Workshop

ORIGINAL SOURCES IN TEACHERS TRAINING POSSIBLE EFFECTS AND EXPERIENCES

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The advantages of using original sources in school mathematics have been discussed widely. Within academic studies for prospective teachers, history of mathematics is also attributed an important role, with a great variety of possible implementations. At Siegen University the integration of history of mathematics is one of the key features: In addition to special courses on history of mathematics, we integrate historical aspects and work with original sources in didactics courses as well as in lectures on mathematics and elementary mathematics. From a normative point of view, working with original sources can contribute to widely accepted aims of teachers' education, such as understanding the process character of mathematics and fostering authentic experiences in mathematical research on an elementary level. Furthermore, the alienation by historical sources can provoke cognitive conflicts that can enhance the perspective on current mathematical processes of conceptualizations. After discussing advantages and problems of an approach to original sources on an abstract and theoretical level, we will present concrete examples and experiences from courses on elementary mathematics as well as didactic seminars, which show the broad use of original sources.

At Siegen University, there is a long tradition of courses on history of mathematics and implementing an historical perspective and historical sources in the context of teachers' education. In our paper we want to present and discuss these experiences. We believe that the general context of the course – university mathematics, elementary mathematics, mathematical didactics or courses on history of mathematics – is of crucial relevance for the impact made. In our paper, we therefore discuss advantages and problems of an approach to original sources first on an abstract and theoretical level. Naturally, our personal experiences influence this account, however they are in accord with empirical results from studies focusing on various levels of mathematics education at school and at university. With that said, we will present concrete examples and experiences from courses on elementary mathematics as well as didactics seminars. In each case, we will focus on the connection between context and aims. By this, we want to thoroughly differentiate between different impact on students, which original sources can have, and propose a possible classification.

USING HISTORICAL SOURCES – GOALS AND OBSTACLES

> "It is unbelievable how ignorant the young students come to the university. If I calculate or do geometrical constructions for only 10 minutes a quarter of the audience will fall asleep."

We started this section with an original source[1], a quote from 18th century – written by Georg Christoph Lichtenberg, mathematician, physicist, philosopher and man of letters. Its content, however, fits well into current debates. These debates articulate a widely spread dissatisfaction with the level of General Mathematical Education (Allgemeinbildung). In particular, this refers to

- a robust and flexible elementary knowledge,
- the propaedeutics: first year students of mathematics and other disciplines heavily relying on mathematics do not seem to be well prepared to their studies,
- but also - which is often forgotten, but important to add - with respect to reflection, to the ability of judging about mathematics and its role for cultural history and for present societies.

For the universities and their mathematics (or physics or engineering) departments, the second type deficits might be the easiest to compensate by just implementing specific first year courses for necessary training. Concerning the other two, it is only the schools (elementary as well as high schools) where this knowledge could be imparted. Certainly, it is the duty of schools to teach the elementary knowledge. But also the ability of reflection *about* mathematics stems mostly from school experiences. All other efforts of popularization are based on these first experiences[2].

An all too easy implication of the above observation could state that our school education is insufficient, or pupils as well as mathematics teachers are simply to stupid for mathematics. However, a slightly more sophisticated analysis might ask for the quality of mathematics teachers' studies – thus the destination of the critique changes to the universities and their professors: The quality of teachers' studies is in fact one key element of the problem.

Here an integration of history as well as philosophy of mathematics3 - within the special focus of our paper based on original sources - can help in various ways to improve these studies. In the following we will substantiate this claim. In the first part of this theoretical introduction we will focus on historical sources as **a mere tool f**or better teaching mathematics. The aim is thus fostering a deeper understanding of the mathematical content. Moreover, we will also shed some light on certain limits and risks of this usage. In the second part, however, we will illustrate why and how knowledge in the reflection discipline is also **a goal in itself**.[4]

History as a tool

We start with describing different ways how history of mathematics could be used as a teaching tool. The most frequent usage of the history of mathematics is **telling anecdotes**. Sometimes also a professor feels the necessity to pep up his boring stuff.

