A CABINET OF MATHEMATICAL WONDERS: IMAGES AND THE HISTORY OF MATHEMATICS*

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Most mathematics teachers agree that the incorporation of mathematics history into their instructional process would enrich the lessons. The one most common objection to this innovation is the expenditure of teaching time. Compatible but nonintrusive ways must be found to overcome this objection. One such method, which has proved successful, is the use of historical images in the classroom: both during the instructional process and as passive reminders of the human involvement with mathematics. This presentation introduces the concept, supplies illustrative examples and renders advice on the securing of appropriate images. In particular, the historical archive, "Mathematical Treasures" maintained by the Mathematical Association of America in its e-journal <u>Convergence</u> *is recommended*

INTRODUCTION

A few years ago while visiting the British Museum, I was attracted to a display case containing cuneiform tablets from the Old Babylonian Period (1800-1600 BCE). One tablet in particular attracted my attention; it was a palm size oval with several columns of characters. Consulting the information supplied about this tablet, I learned that it was a sexagesimal multiplication table. Here, four thousand years ago, a young student, probably a scribe in training was learning his multiplication facts. This realization impressed upon me the continuity of mathematics and its learning tasks over a period of 4000 years. But what would impress be even more deeply, was the imprint of a human finger that accompanied the numerals and was preserved in the hard baked clay surface. It jarred me both emotionally and conceptually. This mark served as evidence of the human involvement with mathematics, it reinforced the fact that a person, an individual, did this mathematics. Despite further years of study and research on the history of mathematics, the impact of this image affirming the need to acknowledge and attempt to understand the persistent human involvement with mathematics has remained with me. An old adage says that "One picture [or image] is worth a thousand words," I certainly believe this. The statement is not merely a cliché as neuropsychological investigations have shown that the majority of reality-based information processed by the brain is obtained visually. Further, much of this processing is affective-it influences attitudes. Educational practice has always utilized this facility. In particular, visual imagery and its interpretation is a vital part of mathematical discourse (O'Halloran, 2005).

One of my major involvements over my career has been to convince mathematics teachers, at all levels, to incorporate the history of mathematics, the record of human involvement, into their teaching. I feel like such an endeavor helps to humanize the subject, that is, remove its aura of mystery and better reveal mathematics as a natural, human, activity. Usually, teachers appreciate the implications of this association but then the practical task of just how to incorporate history into mathematics teaching arises. Teaching time is valuable and examination standards must be met. If historical insights are to be provided they must be effective and minimally intrusive. Working with teachers, we have explored several appropriate strategies including: use of occasional anecdotes, brief stories from the history of mathematics; employing historical related mathematical learning tasks in small group settings (Swetz, 1994) and the classroom assignment of actual historical word problems (Swetz, 2012). Generally, such attempts, if diligently undertaken, have produced favorable results. Still, the expenditure of "valuable" classroom time remains a deep concern for teachers. One effective solution to this issue is the use of visual centered displays: images and posters that pertain to relevant historical personages and achievements. Images can be used as part of the classroom instruction, illustrating, enriching and reinforcing the specific concept being discussed or as a passive learning aid, displayed to attract student attention, arouse curiosity and perhaps prompt further investigation. Extra credit reports can be assigned on an image-"Here is a copy of a page from a 1520, British geometry book. What do you recognize? What unusual things do you see?"

TO KNOW HOW TO SEE

Let me give two examples of instances where the use of historical images resulted in fruitful discussions and prompted further learning interaction. During the ESU4 session in Sweden, Leo Rogers gave a talk, "Robert Recorde, John Dee, Thomas Diggs, and the "Mathematical Artes" in Renaissance England" (Rogers, 2004) in which he employed several illustrations. One simple image, from Robert Recorde's *Pathway to Knowledge* (1551) excited the audience: "Where did you get that? How can I get a copy?" See Figure 1. The situation depicted is elementary: it shows a man ascending a ladder to reach the top of a tower. The geometry reveals a right triangle,

