Workshop

SOME STEPS OF LOCATION IN OPEN SEA

HISTORY AND MATHEMATICAL ASPECTS

FROM ASTROLABE TO GPS

Xavier Lefort

IREM des Pays de Loire, Université de Nantes

Quelques étapes du repérage en haute mer: atelier pour ESU7

Se situer en haute mer, loin des côtes, a toujours été le principal problème de la navigation hauturière. Les seuls repères possibles se trouvent dans le ciel, le soleil, la lune et les étoiles. L'utilisation d'un système géocentrique a perduré jusqu'à aujourd'hui, conduisant à d'intéressants calculs en géométrie sphérique.

L'atelier proposait un historique des méthodes de repérage, en s'arrêtant à quatre étapes: Pedro de la Medina (L'Art de naviguer 1569), Simon Stevin (De l'histiodromie 1634), Etienne Bézout (cours de navigation 1781), Thomas Sumner (Finding a ship's position at sea 1851). Lire et refaire les calculs issus des ouvrages de ces auteurs pour retrouver leurs résultats et les commenter est peut-être un bon moyen d'ouvrir les mathématiques au grand large!

Some steps of the location in open sea: workshop for ESU7

On the open sea, far from land, has always been the main problem of navigation on the high seas. The only possible marks are in the sky -- the sun, the moon and the stars. The use of a geocentric system has endured to the present, leading to interesting calculations in spherical geometry. The workshop presented four episodes in the history of the methods of location, drawing on the works of Pedro de la Medina, (L'Art de naviguer, 1569), Simon Stevin (De l'histiodromie, 1634), Etienne Bézout (Cours de navigation, 1781), Thomas Sumner (Finding a ship's position at sea, 1851). Reading and carrying out the original calculations from the works of these authors to find their results and comment on them is maybe a good way to open up mathematics to the wide horizons.

THE ASTROLABE, THE COMPASS, LATITUDE, LONGITUDE

When did offshore navigation begin to be practised? Not before the fifteenth century as far as we know, although there is possible evidence of earlier open sea navigation from North America and Northern Europe and elsewhere.

To find a position on the Earth, we need a model: in the second century, Ptolemy conceived a geocentric model of the universe, which remained the undisputed standard until Copernicus and Galileo. This geocentric model is still the one used today for the location of any point on the surface of the Earth, including and especially for navigation. We also need fixed points, such as the stars, and some instruments: the astrolabe, the compass, etc.

While the astrolabe has been in use for navigation only since the fifteenth century, and then only in a very simplified version, the compass appeared much earlier, around 1100. It consists of a magnetized needle, originally used in China and in the Mediterranean and then, during the following century in England, before its adoption throughout Europe and Arabia. Initially, it consisted of a needle resting on a pivot above a compass rose (see figure 1) but was later set on universal joints inside a wooden box. The compass rose is graduated in directions or ‘rhumbs’, that is in bearings. The problem is that this graduation is not in degrees, but obtained by successive divisions between the four cardinal points, so that between North (N) and East (E) we have NE and then NNE, ENE and so on.

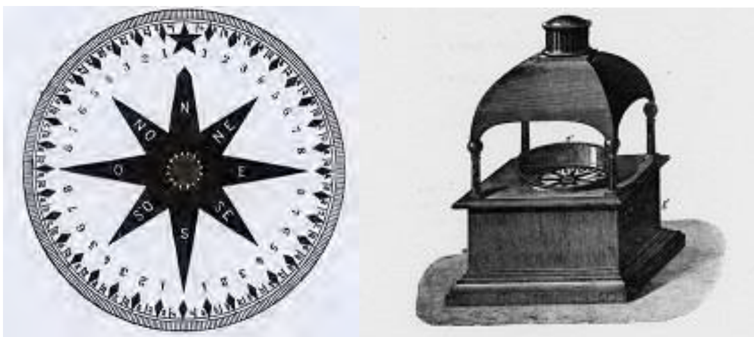


Figure 1: Wind rose and compass, Dubois, E. (1869). *Cours de navigation et d’hydrographie*, Paris: Arthus-Bertrand.

