Workshop

YES, I DO USE HISTORY OF MATHEMATICS IN MY CLASS

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In this article I report on the two parts of the workshop I carried out during ESU7. The first one was aimed at gathering and discussing the opinions of the participants to the conference about the use of history in mathematics classroom. During the workshop, Man Keung Siu's article "No, I don't use history of mathematics in my class. Why?" has been illustrated. People who did not participate in the workshop could post their answers into a specific BOX standing in the hall of the conference site, during all the five ESU7 days. The main outcome was that history is believed to be a resource for students to improve their way to work in mathematics.

In the second part of the workshop, some documents from the work of Italian mathematicians Francesco Ghaligai and Rafael Bombelli were provided to participants who were asked to analyse the originals and to plan some activities, for example laboratories or short exercises, for students aged 14-16. Discussion regarded both historical and pedagogical remarks.

PREFACE

At the beginning of the conference until the end, a BOX and a poster were put in the main hall of the conference. The BOX was aimed at collecting the ESU7 participants' answers to the questions proposed by the following poster, which was hung up aside:

TEN YEARS AGO:

"No, I don't use history of mathematics in my class. Why?" - Observation and thought of school teachers collected by Man Keung Siu (2006)

[The content of column (*) of the table that follows was also reported here]

NOW:

WHY DO WE USE, OR SHOULD WE USE,

THE HISTORY OF MATHEMATICS IN CLASS?

Please, post your answers here

[Part of these answers is available in the Table 1, column **#]**

Fig. 1: The poster standing in the main hall

INTRODUCTION

Before analyzing some excerpts from the participants' answers that tackle the "observation and thought of school teachers", it is meaningful to read what Siu says about the reasons for using the history of mathematics in the class:

"First of all, the basic tenet I hold is that mathematics is part of culture, not just a tool, no matter how useful this tool might prove to be. As such, the history of its development and its many relationships to other human endeavors from ancient to modern times should be part of the subject. Secondly, through my own experience in teaching and learning I have found that knowledge of the history of mathematics has helped me to gain a deeper understanding and to improve my teaching. Now, integrating the history of mathematics with teaching is only one of many ways to do this. Anything which makes students understand mathematics better and makes students get interested in mathematics may be a good way. The history of mathematics may not be the most effective choice, but I believe that, wielded appropriately, it can be an effective means." (Siu, 2014, pp. 27-48). Some articles quoted in references in the same work discuss reasons to use history of mathematics in class (for example: Jankvist 2009, Pengelley 2011; see also: Haverhals and Roscoe 2012).

Inspired by (Siu 2006), Tzanakis (2008; see also: Tzanakis & Thomaidis, 2012) proceeded to classify/structure the various objections by grouping together arguments as follows (see numbers in column (*) of table 1):

A. Objections of an epistemological – philosophical character

A.1: Related to the nature of mathematics: No 2, 9, 13;

A.2: Related to difficulties inherent in the attempt to integrate history in mathematics education: 14, 15, and "Students may have an erratic historical sense of the past, which makes historical contextualization of mathematics impossible without their having a broader education in general history" (Fauvel, 1991).

B. Objections of a practical and didactical character

B.1: Related to teachers' background and attitude: 1, 10, 11, 12;

B.2: Related to difficulties in assessing the impact of a historical dimension in mathematics education: 3, 4 plus 16 (which could merge into one argument);

B.3: Related to students' background and attitude: 5, 6, 7, 8.

Furinghetti (2012) has grouped the objections under these main points:

integration (1, 2, 3, 4);

cultural understanding (5, 6, 7, 8);

looking for meaning (9, 13, 14, 15);

teacher training (10, 11, 12).

Integration refers to the fact that history has not to be considered an additional subject, but it has to be embedded in the teaching. *Cultural understanding* refers to a way of looking at mathematics as a vivid matter embedded in the socio-cultural process. *Looking for meaning* has to be one of the aims of mathematics teaching and learning. *Teacher education* gains from the point of view of cultural understanding and of fostering pedagogical reflection.

PARTICIPANTS' ANSWERS

The following table shows the "sixteen unfavorable factors" collected by Siu and some excerpts from the answers posted during ESU7 (in total, four snippets whose length varies from three to 30 lines).