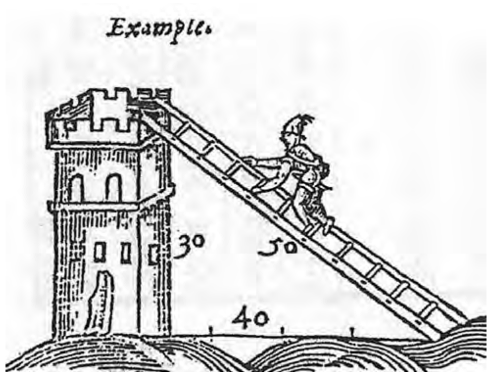


Fig 1: *Pathway to Knowledge*, Tower Problem

actually a 3-4-5 right triangle, and the viewer is asked to determine the length of the ladder when the height of the tower is given as 30 feet and the foot of the ladder is 40 feet away from the tower. What is the pedagogical appeal here?

- The ladder against a wall problem is familiar to almost all young algebra and geometry students.
- The illustration is from the 16th century-years ago, "they were doing the same problems as us five hundred years ago."
- The Pythagorean theorem is being demonstrated: the lengths of two legs of the triangle are given; the viewer must find the length of the hypotenuse.
- A 3-4-5 right triangle is involved.
- The illustration appeals to the imagination of a young viewer-a castle is being stormed.
- The more perceptive student would note that for better climbing safety, the bottom of the ladder should be placed closer to the base of the tower. This is a pseudo-realistic situation, contrived for the mathematical convenience.
- A follow up exercise for this scenario would be "Determine the shortest ladder that could be used to scale this tower."

At a talk I gave in the United States at a national conference of mathematics teachers, I briefly discussed and illustrated the galley method of division using the image from a 1604 manuscript, presented to King James I of England as a gift [1]. See Figure 2. After this talk, the student aids, all secondary school students, solicited me to show them this diagram again

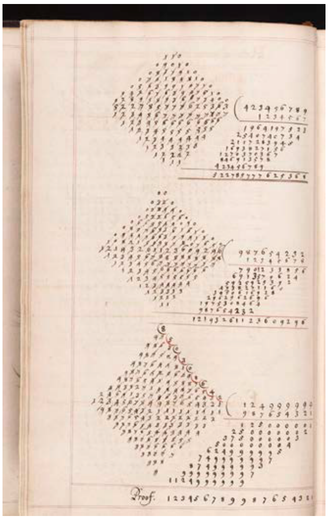


Fig 2: Galley Division in King James I's Manuscript

and to explain it in detail-'Just how was the operation of division being carried out?' Firstly, they were visually attracted to the number configuration: a large triangle of numbers. Then, they became conceptually involved: if this is a division exercise, 'How is the process being undertaken?' 'Where is the divisor, the dividend?" I explained the algorithm, line- by- line, pointing out the significance of the crossed out or "scratched out" terms and noted that due to its appearance, the technique was sometimes referred to a "scratch division". Also in this discussion, I interjected the fact, that at this time in history, there were several division algorithms in use and this one, more popularly known as "Galley division" was the most frequently performed. I left my audience with the provocative comment that I believe this algorithm to be more mathematically efficient than the one we have actually adapted for school teaching, the "downward" or 'Italian division" method. Hopefully some of them would investigate and attempt to substantiate my claim. If the reader has never attempted a "galley" division exercise, I suggest he or she do so. It is a very enlightening experience.

In selecting images for instructional purposes, two requirements should be fulfilled:

1. The material should appeal to the viewer, that is, be visually or conceptually attractive or intriguing. It should seduce the viewer "to want to find out more".
2. It should have purpose and power by pertaining to the mathematical situation, topic or concept, you wish to promote and clarify.

View the object in Figure 3. Can you see any mathematics? Ask a companion to

perform the same task and compare what each of you see. Hopefully, each of you has perceived some similar images: a circle, trapezoids, hexagons, a star, etc. and would probably eventually query: "What is this object? Where did it come from?" If so, I have set you (the viewers) up, prepared you, to pursue the topic of ethnomathematics.