Apart from the lack of refinement, a compass bearing is not the best solution to go from A to B except for short distances. Indeed to travel on a constant course is an excellent way not to arrive there. The trajectory determined in this way is a loxodrome or rhumb line, which will wind around the pole, as shown in figure 2. The most direct trajectory is an orthodrome, a section of a great circle, but this requires regular adjustments to the course.

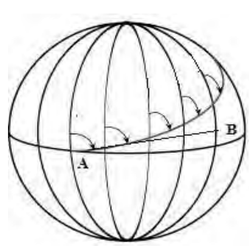


Figure 2: Rhumb line on terrestrial sphere

Any attempt at navigation must begin by identifying a position on the globe. A spherical coordinate system uses latitude (the distance from the Equator), and longitude (the distance from a meridian, or fixed great circle through the poles). In fact these ‘distances’ are measurements of angles, given by α and β in the figure. Lines of latitude are Northern or Southern as required and longitudes are East or West of an agreed prime meridian (universally adopted as the Greenwich meridian in 1884).

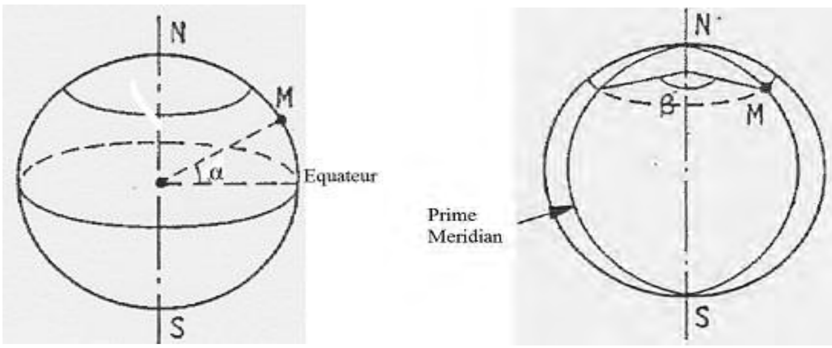


Figure 3: latitude and longitude of a point M

We can determine latitude from observing a fixed point in the sky, the polar star or the sun at noon, say. The polar star was a popular choice because when it is at its maximum the angle the North Star makes with the horizon is the same as the latitude of the observer. For other fixed points we need to take account of the apparent tilting of the polar axis with respect to the plane of the ecliptic, called the declination, which changes throughout the year (see figure 4). For the sun at noon at any point P we have: Latitude = 90° - altitude + declination

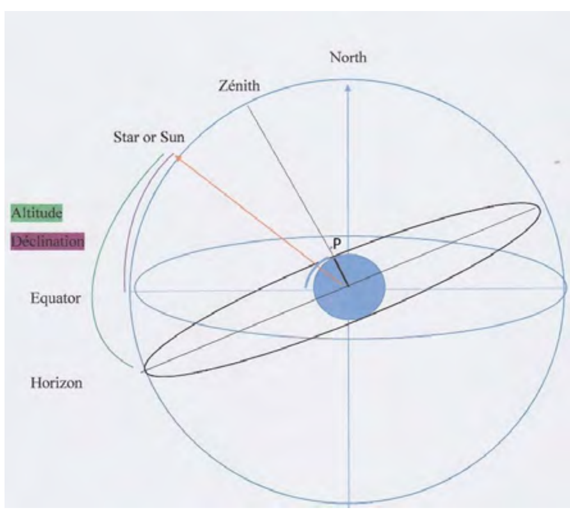


Figure 4: Declination on terrestrial sphere

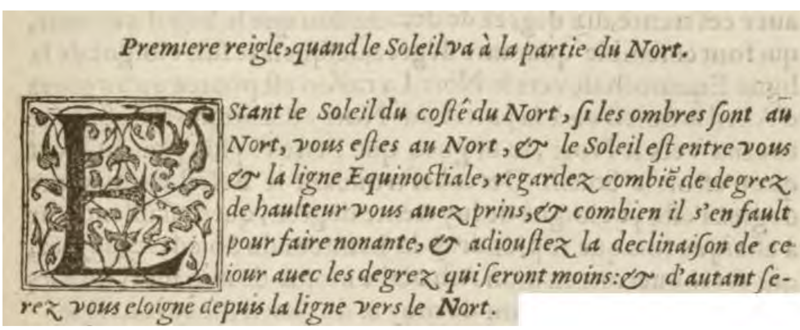
The declination being taken as negative when the sun is north of P. When using this method, the values of declination for each day would be have to be found from previously calculated tables.