(*) A list of sixteen unfavorable factors	**(#)** To tackle the "sixteen unfavourable factors" From the participants' answers
1. I have no time for it in class! 2. This is not mathematics! 3. How can you set question on it in a test? 4. It can't improve the student's grade! 5. Students don't like it! 6. Students regard it as history and they hate history class. 7. Students regard it just as boring as the subject mathematics itself! 8. Students do not have enough general knowledge on culture to appreciate it! 9. Progress in mathematics is to make difficult problems routine, so why bother to look back? 10. There is a lack of resource material on it! 11. There is a lack of teacher training in it! 12. I am not a professional historian of mathematics. How can I be sure of the accuracy of the exposition? 13. What really happened can be rather tortuous. Telling it as it was can confuse rather than to enlighten! 14. Does it really help to read original texts, which is a very difficult task? 15. Is it liable to breed cultural chauvinism and parochial nationalism? 16. Is there any empirical evidence that students learn better when history of mathematics is made use of in the classroom?	I. I start teaching probability with the problem of dividing the bet if the game is interrupted. Success is sure. I can find in old textbook a lot of interesting tasks. II. Studying history of maths can open up a student's mind to new ways of thinking about and solving problems. III. Using historical problems may bring even fun to the lectures. Humour and poetry can be brought through the stories from the life of scientists (Archimedes run naked through Syracuse, Newton was hit by a falling apple...). IV. I use history of mathematics in my class because it allows to work in interdisciplinarity. V. I agree that such an integration is helpful to the attainment of the so-called "3-D teaching objectives", namely, knowledge and skill, process and method, affective attitude and value judgement. VI. In my opinion, history of specific topics is a necessary part of the whole subject (not only in mathematics): teacher who does not know history is not a good teacher. VII. History makes more sense of curriculum contents, as well. Many scientists were international several centuries ago. Was Euler Swiss or Russian? Kepler was German but he stated his theorems in Prague... If we use the historical approach, we can explain to the students why was the formula (or method etc.) developed. Example: using historical tasks to solve quadratic equations students understand that the formula does not fall from the sky.

Table 1.

Many of the "sixteen unfavorable factors" constitute real objections participants could, at least partially or indirectly, contribute to refute. Here I propose a correspondence of them and the seven excerpts from participants' answers, accompanied by my synthetic interpretations.

1. I. (Problem solving is part of school mathematics) II. (History of mathematics can facilitate students' reasoning)
2. II. (By means of history, students do mathematics) V. (Attitudes and values have to be taken into account for a better understanding of mathematics)
3. VII. (See the last part of the excerpt as an example)
4. II. and V. (History creates premise to improve the grade) VI. (Teacher's competence helps students' achievement)
5. III. ("Humor and poetry" with history) I. (Students like to be successful in problem solving!)
6. III. (History of mathematics can be a lovable part of general history)
7. IV. (Students can not find only "the subject mathematics itself" in history) I. (History of mathematics offers various opportunities to involve different students) III. (Mathematics in history gives more resources then pure mathematics)
8. VII. (Students appreciate that mathematics can acquire more sense) IV. (History of mathematics is culture)
9. II. (Students who believe that mathematics is "routine" do not have an open mind)
10. I. (Ancient texts, such as their modern revisions, are rich of ideas for mathematical activities)
11. VI. and I. (Teachers can autonomously find material in ancient texts)
12. I. (Originals are the best source, not only for researchers but also for teachers)
13. V. (History clarifies the aspects of mathematics which is an intrinsically complex science) II. (By means of history, students get instruments for mathematical thought)
14. I. (Finding problems in the originals is not an impossible task)
15. III. and VII. (Mathematics is international)
16. I. (Teachers who use history in class can say 'yes')

STUDENTS' OPINIONS

Table 1 reports some of teachers' points of view. During the workshop we also briefly discussed students' opinions about the history of mathematics in class. Comparing some hints from Hong-Kong and Italy we have agreed that students who usually get high assessment levels in mathematics tend to criticize the use of history. Here I report the opinions of a female student, Anna. Her level of competence in mathematics was good but not always satisfactory in the other school subjects. Anna and her classmates were requested to write a report (at least 10 lines) imagining to talk about the use of history in math classes to a friend of another school. In brackets, you can see a suggestion of correspondence to some points of the "list of sixteen unfavorable factors".