The object in question is a container for sticky rice as served during a meal in northern Thailand. The tribal people who made the basket never had formal schooling

Fig 3:Topview of Thai Basket for Serving Rice

or studied geometry and yet they exhibit knowledge of the geometric shapes you have discovered and are certainly concerned with the concept of volume, in this case, the volume of a circular cylinder of rice.

SELECTING IMAGES

Appropriate computer searches reveal an abundance of images: of instruments for measuring and computing; of people who contributed to mathematics and actual mathematical works such as books and notes, that can be used to enrich mathematics teaching and learning. If known by name, they can be sought out, for example, the Ishango Tally Bone, the oldest existing human mathematics artifact. See Figure 4. Its age, 20, 000- 25,000 years, testifies to the long human involvement with mathematics and the need to record mathematical data and its function prepares the audience for a discussion on numeration in general. In observing this image two questions immediately arise: "How do we know these are human markings?' and 'What might these notches be enumerating?' The latter question has actually resulted in quite a controversy. Of course, the situation opens the topic of tally sticks, European

Fig 4: Ishango Tally Bone [2]

experiences and the rather interesting/amusing story of the British Exchequer's Office 1826 fire. Due to their disposal of centuries of accumulated tally sticks by burning, the House of Lords was set afire and burned down. A very effective and information laden image which has appeared in several books on mathematics and the history of mathematics is YBC 7289, i.e. Yale Babylonian Cuneiform 7289, a cuneiform tablet from the Old Babylonian period. [3] See Figure 5.First, one notes the physical size, it fits into a man's hand, and indeed it was a palm tablet, held conveniently in the palm of one hand while the scribe, using his other hand,

Fig 5: YBC 7289 [4]

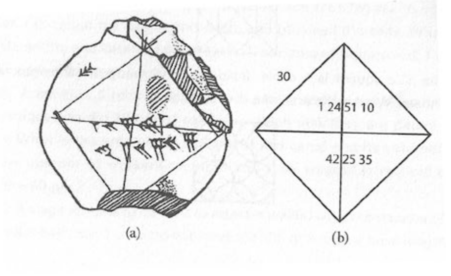


Fig 6: YBC 7289 Deciphered

wrote on it. Modern students visualize cuneiform clay tables as being quite large-perhaps they are influenced by the stylized biblical illustrations of Moses holding the tablets of the Judaic-Christian Ten Commandments. But, the most striking feature, to the novice viewer, is that the diagram clearly shows a square with its inscribed diagonal. Yes, the ancient Babylonians drew accurate geometrical diagrams in clay as part of their mathematical discussions. Once, when I offered this illustration in an article for publication, I was personally challenged by the journal's editor that I had drawn the diagram myself and was falsely attributing its origins to the Babylonians. Even very sophisticated people do not appreciate the fact that ancient mathematicians drew geometric diagrams to assist in problem solving. Next, a deciphering of the cuneiform inscription reveals two numbers in sexagesimal notation: what we know as $\sqrt{2}$ and $\sqrt{2}$ X 30, the specified length of the side of the square. See Figure 6b. At this point in the inspection, two implications emerge: at this early period before the rise of Greece and the Pythagoreans, the Babylonians possessed what we generally know as "the Pythagorean Theorem" and that these ancient people could extract the square root of a number to several decimal points accuracy. Another obvious question then arises: "How did they do it?" Further, if one looked closely at the image, the viewer would see the "tell-tale" human fingerprint!

The finding of some appropriate images for instruction may require a more formal search and research on the part of the instructor. In an instance where I desire to talk on the subject of linear measure and would like to use actual examples of societal devised units of measure, the image shown in Fig.7, taken from the German humanist, Peter Apian's, *Cosmographia* (1524), Folio 15r, would provide a learning inducement. It illustrates and demonstrates the use of fingers and feet to designate distance measures of the 16th century, ones not usually known or recognized by the modern viewer: one foot or *pes* equals four *palmus* or palms of the hand; a *leuca, or league,* the distance a person can walk in one hour, equals 1500 *passus* or paces


GRAPH. PETRI AP. Fo.15.
DE PARTIB. MENSVRAE, SEV SPEciebus Geometricę practicę. Cap. XI.