Anecdotes are living from the contrast to the “normal presentation“, the emphasis lies, e.g., on the biographic and personal, the human touch. **Original sources** in this function usually come along as **longer or shorter quotes** (see, e.g., our initial quotation). A specific form is the **comforting usage**. Here one might tell the students about great mathematicians struggling for years and years before they found a solution or the fruitful concepts. So, the beginner should not be too much frustrated if it takes weeks or even months for him to understand the given solutions.

A problematic variant of the anecdotal type presents a purely **heroic picture**. Mathematical development is thus created by inimitable geniuses. Living in the shadow of these giants there is no way for own mathematical activity. The contrary picture is equally unsatisfactory. We will call it the **jovial picture**. Here the historical progress is overstressed, we look back from our state of the art and describe in a more or less patronizing way what former times “already knew“. We thus completely forget the historical context.

The next form is much more ambitious. Going back at least to Otto Toeplitz (1881-1940) it might be called the **genetic use**. Here the historical-genetic approach is only one facet of a more general concept being in contrast to a ‛deductive’ or ‛formalistic’ presentation of mathematics. Within the historical-genetic approach one can distinguish an explicit and an implicit form. An **implicit historical-genetic** approach relies on some historical knowledge on the side of the teacher but does not explicitly present the history of the mathematical subject. The main function of historical awareness is becoming sensitive to barriers of understanding which were historically effective and therefore probably also for the individual learner. In contrast, the **explicit form** will present the mathematical subject by a presentation of its historical development. Using **original sources** can thus almost automatically be classified as explicit. If, however, the historical process is presented all too exact this form can become rather problematic if not counterproductive. The complexity of the history of mathematics – containing errors, dead end streets, deserted roads (from our point of view!), intended and now forgotten applications etc. - might become quite *confusing*, especially for the beginner. The explicit-genetic tool may then be switched into an autonomous goal. Though it does not make the *learning* of mathematics easier, we could obtain a more substantial picture of *mathematics* by knowing the historical way. We will come back to this later.

A specific function for students’ future teaching duties is the **de-familiarizing effect** of historical sources. Especially with respect to the elementary school a student normally does not have a vivid memory of his own learning processes. Here the historical context can change the appearance of elementary mathematics in a way that students are forced to repeat the learning of the basic concepts. Moreover, knowing the historically realized alternatives we see that, e.g., our way of denoting numbers and calculating is by far not the only or even the canonical way of doing it (cf. the aspect of “reorientation” (Jahnke et.al., 2000)). And, finally, we learn to appreciate the value of a historically grown clever notation.

Finally, we would like to mention the **paradigmatic use** of history. In most cases teachers’ students will hardly come into contact with authentic, actual mathematics –

or if so, they will probably experience only total ignorance. Intensively discussing a historical example - based on original sources - can bring them into successful contact with authentic, though not contemporary, mathematical research. In general the mathematical difficulties will shrink with the time distance but the historical difficulties will grow. Therefore one will make some concessions with respect to the historical precession, e.g., will use **translations** and **secondary sources**. Within this paradigmatic use another most important experience is possible namely learning about interdisciplinary connections of mathematics within authentic settings (e.g. Habdank-Eichelsbacher B. & Jahnke H.N. (1999)).

History as a discipline for reflection

Besides the above discussed variety of supporting function there is an independent role of history of mathematics for any teacher's studies. First of all, it is a mere triviality that any faithful picture of mathematics encompasses the genesis of its problems, its concepts and results thus the history of the discipline. Really *knowing* mathematics in a proper sense implies knowledge of its history and this is more than a mere collection of historical facts – a systematic orientation about the history is needed. Besides a training of mathematical skills any school education should provide also this type of knowledge[5]. In that spirit a **historico-critical perspective** on mathematics will present a historically grown discipline and therefore the history of mathematics as an autonomous subject. Among others one goal will be showing that there is usually a large contrast between the canonical, systematic and often formalized version of mathematics taught in our lecture halls and the much more involved, in part also quite chaotic history of its genesis. Only with respect to that background we can appreciate the huge accomplishment of that systematization, but also the losses[6]. Moreover, it is our conviction that questions for motivation and heuristics within mathematics itself can hardly be discussed without reference to history.[7] The adequate presentation and discussion of any subject within the history of mathematics will require much more than telling a nice anecdote, it takes time and a special sort of diligence - and it will almost necessarily refer to original sources. On the side of the professor at least some professionalism is needed and on the side of the students a solid and flexible, though not yet a historical knowledge of the mathematical subject.