Thus calculating latitude is a fairly simple procedure but determining longitude is much more difficult. It is proportional to the time difference between a point P and a reference point, originally the port of departure. But to do that requires carrying an accurate timepiece set to the reference point. Before that time some astronomical measurements were in use. The general idea was to know the time when an astronomical phenomenon seen from a reference place occurs, and to note the local time when one observes the same phenomenon. There were at the end of the fifteenth century tables giving this information, along with the declination of the sun. The time difference in hours multiplied by 15 will give the difference of longitude.

THE AGE OF DISCOVERY

Determining position at sea was always a problem. In 1487, the Portuguese sailor Bartolomeu Diaz sailed south along the African coast. He went too far south to avoid storms and then sailed east. He went back north and found land, having passed the Cape of Good Hope unawares. In 1492, the first voyage of Christopher Columbus was along a fixed latitude from the Canary Islands. Columbus described many difficulties in determining his longitude position, and his calculations are very erroneous. Amerigo Vespucci, later, explained his methods for determining latitude and longitude, but the results he gave were obviously wrong and his explanations not very convincing. (He made four trips between 1497 and 1504.)

A contemporary of Christopher Columbus, Pedro de Medina (1493--1567) is the author of one of the first books about navigation in the open sea. His *Ars de Navigar* (1569), was very quickly translated into Portuguese, French, English and Italian.


Premiere reigle, quand le Soleil va à la partie du Nort.

Estant le Soleil du coſté du Nort, ſi les ombres ſont au Nort, vous eſtes au Nort, & le Soleil eſt entre vous & la ligne Equinoctiale, regardez combiẽ de degrez de haulteur vous auez prins, & combien il s'en fault pour faire nonante, & adiouſtez la declinaiſon de ce iour auec les degrez qui ſeront moins: & d'autant ſerez vous eloigné depuis la ligne vers le Nort.

Figure 5: First rule when the sun is going to the North.Pedro di Medina. (1569) *L'Art de naviguer*, trans. Nicolai, Lyon: Roville (ed), pp 97-98.

The sun being on the North side, if the shadows are at the North, you are in the Northern hemisphere, and the sun is between you and the equator; watch how much degrees in altitude you've measured, how much it needs to make ninety and add the

> declination of this day to the degrees which will be less: you will be as far away from the line northwards. (pp. 97-98)

This is pretty self-explanatory. The sun being on the north side meaning north of the equator and the declination of the day would have to be found from tables.

> Example
>
> The sixth of April, taking the altitude of the sun, the shadow comes to fall to the North, and finds the sun in sixty degrees on the astrolabe, ten degrees of declination which the sun has this day. (pp. 97-98)
>
> What is the latitude? (Solutions to all problems are at the end.)

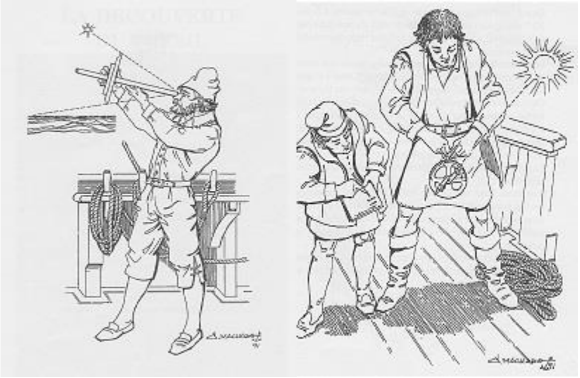


Figure 6: Use of the cross staff and astrolabe (private collection)

To measure the altitude of a celestial body, different instruments were used (see figure 6), some taking a sighting of the sun and so dangerous for the eyes; observers were frequently affected by blindness.

Figure 7: (By courtesy from the association *La Méridienne*, Nantes)

It is also important to know how far the ship has voyaged. Until the nineteenth century, speed at sea was measured using a ‘chip log’ (see figure 7) which was a wooden board cast over the side. The board was attached with a rope and knots were made in the rope. To measure the speed of the ship, one sailor counted the knots as the rope passed between his hands while another marked the time using a sand glass. The number of knots in 30 seconds was the speed of the vessel.