"HISTORICAL COURSE IN MATHEMATICS

About the program implemented this year in mathematics, I believe that all the mathematical-historical course we learned is not useful [No 4] because I do not see it as

> mathematics [2] and, considering that it is quite a difficult subject [9], if I have something to suggest to the teacher, in the years to come he should try to engage in real mathematics, that is exercises with expressions, equations etc., everything in a mathematical way and without inserting history because if we have to make reckonings, which I believe it is a fundamental thing for everyday life, neither history nor those problems with strange words inside will remain [8, 14].
>
> Happily enough, past years my teacher was much engaged in his work and he taught us the real mathematics [2], and I believe that this must be done at school; then, if you are interested in something regarding ancient mathematics, everybody can do it by oneself using a computer or something else.
>
> Numbers learned this year are:
>
> EGYPTIANS, BABYLONIANS, MAYANS [capitals in the original].
>
> May be, it is because I do not get on well with history [5, 6], in contrast to another person who likes the history and could find it funny".

Another female student, Anny, hardly got a positive level of competence in mathematics. She answered the same request.

> "This year, at school we learnt mathematics, applying history to it. In my opinion, it is a thing which is useful in order to understand the bases and is interesting, too, but also a little difficult because I was not always able to apply those rules to problems of today. We have learnt methods that have been discovered by famous mathematicians like Bombelli and Fibonacci, ancient problems and equations and also the number systems of Egyptians, Babylonians and Maya. As I said before, it is a very useful but also a bit difficult and I was not able to understand why they studied the things of the past but not ours".

I consider the last line as the most obscure in Anny's snippet. Anyway, I like to interpret her remark as a consequence of her personal work with historical documents. She looks like having found more difficult working with - I believe - Mayas numbers than with ours. I consider her naive historical question a consequence of *replacement* caused by using original documents (Jahnke et al., 2000).

MATHEMATICS WITH GHALIGAI AND BOMBELLI

I think that some teachers' beliefs on the use of history could be modified if the problem of identifying materials suitable to work in the classroom is weakened. For this reason I complemented the workshop on the theoretical discussion with the presentation of materials that can be used in the classroom. Then, the attendants the field what the use of history implies.

Fig. 1: Ghalighai, *Pratica d'Arithmetica*, frontispiece

Ghaligai's *Pratica d'Arithmetica* was first published in 1521. The author does not preface his treatise with an autobiographical introduction, in fact after the frontispiece we only know that he was a Florentine.

Like other medieval or renaissance manuals, its audience was merchants. The treatise is divided into thirteen books. The last four are devoted to algebra, which includes explanations of methods for the extraction of roots. It also includes operations with binomials regarding radicals, as found in other 15^{th} and 16^{th} century's works. Ghaligai quotes from other authors' writings, for instance Euclid, Fibonacci and Pacioli. In regards to the symbols he introduces, he says they belong to "Giovanni del Sodo who uses them for his algebra".

Rafael Bombelli (Bologna 1526 – Rome 1572) in his ***L'Algebra***, gives an account of the algebra known at the time. The treatise is divided into three books and includes his contributions to complex number theory.

Bombelli's *L'Algebra* was intended to be in five books. The first three were published in 1572. Unfortunately Bombelli died shortly after the publication of the first three volumes. In 1923, however, Bombelli's manuscript was discovered in a library in Bologna by the Italian mathematician and historian Ettore Bortolotti. As well as a manuscript version of the three published books, there was the unfinished manuscript of the other two books. Bortolotti published the incomplete geometrical part of Bombelli's work in 1929 (http://www-history.mcs.st-andrews.ac.uk).

In the preface "to the readers" Bombelli makes mention of al-Kwarizmi, Diophantus and Pacioli. Leibniz praised Bombelli as an "outstanding master of the analytical art".