Widening the horizon we observe that mathematics is not just important for the history of sciences – but it is an essential power influencing the whole **history of ideas**, the political, cultural and social history (cf. the aspect of "cultural understanding" (Jahnke et.al, 2000) in a broad sense). This aspect is often neglected or dramatically underestimated. Certainly this is due to the fact that historians in general do not look at mathematics and mathematicians and also historians of mathematics do not transgress the boundaries of the mathematical discipline. A central question for teachers-to-be, namely a justification for the choice of the mathematical subjects and the form of presentation in the classroom, however, can only be answered with reference to this cultural history. Of course, neither the school curriculum nor the university studies could encompass a complete picture of the role of mathematics for cultural history. Therefore we should teach and learn by examples or case studies. For this „macro-perspective“ probably the use of original sources might be a to narrow look and **secondary texts** might be more suitable.

USING HISTORICAL SOURCES – EXAMPLES AND EXPERIENCES

In Germany, courses for prospective mathematics teachers do not only differ in content (geometry, algebra, calculus and so on) or method of teaching (lecture, seminar, working group). They are also very different concerning the perspective on the mathematical content and its impact for the teaching and learning of mathematics. In Siegen, alongside to the regular courses on advanced mathematics, we present courses on elementary mathematics – e. g. school-mathematics from an advanced point of view[8] –, courses on mathematics education as well as courses on history and philosophy of mathematics.[9] In each of these parts of teachers' education a fruitful integration of work on original sources is possible – of course with different goals, impact and obstacles like described above.

In courses on history and philosophy of mathematics original sources are used quite naturally, e.g., to train text apprehension. At the university of Siegen, a seminar in history of calculus will require the students to present the important steps in the development of analysis by reference to sources from Euclid, Archimedes, Torricelli, Leibniz, Riemann and others …

Students will work intensively with original sources, when writing a master thesis on history or philosophy of mathematics. Among others, Siegen's students wrote theses on Bernard Bolzano's *Paradoxes of Infinity*, George Berkeley's *Analyst*, Blaise Pascal's *Wager*, Lewis Carroll's (1972)[10] *Symbolic Logic* and *The game of logic* or infinite series in the middle ages using the example of Nicole Oresme.[11]

In the following we will describe our experiences by using original sources. We will present an example for each of the different parts of teachers education mentioned above, to show the wide range of possible applications and will discuss the similarities and differences concerning the sources impact on teachers' education. In order to avoid problems with different degrees of difficulty or different periods of development, we focus on the same mathematical discipline throughout all examples. Thus all original sources used are well known historical texts concerning real analysis. Therefore the sources will not be analyzed and described in detail. Our focus will lie on their function in teachers' education and their possible impact according to the different types of courses.

Euler's Introductio in a lecture on real analysis

Infinite sequences and series are an established content in lectures on real analysis. For students, however, it is the first time they are confronted with the mathematical concepts of infinity and convergence in a rigorous way. Due to its prominent position in curriculum, raising the students' awareness for possible obstacles and ways of handling them (mathematically) has to be the main goal. One possibility to achieve this aim is to take a glance at the concept's history. Original historical sources give authentic impressions of history (of mathematics) and an illuminating contrast to the current and rather technical ways of handling the concepts. Moreover they emphasize the importance of a more or less intuitive way of grasping a concept, especially if the sources mark the state of the art at their time.