As long as the navigation remained coastal, it was essentially a question of giving the pilot enough information about his position to avoid the dangers of the coast. This need gave rise to numerous maps, so called portolans, with information about ports and straight line directions. They were not very accurate and were often ornamental. They were mostly useful for coastal navigation. These portolans did not take account of the curvature of the Earth and to correct this Mercator proposed a map where meridians are parallel and spaced out so as to preserve the angles. Mercator's projection is still in use today.

In straight line navigation, called dead reckoning (also dead reckoning from deduced), the navigator finds his position from the course taken and the distance sailed from the starting point, whose location is known. The differences of longitude and latitude are then easy to calculate (assuming a plane surface).

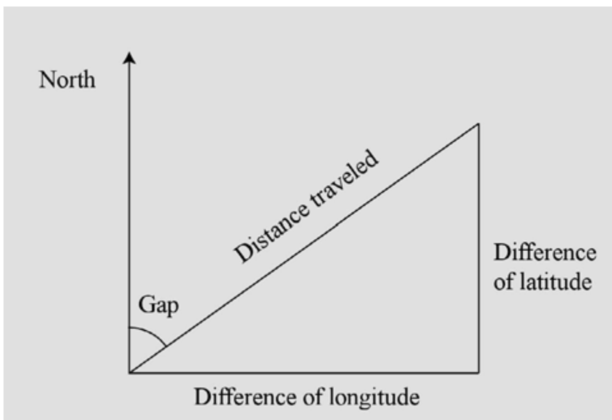
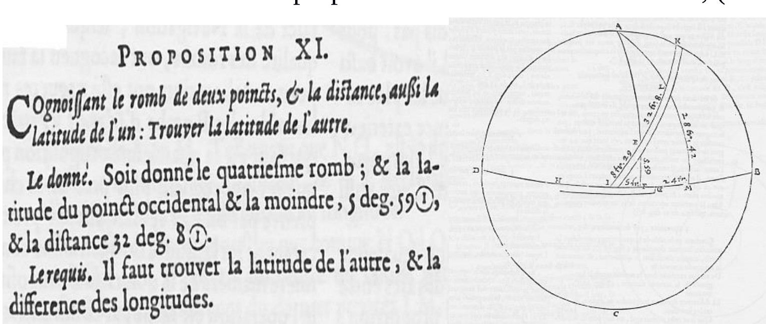


Figure 8: Dead reckoning

THE SEVENTEENTH CENTURY

Simon Stevin, more scientist than sailor, is the author of a work on navigation where the triangle of position appears for the first time, which will later be the foundation of the calculation of location. This is proposition XI from *L'histiodromie*, (1634):



PROPOSITION XI.

Cognoiſſant le romb de deux poincts, & la diſtance, auſſi la latitude de l'un : Trouver la latitude de l'autre.

Le donné. Soit donné le quatrieſme romb ; & la latitude du poinct occidental & la moindre, 5 deg. 59①, & la diſtance 32 deg. 8 ①.

Le requis. Il faut trouver la latitude de l'autre, & la difference des longitudes.

Figure 9: Stevin, S. (1634). De l'Histiodromie ou cours des navires, *Géographie, Livre 4*, trans. A.Girard, Leiden: Elsevier.p163

> Knowing the rhumb between two points, their distance and the latitude of the first one, find the latitude of other one. We give the fourth rhumb, the latitude of the first point, 5° 59', and the distance 32° 8'. Find the latitude of the other one and the difference of longitudes. (pp. 163-164)

See the figure. The fourth rhumb is the fourth direction starting from north towards the east, in divisions of one-eighth of a right-angle, that is bisecting NNE and NE. For 42° 37' of distance, the same table gives 28° 42' of latitude. In fact, according to the latitude, the degrees of longitude do not represent the same distances. (Solution to the end of the article)

During this century graduated paper, called a ‘quarter of reduction’, (see figure 10) began to be used. If the latitude and the longitude of the starting point are known and the distance sailed on a constant course has been measured, the differences of latitude can be found. For the fourth rhumb, the table of Stevin gives 6^o of longitude and 8^o 29' of distance from which the longitude is easy to obtain, in plane approximation, from graduated paper.