Mensura est longitudo finita, quę ignotam locorũ distantiam sensibili experimento metitur. Cuius partes seu famosę quantitates, quibus Geometer vtitur, sunt: Granum hordei, Digitus, Vncia, Palmus, Dichas, Spithama, Pes, Sesquipes, Gradus, Passus simplex, Passus duplex, quem Geometricũ appellare libuit, Cubitus seu vlna, Pertica, quam plures Radium vocant, Stadium, Leuca, Miliare Italicum, Miliare Germanicũ. &c. Granum igitur hordei est minima mensura.

Digitus habet per latera contiguè disposita.	4 grana,	Passus geometricus quo vtitur Cosmometra, habet	5 pedes.
Vncia habet	3 digitos.	Pertica habet	10 pedes.
Palmus habet	4 digitos.	Cubitus habet	6 palmos.
Dichas habet	2 palmos.	Stadium habet	125 passus.
Spithama habet	3 palmos.	Leuca habet	1500 passus.
Pes habet.	4 palmos.	Miliare Italicũ habet	1000 passus
Sesquipes habet	6 palmos.	Miliare Italicum habet	8 stadia.
Gradus habet	2 pedes.	Miliare Ger. cont.	4000 passus.
Passus simplex habet	2 pedes cũ dimidio.	Miliare Ger. magnũ	5000 passus
		Miliare Ger. cõmune	32 stadia.

¶Latini mensurãt terrestre spacium p miliaria, Græci p stadia. Franci aliis Galli, atq; Hispani, per leucas. Aegyptij per signes. Persæ p parasangas. Et secundũ aliquos. 480. stadia vni gradui æquinoctialis correspõdent. Quæ. 15. miliaria Germ. aut. 60. Italica mensurant. Galli siue Franci. 25. leucas vni gra. tribuũt. Hispani vero, leucas 18.

DIMENSIO MANVALIS.

Fig 7:Distance Measures Given in *Cosmographia* (1524) [5]

and an Italian military mile is 1000 paces whereas a German military mile comprises 4000 paces. If appropriate, students could perform these measuring postures, record and compare results. Through such activity, the personal, human origins, of measurement are recognized. Another image that I have employed with great success in working with secondary school students is page 60 from the Swiss mathematics teacher, Johann Alexander's algebra book, *Synopsis Algebraica* (1693). See Figure 8. In the history of mathematics texts, this book is rather unique; it is a text/workbook, where instruction is provided on one page and the opposite page is left for notes and computations. It is the first such book of this kind that I know of. On page 60,

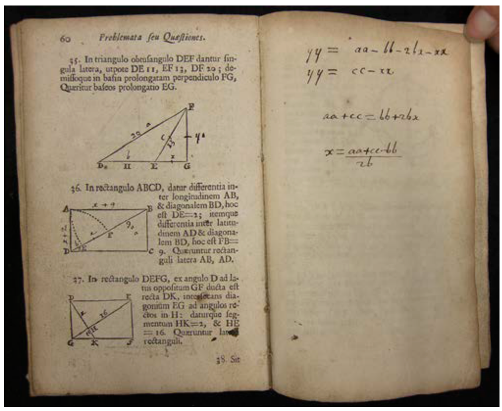


Fig 8; Student Exercise, 1693

three geometric situations are given requiring algebraic solutions. Although the book, including its exercises, is written in Latin, an astute student of today can understand the problems. A student from the 17th century offers a solution for problem 37. Is he correct? My audience checks the work and finds out, there is a mistake! 'Ah, this privileged student of the 16th century made a mistake. What is it?' The students realize that the mistake is similar to one many of them would make themselves; even the mistakes of problem solving hundreds of years ago are familiar to a modern audience. I have them correct the problem. . Then I ask them to solve the remaining problems. They are thrilled to be interpreting and solving "400 year old problems".