Getting to know the genesis of key concepts is not only important for prospective teachers, like the discussion about parallels of historical and individual epistemic "roots" may show (cf. Furinghetti, 2007, p. 133) , but for all mathematics students. We are convinced that - on the one hand - there is a positive effect on the process of understanding. On the other hand as discussed above, knowing at least a little about mathematics' history has to be part of *Allgemeinbildung*. So, historic facts as a self-contained content should be compulsory within every lecture on advanced mathematics. One way of initializing resp. supporting a deeper understanding as well as a mathematic-historical *Bildung* is using original sources (cf. Jahnke et. al., 2000). For example, within a lecture of real analysis, an extract of a textbook by Leonard Euler (and additional information on the historical background) can be used as a "tool" to enlarge the comprehension of the modern notion of infinity as well as a "goal" for mathematical *Bildung* at university (cf. Jankvist, 2009). The following extract of a homework on infinite series and the logarithm may concretize both:

Read chapter 7 of Leonard Euler (1748): Introductio in analysis infinitorum[12].

a) Investigate the historic background of Euler's book.
b) Read chapter 7 carefully with 'paper and pencil': Which assumptions does Euler make? Explicate the goal of the chapter.
c) Which series expansion does he use for the logarithm? What does the text say about the connection between exp and ln?
d) Mark at least two passages in which Euler's handling of infinitely small resp. big quantities could be described as intuitively (e.g. in which Euler's handling doesn't work correctly with respect to our contemporary rigor).

In part a) the students are asked to do, what a historian would do quite naturally when analyzing an original source. However, within a lecture on advanced mathematics, this modus operandi is quite unusual and therefore has to be initialized carefully. By embedding Euler and his oeuvre in mathematics history in general and the given text in the history of real analysis in this particular case the students get valuable insight in the genesis of their discipline. The autonomous research of the historical background within an exercise focusing on the mathematical content as well, will naturally link the historic facts with the mathematical notions of the lecture. Furthermore it will raise awareness for the differences to the modern presentation and therefore will help to avoid treating the text anachronistically or even in a completely ahistorical way.

While part a) is a contribution to the aspect of mathematical *Bildung* as described above, parts b) and c) focus on the underlying mathematical content. Due to the historic and therefore unfamiliar notation it will be difficult for the students to identify the concepts known from the lecture or current textbooks. They are forced to go through the text step by step – using the modus of a historian once again. And it won't suffice to grasp the modern notions only in an algorithmic way. The effect of alienation by the original source is used to initiate a process of repetition and deeper comprehension of the subject matter. Reading carefully through the text, the students will face obstacles, since Euler's approach to the infinitely small resp. large quantities contains some confusing passages. It can be described as quite informal, however successful – i.e. his use of infinitesimals and his notion of convergence doesn't correspond to today's rigor but the results do. Part d) should draw attention to these differences. To identify the crucial passages the students need a flexible comprehension of the current concept of infinite series and their convergence. Conversely, Euler's way of handling provides informally accessible examples for the rigorous definitions but underlines the benefit of today's rigor as well – the students will meet examples where Euler's reasoning won't work. Altogether, working with the original source is supposed to initiate a deeper comprehension of one of the key concepts of real analysis.

Lecture on school calculus from an advanced point of view

The curriculum of the university of Siegen contains lectures on school mathematics from an advanced point of view. Similar to Felix Klein's (1932) concept of his *Elementary mathematics from an advanced standpoint*, the aim of these lectures is to embed school mathematical contents into its mathematical background on a higher, more rigorous level. But contrast to Klein, the intention is not to dig into advanced mathematics deeply and to show possible contiguous research fields, but stick very closely to the actual school mathematical content and to focus on the multiplicity of various aspects and points of view.

A main part of the lecture on school calculus deals with extreme problems and the variety of methods to solve them – with the differential calculus as well as by geometric and algebraic methods. In order to compare the slope and the opportunities of the different methods, the isoperimetric problem for rectangles came in handy as a productive and versatile example. After presenting and discussing this well known antique problem, the students were given an extract of a translation of Pierre de Fermat's *Oeuvres I*, accompanied by the following assignment:

1. Explain, how the example, that Fermat uses to illustrate his method can be identified as the isoperimetric problem for rectangles.
2. Read Fermat's solution carefully and describe his method in your own words, using the modern notation.
3. Fermat is led by the intuition that function values hardly differ in the neighbourhood of a maximum (or minimum).
    a) Is Fermat's intuition viable – that is, does it characterize extrema sufficiently?
    b) In some way, Fermat's method and the modern method of differential calculus are led by the same intuition – explain!