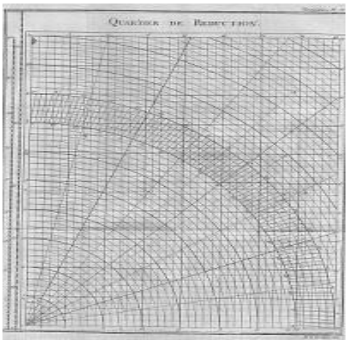


Figure 10: Engraving from Bézout, E. (1781). *Traité de Navigation*, Paris: Pierres (ed).

However, by the end of the century, the problem of determining longitude had still not been resolved.

THE EIGHTEENTH CENTURY, THE CENTURY OF INSTRUMENTS

On 22 October 1707, an English fleet ran aground on rocks close to the Scilly Islands off Cornwall; the cause was due to an inability to calculate its position. This accident led the British Government to propose a prize of £20000 in July, 1714 to anyone who could find an infallible method to determine longitude at sea. The solution lay in having an accurate timepiece. In 1735, the Englishman John Harrison invented the first clock capable of keeping at sea the time of the starting point. Its difference with local time allows one to determine the longitude. At the end of the century, the use of chronometers improved the determination of longitude.

Concerning the measures of angles, the introduction of another instrument, is also important.

We imagined in England a new instrument incomparably more perfect than those we have just talked about. The late Mister Hadley proposed it to the Royal Society of London in 1731; the use has already been introduced in France and it would be relevant if it could be even more common: because this instrument can give the altitude of Celestial bodies within one minute error as I checked it several times. It is a simple portion of circle of 45 degrees: we called it an Octant, because it is the eighth part of the circle, but it is divided into 90 parts, and it is equivalent to a quarter circle because of the property common to the mirrors that are used in its construction.
(Bouguer, (1753) *Nouveau traité de*, Paris: Guérin-Delatour. p.46.)

Why is the Octant equivalent to a quarter of a circle? (see figure11, solution at the end.)

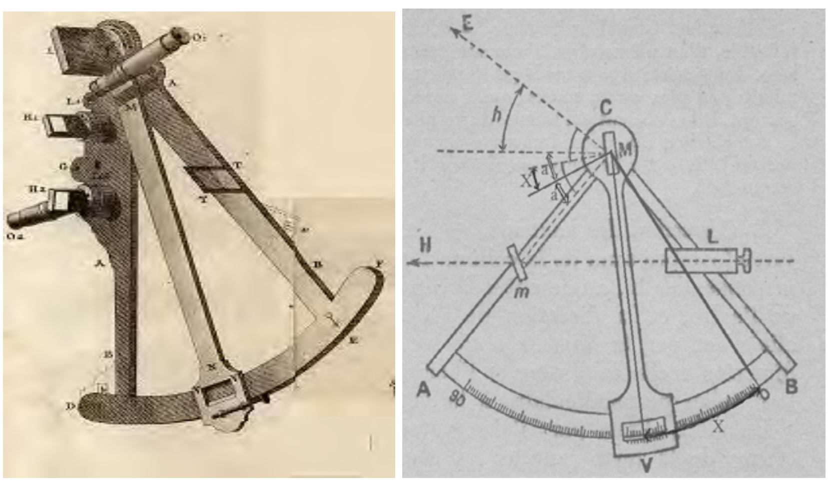


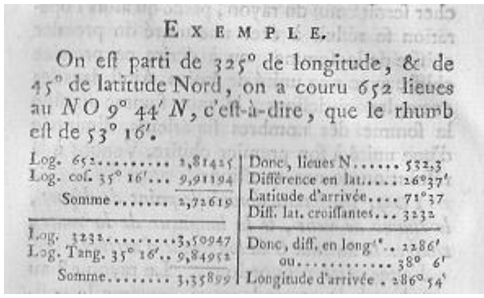
Figure 11: Sextant and octant , engraving is from Lévêque P. (1779). *Guide du Navigateur,* Nantes: Despilly.

By 1780, the octant and sextant (a sixth of a circle) had almost completely eliminated all previous instruments. With the sextant, the precision of the measures allows us to take into consideration the height of the observer, refraction and parallax.

In 1781 appeared the last volume of a mathematics course ‘à l’usage des gardes de marine’, of *Traité de Navigation* from Etienne Bézout. This remained a standard reference in France for many years.