Throughout history, millions of such diagrams and illustrations have been devised to promote mathematical learning and understanding, they can be resurrected to serve immediate teaching needs and, further, they bear the added feature of an historical dimension which increases their attraction for the modern viewing audience. One last example of an image inspired encounter, a calculus student in a colleague's class was directed to an image of Isaac Newton's *The Method of Fluxions and Infinite Series with its Application to the Geometry of Curved Lines* (1736). She was enthralled to see the actual book that introduced calculus, albeit with fluxions, to her historical peers and instigated the mathematical field of analysis. Interpreting the frontispiece reveals an interesting story [6]. See Figure 9. But reading the title page

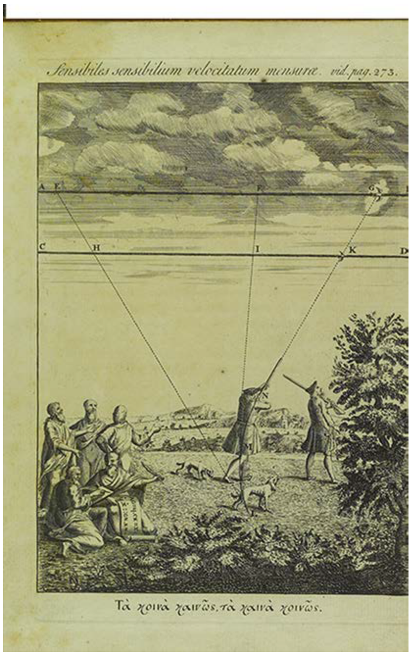
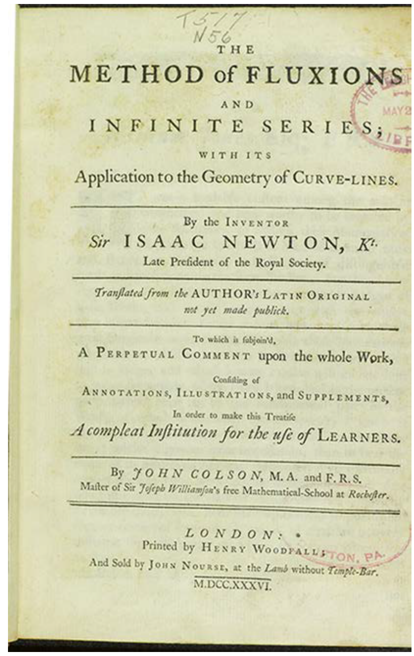


Fig 9:Gentleman Bird Hunters



THE
METHOD of FLUXIONS
AND
INFINITE SERIES;
WITH ITS
Application to the Geometry of CURVE-LINES.

By the INVENTOR
Sir ISAAC NEWTON, Kt.
Late President of the Royal Society.

Translated from the AUTHOR's LATIN ORIGINAL
not yet made publick.

To which is subjoin'd,
A PERPETUAL COMMENT upon the whole Work,
Consisting of
ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,
In order to make this Treatise
A compleat Institution for the use of LEARNERS.

By *JOHN COLSON*, M.A. and F.R.S.
Master of Sir *Joseph Williamson*'s free Mathematical-School at *Rochester*.

LONDON:
Printed by HENRY WOODFALL;
And Sold by JOHN NOURSE, at the *Lamb* without *Temple-Bar*.
M.DCC.XXXVI.

Fig 10: Title Page

information, she was prompted to investigate the circumstances resulting in the appearance of this text. See Figure 10. It was attributed to Newton as the author but was actually composed by John Colton after Newton's death. 'Who was Mr. Colton and what was his association with Isaac Newton?' 'Why didn't Newton, himself, publish the book while he was alive?' Such questions and the resulting revelations their answers supply add much to the understanding of how mathematics is developed, transmitted and refined. Just seeing this image of Isaac Newton's work makes him and his accomplishments less remote- "Yes, Newton lived and he wrote this book!" Such a visual encounter reinforces the perception of mathematics as a dynamic, evolving discipline.