As a main goal, in part 1 and 2, students will become familiar with a historical argumentation that varies from the rigorous explanation, they are used to. Similar to the usage of original sources in lectures on advanced mathematics, the effect of alienation by the original source is used to initiate a learning process on another level. It wasn't easy for the students to identify Fermat's example with the already known isoperimetric problem, simply because the notation used is quite unusual.

By making the students describe the method used by Fermat, we accepted the hazard of an anachronistic interpretation. Indeed some students interpreted Fermat's procedure in the modern way. However, the assignment aims to focus on the benefit of intuition and core ideas that have been developed over the years (especially in part 3)[13]. The students were able to experience mathematics as a human enterprise and a developmental process on the one hand, but learned to value the intuition itself. This led to a fruitful discussion on the core ideas and the final (rigour) elaboration.

Seminar on subject matter didactics of calculus

Seminars on the teaching and learning of mathematics can discuss general topics like mathematical problem solving or mathematical learning processes as well as a special mathematical discipline. In either case, original sources can be used fruitfully: First of all students get to know examples of original sources that can be used in mathematics classroom as well.[14] Within seminars on general topics in addition, they can be used to show the procedural character of mathematics or to give authentic examples for mathematical modeling for example. Within subject related didactic courses, original sources are an appropriate tool to initiate discussions about the nature of the subject and the consequences for the teaching and learning. Original sources force to link mathematics to the situation at school and therefore persuade the students to concentrate on the subject matter *didactically*. The following assignment was given in a seminar on the teaching and learning of school calculus:

Students got a translation of *exercise one* from Johann Bernoulli’s *Lectiones de calculo differentialium (1691/92)*[15] (How to find the tangent on a parabola) and as additional information his *postulates* and the *rules about sums and products of differentials.* Some biographical information about the author and his oeuvre[16] and the following tasks completed the worksheet, on which they had to work during the seminar lesson in groups of three or four:

1. Read the excerpt from Johann Bernoulli’s „Vorlesung über das Rechnen mit Differentialen“ (1691/92) carefully (with ‘paper and pencil‘):
   a) Draw your own sketch while reading. Use the one of Bernoulli for aid.
   b) How can the quantities dx and dy be interpreted within this construction? What is the so called ‘subtangent’? Which geometrical considerations are used?
2. Compare Bernoulli’s text and current textbooks in school:
   a) What is the starting question resp. the starting problem in each text? How is the problem formulated and motivated?
   b) Which mathematical ‘tools’ (geometrical or algebraic considerations, a coordinate system, sketches, computer, …) are used?

Questions one a) and b) should help to analyze the source and to understand the underlying mathematical content. The questions help to focus on the geometrical character of Bernoulli's formulation of the problem and the solution. Even if all of the students know how to find the slop of a tangent in a given point of a quadratic function, it seemed to be this geometrical character of argumentation which make it difficult to grasp the arguments given by Bernoulli: Most students started to draw a coordinate system and reformulated the arguments by their notions of functions – which in fact is more complicated. Other difficulties arose when dealing with the infinitely small triangles.