> Given the starting point, the rhumb line, and the distance, find the latitude and the longitude of the arrival. The radius is for the distance as the cosine of the rhumb is for a fourth term which will be the way made on the north-south line. Reduce to degrees and you will have the change of latitude, thus the latitude of arrival. Look in the table of the increasing latitudes, the difference of the increasing latitudes between arrival and

departure, then do: the radius is for the tangent of the rhumb as the difference of the increasing latitudes is for the difference of longitude. (Pp.101-102)



EXEMPLE.

On eſt parti de 325° de longitude, & de 45° de latitude Nord, on a couru 652 lieues au *NO* 9° 44′ *N*, c'eſt-à-dire, que le rhumb eſt de 53° 16′.

Log. 652.......... 2,81425	Donc, lieues N...... 532,3
Log. coſ. 35° 16′... 9,91194	Différence en lat..... 26°37′
Somme....... 2,72619	Latitude d'arrivée.... 71°37
	Diff. lat. croiſſantes... 3232
Log. 3232..........3,50947	Donc, diff. en longᵈᵉ.. 2286′
Log. Tang. 35° 16′.. 9,84952	ou............ 38° 6′
Somme........ 3,35899	Longitude d'arrivée . 286° 54′

Figure 12: Bézout, E. (1781). *Traité de Navigation*, Paris: Pierres (ed), p102

By the way, there is a typographical error, it should be 35°16 and not 53°16.

For the latitude: take R=109, 1°=20 miles and 1'=1/3 mile.

For the longitude: from the table 71°37' gives 6262 and 45° gives 3030 with a difference: 3232.

R/tan35°16' = 3232/(difference of longitude), gives 2286 for this difference, 38°6' on the table.

With the sextant, the precision of the measures allows to take into consideration the height of the observer, the refraction and the parallaxe.

THE NINTEENTH CENTURY

With better educated sailors it became possible to use spherical trigonometry for navigation. To find the latitude and the longitude of a point C of the terrestrial sphere, a method is to resolve the spherical triangle ABC, where A is the north and BC the distance travelled. This is the ‘triangle of position’ (see figure 13).

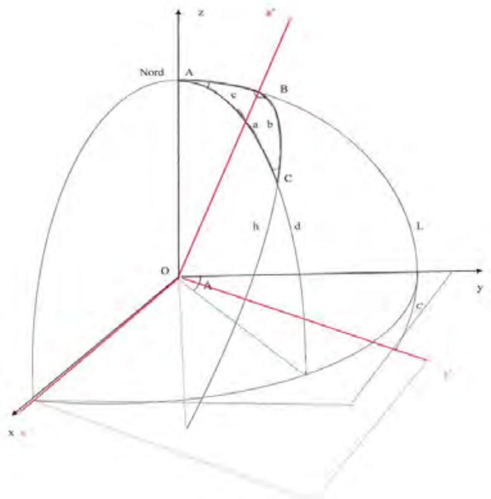


Figure13: Triangle of positio

Such a spherical triangle ABC satisfies the following angular formulae.

sinA.sinb = sina.sinB

sina.cosB = cosb.sinC – sinb.cosA.sinc

cosa = sinbcosAsinc + cosb.cosc

sinh = cosd.cosAcosL + sind.sinL

Show that cosA= (sinh – sind.sinL)/cosd.cosL (Solution at the end.)

In 1837 the American captain Thomas Sumner realized that when he observes a celestial body (the sun) under a certain angle, he is located on a circle, whose representation of an arc on a map can be assimilated to a straight line segment, called the position line.

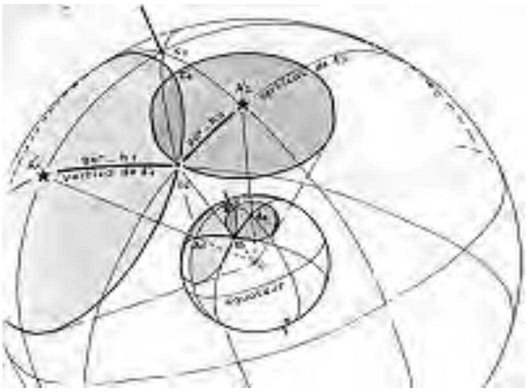


Figure 14: From *Revue Maritime*, (1975) 307, p.102, 1975

When repeating the operation, we can obtain a second segment, and at its intersection with the previous one, the place of the observer. The method was published in *A New and Accurate Method of Finding a Ship's Position at Sea* in 1843 (pp.16-17, see figure 15).