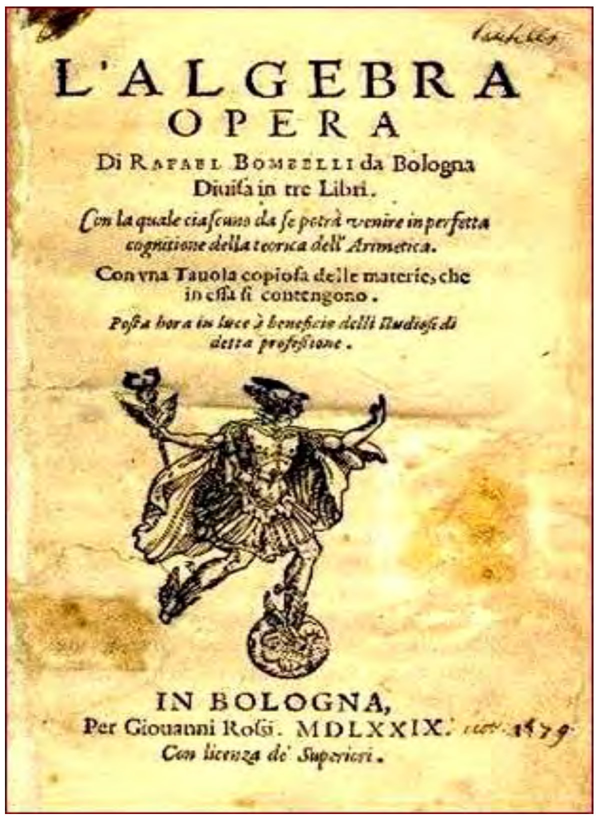


Fig. 2: Bombelli's *L'Algebra*, frontispiece

WORKSHOP MATERIALS

From Bombelli's *L'Algebra.* Bombelli's documents that were proposed for the workshop aimed to give examples of addition, subtraction and multiplication of polynomials. Opening examples regarding powers have to be considered prerequisites (figure 3).

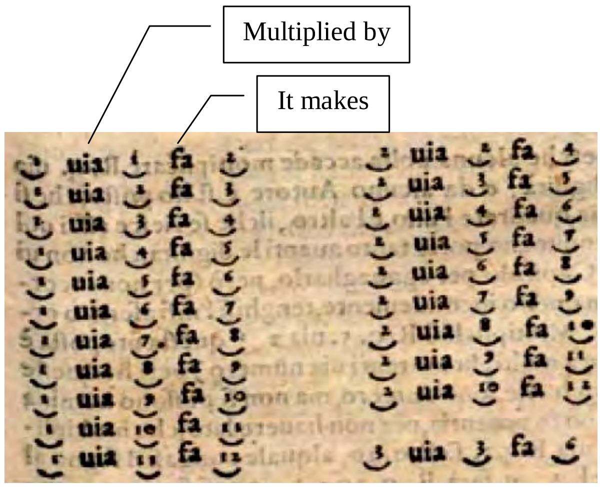


Fig. 3: *L'Algebra*, Book second, 205, examples regarding powers
Fig. 4: *L'Algebra*, Book second, 206, examples regarding monomials