In the previous lesson the students discussed different established approaches to the introduction of the derivative at school[17] and identified them within popular German textbooks. Question two leads the students to recapitulate these and to compare them with Bernoulli's approach. The aim of Bernoulli (finding a second point on the sketchpad to draw the tangent vs. determine the slope of the tangent in any point to get the derivative function) as well as the discussed objects (curves on a sketchpad vs. functions or graphs of functions) and the mathematical tools (Euclidian theorems about congruence of (infinite small) triangles vs. algebraic rules and the limit value of the ratio of the differences) are very different from the modern differential calculus. But on the second glance there are also parallels: In both cases, the parabola resp. quadratic function is used as an introductory example. Maybe less obvious, a deep comprehension of the concept is complicated by dealing with infinitely small quantities – actually this is often disregarded by prospective teachers because of being too familiar with the concepts in a way and especially because of the rather schematic and technical use of rules for calculating derivatives. The effect of alienation by the original source can help them making these obstacles relive and becoming aware of possible difficulties, their prospective pupils might have. Another parallel can open a discussion about different domain-specific beliefs of mathematics: Even if current textbooks often start interpreting the derivative as instantaneous rate of change, the argumentation is prominently supported by pictures of graphs of functions with their secants and tangents. This may lead to an "empirical belief system" (cf. Schoenfeld, 1985, p. 161), in which the function is identified with the sketch of the graph in the way Bernoulli did by arguing by means of the curve on the sketchpad. (Cf. Spies & Witzke, forthcoming) Of course a profound reflection about such topics needs additional information and quite some time of plenary discussion after working with the original source.

Main differences of the usage in the different types of courses

Some impact can be equally obtained in all kinds of mathematical courses. Whenever original sources are used in university courses, they contribute to mathematical *Allgemeinbildung*, as described above, by giving an insight into the historical development of the studied subject. Students will also always train the text comprehension and experience the alienation, when comparing the different representations, verbalizations and sometimes different underlying intuition.

However, we believe, that in every course the usage of original sources may have a different focus and therefore different impact:

In courses on *advanced mathematics* original sources will help to contrast the often deductively presented content by underlining the procedural character of mathematics. This might have a positive effect on the process of understanding the instructed notions.

Courses on *elementary mathematics* not only aim to embed the school-mathematical content into the mathematics taught at universities and to present it in an abstract and mathematically correct language, but to call the students attention to the multitude of different approaches to an object, definition, theorem or prove. History of mathematics provides us with a number of different approaches. The often intuitive presentations, help to emerge the core ideas, that may be concealed in the modern, deductively structured and abstract notation.

Within courses on *didactics of mathematics*, original sources can initiate discussions about the nature of the subject. In these courses the focus will lie on the discussion of consequences for the teaching and learning. On the one hand the occupation with original sources might help to allude to some epistemological obstacles (cf. Brousseau, 1997) and possible consequences for the mathematics education. On the other hand original sources force to link mathematics to the situation at school and therefore persuade the students to concentrate on the subject matter didactically.

HISTORY OF MATHEMATICS – A DEMAND WITH TRADITION AND FRUSTRATION

The positive impact on teachers' studies is well known and intensively used at certain places. However, these places are still rare and the history of its use is as long as the history of the resigned comments about the ignorance of the "rest of the world". It suffices to quote just a few examples: Vollrath (1968) pleads for a direct genetic method (see our examples for thesis subjects) and mentions a lack of time and knowledge on the side of the professors[18], Artman et al. (1987) complain that mathematics fails to use the excellent historic examples[19], while Jahnke et al. (1996, p. VIII) state that "[h]istory [...] has an indispensable role [...] However, in spite of these well-established reasons in favour of introducing history into the mathematics classroom, both in schools and universities, the idea has not been very successful, as yet. [...] Somehow, history is considered alien to everyday classroom work".

Finally, Mosvold et al. (2014) look back at Arcavi (1982) and state: "Furthermore, the role of history of mathematics in general mathematics education research appears equally limited today as it was back then." So it seems important to realize and address the question: Why is the role of history of mathematics in general so much underestimated for teachers' studies? Why is it so plausible for most mathematicians *and* mathematics educationalists to abstain from an historical approach? Of course, the spectrum of reasonable answers will be out of proportion to the scope of our paper. One important aspect, however, might be that it is indispensable to reflect about the various specific aims one could pursue by presenting historical sources to teachers' students. Here, the general context of the course – university mathematics, elementary mathematics, mathematical didactics, or history – is of crucial relevance as we tried to show by our examples.

NOTES

1 Quoted from Georg Christoph Lichtenberg (1983). *Aphoristisches zwischen Physik und Dichtung.* Vieweg, Braunschweig, p. 135 (author's translation).