On 17th December, 1837, sea account, a ship having run between 600 and 700 miles without any observation, and being near the land, the latitude by dead reckoning was 51° 37′ N., but supposed liable to error of 10 miles on either side, N. or S.; the altitude of the sun's lower limb, was 12° 02′ at about 10½ A.M., the eye of the observer being 17 feet above the sea; the mean time at Greenwich, by chronometer, was $10^h\ 47^m\ 13^s$ A.M.

Required, the true bearing of the land: what error of longitude the ship was subject to, by chronometer, for the uncertainty of the latitude: the sun's true azimuth.

dip.	— 4′ 3″	semi-dia.	+ 16′ 8″
refra.	— 4′ 23″	px	+ 8″
	— 8′ 26″		+ 16′ 16″
			— 8′ 26″
		correction of alt. obs'd.	+ 8′ 00″

Obs'd alt. ⊙ L. L.		12° 02′
Correction	+	8′
True alt. ⊙'s centre,		12° 10′

1st. The latitude by dead reckoning was 51° 37′ N.; he latitude the next degree *less*, without odd minutes, is ›1° N.; and that, the next degree *greater*, is 52° N.

2d and 3d. Find the longitude of these two points, as ollows:

⊙'s ALTITUDE 12° 10′.

For the point A in latitude 51° *N.*

Lat.	51°	N.	- - - - -		sec. 0·20113
Dec.	23 23	S.	- - - - -		sec. 0·03722
Sum.	74 23		nat. cos.	26920	
⊙ Alt.	12 10		nat. sin.	21076	
			diff. - -	5844	log, 3·76671
H. M. S. 1 43 59		from noon =	log rising	=	4·00506
12 hours.					
10 16 01		app. time at ship.			
— 3 37		equa. time.			
10 12 24		mean time at ship.			
10 47 13		do. by chro.			
34 49		= 8° 42¼′ west of Greenwich.			

Figure 15: Sumner T.H. (1851). *Finding a ship position at sea*, Boston: Groom & Co, pp. 16-17

After that, he carries out the same calculation for the point of 52^{o} latitude, to obtain his longitude; the two points give a right segment.

In 1875, Marcq de Saint-Hilaire proposed a simpler method to draw a position line, having noticed that this one was to be perpendicular to the direction (the azimuth) of the observed celestial body. By taking an estimated point, the difference between the altitude of the celestial body observed and the one which would be obtained from the estimated point, we obtain the distance between the estimated point and the straight line that can thus be drawn. We can draw a second straight line and obtain the position of the ship on the chart. This method remained in use until the end of the twentieth century.

THE TWENTIETH CENTURY

At the beginning of the century there was an investigation into the use of radio waves for determining position. Broadcasting antennae, with a reach of 200 miles were erected. For example, the antenna of the island of Sein, Brittany, remained in use until 1911. Every antenna transmits a signal on a certain wave length. By directing the receiver, not to have the maximal reception, but on the contrary but to get no reception, one obtains by alignment the direction of the antenna, thus a first straight line. A second radio transmitter will give a second line. The method is called radiogoniometry. For confirmation, a third one will determine, at worst, a triangle. Radio transmission is no longer used for navigation, but other systems are still used, such as Decca.

The `beep-beep' emitted by the Sputnik gave the idea of using such emissions to find the direction of the station. From 1964, the American TRANSIT system could provide a location of a receiver from a satellite. However this system and the others of the same kind used a restrictive number of satellites, and their passages were spaced out too much.

For military purposes, the department of the defence of the United States envisaged from 1968 a system allowing them to localize any point on the earth, all the time and in real time. Its conception dates from 1973, and has been developed around 24 satellites (2 more in reserve) which constantly emit signals allowing the determination of the location of every receiver, as well as the receiver of the administrator of the system. The first satellite was sent into orbit in 1978 and the system has been operational since 1995. The satellites are periodically renewed.

The receiver measures the distances between it and the various satellites from which it receives information. These distances position it on circles whose intersection provides its position. This ‘Global Position System’ (GPS), makes all calculation methods redundant but they are still taught today. For safety?