In each of the visually focused examples considered above, a different picture of mathematics was presented. Each image allowed us, the viewers, to briefly travel back in time, to realize that mathematics has a past. It came from somewhere and was devised for a purpose. Mathematics students introduced to similar visual excursions, when confronted with a new mathematical concept or idea, are less apt to inquire: 'What is this good for?' or 'When are we ever going to use this?' They have seen the past and know the answers. Of course, before introducing an historical image to a class, an instructor, himself or herself, must be historically knowledgeable of the significance of the item in question.

OBTAINING IMAGES

As the previous discussion has revealed, there are many available resources from which to seek historical mathematical images for instructional purposes. We all have our personal collection of mathematics texts and reference books that can supply us with images. Library searches, especially among large university holdings, also prove fruitful; however, computer searches of existing digital collections similarly provide many visual treasures [7]. In such search efforts, one must be mindful to comply with all legal and/or cost requirements attached to items. However, the Mathematical Association of America, MAA, has complied an archive of historical images specifically intended for instructional purposes that is available to all mathematics teachers and researchers free of cost. It is found online in the Association's e-journal *Convergence* as the feature "Mathematical Treasures". All the images discussed here, together with over a thousand others, are available for use from 'Mathematical Treasures". I sincerely hope that many readers of this article will avail themselves of this wonderful resource.

* The initial workshop paper was entitled "Pantas' Cabinet..."; *Pantas* is an old Indo-Malayan word meaning "teacher". It was the author's nickname when he taught in Southeast Asia and was used to personalize the title: however, it caused confusion and has been deleted from the revision.

NOTES

1. In 1604, George Waymouth, English explorer and navigator presented his King, James I, an illustrated manuscript: Jewell of Artes. Written and illustrated by Waymouth, the manuscript showed all the novelty, beauty and utility of the mathematical arts of this period. In 1605,Waymouth did receive royal funding for an expedition to the East coast of North America.

2. Unearthed in 1950 in the Belgian Colony of the Congo, this tally bone is the oldest known, existing, mathematical artifact. Discovered by the Belgian anthropologist jean de Heinzelin de Braucourt (1920-1998), it is named after the region in which it was dug up. The Ishango Bone is now housed at the Museum of Natural Sciences in Brussels, Belgium.

3. For example, David burton *The History of Mathematics: An Introduction*, New York: McGraw-Hill, 1997, p.77; Hodgkin *A History of Mathematics: From Mesopotamia to Modernity*, Oxford: Oxford University Press, 2005, p.25.

4. Yale Babylonian Cuneiform 7289 (ca.1800-1600BCE) is contained in the Yale Babylonian Collection. It was donated to the University in 1912 by the American industrialist J.P. Morgan. An excellent presentation on this tablet can be found @ www.math.ubc.ca/~cass/Euclid/ybc/ybc.html .

5. Peter Apian's (1495-1552) *Cosmographia* was intended to supply student instruction in astronomy, geography, cartography, navigation and instrument making. This image of Folio 15 recto was obtained from the History of Science Collection, University of Oklahoma Libraries.

6. The scene shows two country gentlemen bird hunting. They judge the flight of the birds and allow a lead in their aim, that is, they judge the velocity correctly. At the lower corner of this scene is a group of classical philosophers debating the action. The literal translation for the Latin inscription pertaining to the hunters is:

"Velocities perceived by the senses are measurable by the senses." While the lower classical Greek reads as:" What was common then is the same now [Things don't change]. Colton was mindful of the controversies surrounding the calculus, particularly George Berkley's charges in the *Analyst* (1734).

7. Some rich sources for images are:
   a. The US Library of Congress: lhttp://www.loc.gov/library/libarch-digital.html
   b. The British Library: http://www.bl.uk/
   c. Schoenberg Coll.,The University of Pennsylvania: http://www.library.upenn.edu/collections/rbm/bks/
   d. History of Science Collection, UO: http://ouhos.org/2010/06/19/digitized-books/
   e. Bielefeld U.: http://www.mathematrik.uni-bielefeld.de/~rehmann/DML/dml_links_author_A.html

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