive and demonstrative method.

Admittedly, such a mathematical fundamentalism, without any idea for what is achievable in school teaching, could not be maintained for a long time. Soon, the Ministry felt obliged to attenuate somewhat this teaching conception. Also other textbooks than Euclid became allowed, and mathematics was extended to all grades of the *ginnasio*. However, being in general restricted to two weekly hours in all the five grades, it was clearly understood as a minor teaching subject. This week role had the harmful effect that the students upon entering the *ginnasio* were not sufficiently prepared to pass the exams. In fact, students used to fail in the final exam, and since these were central common exams for entire Italy, this soon became a public calamity. From 1878, the Ministry tried to avoid it by simply excluding mathematics as a subject of the final exam (Scarpis 1911, p. 8). Although one tried to find various compromises, the basic message for the students (and their parents) was: mathematics is not of equal status to the humanities.

CONCLUSION

The history of mathematics teaching and learning reveals it as a highly pertinent and rich source for interdisciplinary studies on the role of mathematics in society. Already the double-faced nature of mathematics as a pure science and as an applied science allows to study the functioning of school systems and their evolution along differing needs of societies. One remarks breaks in the legitimation of mathematics as a school discipline, which reveal epistemological dimensions, differing according to cultures, but also religious motivations, which refer to sociology of religion. And one remarks breaks in the status of mathematics teaching due to political breaks. Less studied still is the modernizing function of mathematics for societies endangered politically – like in the case of traditional powers but having become stagnant and thus weak in confrontation with expanding Western powers.

Pertinent factors and dimensions proved to become better and more profoundly understandable by a methodology relying on comparative education. Investigating the history in one country or culture, one will thus be able to discern between what is particular and characteristic for that case and what is revealing general patterns – or what is now called 'local' versus 'global' in history of science.

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[1] I am in general using the term "history of teaching and learning Mathematics", since "history of mathematics education" has a double meaning: history of teaching mathematics and history of the scientific discipline mathematics education, in some languages called, e.g., *didactique des mathématiques*.

[2] In the three-hour workshop in the afternoon, the participants worked on various aspects of the historical reality of mathematics teaching: comparing the contrasting formats of the first syllabi for mathematics teaching: in France (1802) and in Prussia (1810) and deducing differences in the school systems; analysing the official commentary for the 1867 syllabus in Italy and discussing its epistemological implications for school mathematics; commenting upon a rather critical remark on Italian use of rigor in teaching the calculus in secondary schools, in a report by Emanuel Beke in 1914; the French conception of 'history of school disciplines' (Chervel) and its role in the methodology for history of mathematics education.

[3] Die Mathematik [ist] kein modernes, sondern ein antikes Bildungsmittel.