SOLUTIONS

Example from Pedro de Medina:

The sixth of April, taking the altitude of the sun, the shadow comes to fall to the North, and finds the sun in sixty degrees on the astrolabe, thirty degrees are needed to reach ninety; I close with these thirty degrees, ten degrees of declination which the sun has this day, which are together forty degrees of which I am away from the equator to the North.

Exercise from Simon Stevin:

I look in the table of fourth rhumb and find, for the given latitude, the longitude 6^o and the distance 8^o 29'. I add the given distance (add, because the given latitude is the less, the start), it is 40^o 37'. I look in the table of the fourth rhumb, I find that it agrees with the required latitude of the second point 28^o 42'. By joining the longitude 30^o, among whom the difference with 6^o (first one in the order) we obtain the difference of the required longitude 24^o. The proof is obvious. Conclusion: knowing the rhumb of two points, the distance and the latitude of the one, we found the latitude of other one, which was asked.

Geometry of the sextant:

When the mirror goes from the initial position (measure 0) to the position of measure of altitude of the sun (or the star), it makes an angle X. That also makes the perpendiculars.

With the refraction property:

$$X+a = h+a-X \quad \text{and} \quad 2X = h$$

Calculation of the latitude by Bezout:

R/652= cos35°16'/NS with R = 109 and NS = difference of latitude

If NS = 532.3 with 1° = 20 miles and 1' =1/3 mile, result = 26°37'

Formulae in a spherical triangle:

By writing the coordinates of the point C in the position Ox'y'z' directly then from coordinates in Oxyz and from change of position by rotation of angle C around the axis Ox, the identification gives the three formulae.

The last gives: $\cos A = [\sin h+\cos(d+L)].\sec d.\sec L -1$

Text from Sumner:

In the first part, the author applies to the measure of the altitude of the sun the used corrections. Then, he calculates the secants of the latitude and the declination, the cosine of their sum, the sine of the height to apply the previous formula. A last correction allows to have the time of the boat, thus the longitude, if the latitude is 51°.

The author thanks Chris Weeks for his help in preparing this paper.

REFERENCES

Barbin, E. (1987). Le secret des longitudes et le pendule cycloïde de Huygens in *les Mathématiques dans la culture d'une époque*, (pp.143-163), Strasbourg: IREM.

Bouguer, P. (1753). *Nouveau Traité de Navigation*, Paris: Guérin-Delatour.

Bézout, E. (1781). *Traité de Navigation*, Paris: Pierres (ed).

Célérier, P. (1951). *Technique de navigation*, Que sais-je 498, Paris: PUF.

De Medine, P. (Pedro di Medina). (1569) *L'Art de naviguer*, trans. Nicolai, Lyon: Roville (ed) 97-98.

Dubois, E. (1869). *Cours de navigation et d'hydrographie*, Paris: Arthus-Bertrand.

Grégoire, M. (1996) Qu'est-ce qu'un rumb? in *Mémosyne* 11, (pp.4-8), IREM Paris VII: Paris.

Hébert, E. (2004). *Instruments scientifiques à travers l'histoire*, Paris: Ellipses.

Lévêque, P. (1779). *Guide du Navigateur*, Nantes: Despilly.

Marguet F. (Cdt) (1931) *Histoire générale de la navigation du XV^{eme} au XX^{eme} siècles*, Paris: Editions maritimes et coloniales.

Raymond, F. (1949) *Radionavigation*, Que sais-je 41, Paris: PUF.

Sacaze, (Contre-Amiral). (1975). La navigation astronomique, in *la Revue maritime, 301-305*.

Scot, C. & Malin, S. (1989). *The Greenwich Meridian*, Southampton: Ordnance Survey

Ségéric, J-J. (2013). *Histoire du point astronomique en mer*, Rennes: Marines Editions.

Sobel, D. (1995). *Longitude*, Paris: Points Sciences S126.

Sonnet, H. (1867). *Dictionnaire des Mathématiques appliquées*, Paris: Hachette.

Stevin, S. (1634). De l'Histiodromie ou cours des navires, *Géographie, Livre 4*, trans. A.Girard, Leiden: Elsevier.

Sumner, T.H. (1851). *Finding a ship position at sea*, Boston: Groom & Co.