2 For further reading about the role of reflection on mathematics and its integration into school curricula see for example Ole Skovsmose (1998) or Roland Fischer (2006).

3 In this paper we focus only on history of mathematics. In our workshop at ESU 7, however, we discussed also the impact of studying philosophical sources. For further reading cf. Nickel (2015).

4 With respect to this differentiation cf. Jankvist (2009).

5 This conviction is in accordance with the widely shared concept of Allgemeinbildung. With respect to mathematics see, e.g. Winter (1995).

6 It may be necessary to emphasize here that approaches to transform the bulk of mathematical knowledge into a systematically ordered form are almost as old as the history of mathematics itself (Euclid's elements being the most prominent example).

7 In accordance to this general perspective working with historical examples and reflecting on contrasts is also supposed to be helpful for getting aware of ones own strategies of doing mathematics - from a theoretical point of view (cf. Fried, 2007) as well as by empirical evidence (cf. Kjeldsen and Petersen, 2014).

8 This special course, established in Siegen, is based on the curse of Felix Klein (1908): Elementary mathematics from an advanced standpoint. Among a detailed analysis of Klein's oeuvre, differences of these two concepts are discussed in Allmendinger (2014).

9 The curriculum is not the same all over Germany. Elements of the described perspectives are established in nearly every German university, but with very different weights of the different parts and organizational forms. So for example there are universities where history and philosophy of mathematics are only optional parts of courses in advanced mathematics. And sometimes elementary mathematics and mathematics education are toughed together in the same course. Never the less the authors are convinced of the necessity of an equilibrated role within teachers education of all these perspectives (cf. Beutelspacher et. al., 2011).

10 Originally the *Symbolic Logic* was published 1862 and the G*ame of Logic was* published 1887.

11 Some more examples can be found in Beutelspacher et al. (2011).

12 We use the German translation of Maser (Berlin, 1885), made available as copy for the students.

13 This part of the assignment was inspired by the interpretation of Fermat in Danckwerts & Vogel (2005, pp. 94-97). Going deeper into the details of Fermat's approach one can argue that there are alternative - and even by today's standards rigorous - heuristics which could account for his procedure (we owe an anonymous referee this remark).

14 For examples of successful usage of original sources in mathematics classroom and the discussion of didactical strategies and possible impact see Jahnke et. al. (2000, pp. 307-316) or Glaubitz (2010).

15 Schafheitlin, Paul (ed.) (1924): Die Differentialrechnung von Johann Bernoullie aus dem Jahre 1691/92. Oswalds Klassiker der exakten Wissenschaft. Leipzig: Akademische Verlagsgesellschaft.

16 Due to lack of time within the seminar lesson the students are not asked to find this information on their own – even if this would be more helpful by means of the aim of Allgemeinbildung coupled with the use of original sources (see above).

17 To introduce the different interpretations again original sources can be helpful. Comparing the definition by August Louis Cauchy (Cours d'Analyse (1815)) with the one of Karl Weierstrass can open up the horizon on a well known subject and help to subsume the schools' notion of derivation into the historical development (cf. Danckwerts/Vogel 2006, pp. 45ff.). Furthermore the students get to know two important protagonists in the history of rigorous analysis.

18 "Die direkte genetische Methode erfordert natürlich viel Zeit im Unterricht. Sie setzt auch einen Lehrenden voraus, der selbst in der Entwicklungsgeschichte der Mathematik zu Hause ist. Darin sieht H. Behnke (1954) die Ursache dafür, da diese Methode an der Hochschule kaum praktiziert wird. (...) Auch die Schule praktiziert kaum die genetische Methode. Sie vergeudet damit eine Quelle für Motivationen von der Sache her."

19 "Weder der Musiklehrer noch der Deutschlehrer lassen sich von den Schwierigkeiten abschrecken: Die Schüler lernen an erstklassigen Beispielen unserer kulturellen Tradition, ihren Geschmack zu bilden. Hier hat die Mathematik ein Defizit, in der Schule wie auch in weiten Bereichen des Hochschulunterrichts."

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