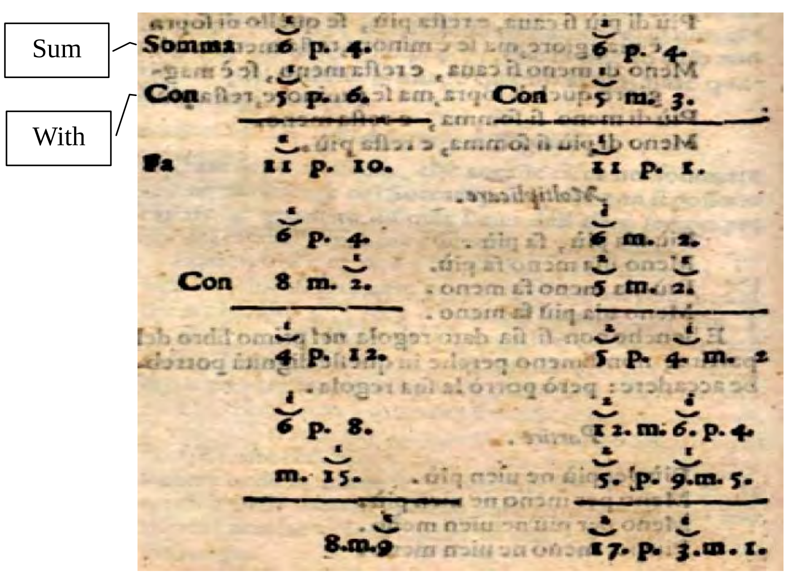


Fig. 5: *L'Algebra*, Book second, 212, examples regarding sums

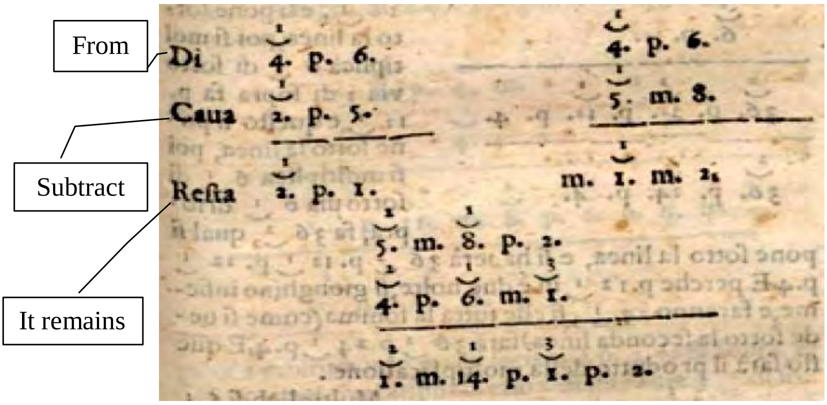


Fig. 6: *L'Algebra*, Book second, 213, examples regarding subtractions

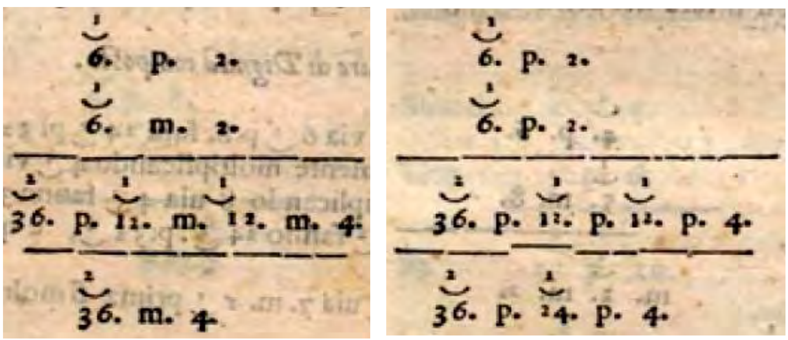


Fig. 7: *L'Algebra*, Book second, 214, examples regarding multiplications

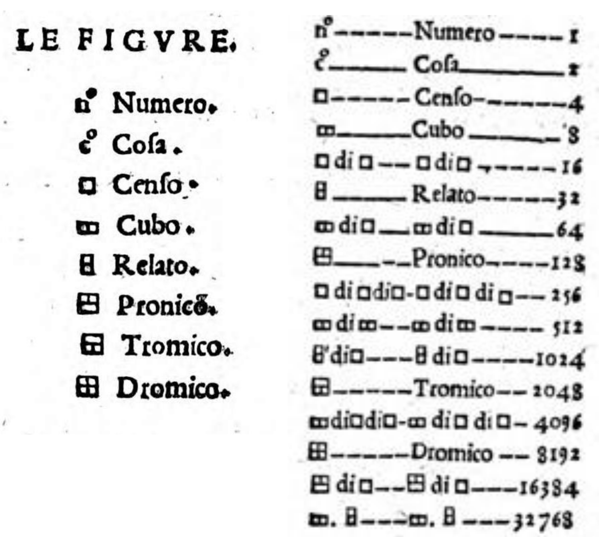
This material from *L'Algebra* was previously used in class, in an Italian Liceo delle Scienze umane (Human Sciences Lyceum, 9^{th} grade, with 14-15 aged students). In this experiment we observed an outcome similar to that regarding the interpretation of a Pacioli document, illustrated in (Demattè, to appear). In this case students often get stuck when they were not able to read some symbols, for example the exponents written in small dimension, seen in the column on the left, in the first document from *L'Algebra,* presented in Fig. 3. Note that they are widespread in the same document so students could get an opportunity to infer the right value reading all the original text (*hermeneutic circle*). In that situation they often asked the teacher. This shortcut suggests some reflections regarding which previous educational circumstances made this choice more preferable for students, instead of insisting on finding the correct answer by themselves. This issue is particularly relevant considering that, among the possible answers, one regards aspects closely inherent to mathematical reasoning. In fact, when students conjectured the right answer, they were in a situation of uncertainty: they could foresee it or not. The choice between these two alternatives was left to them. Asking the teacher allowed students to avoid any risks with respect to the teacher's approval. The conclusion is quite disconsolate: necessity to have an authority who masters education (the teacher) reinforces performances that are different to those which qualify mathematical reasoning, for example: pleasure in personal endeavour, conjecturing and checking conjectures.

The students' task analysed here has to be considered rather easy because it does not require a wide amount of prior mathematical knowledge. This suggests that among different reasons for getting stuck, their lack of self confidence in mathematics should not be considered. I prefer to address the problem of how a teacher has to act in order to improve students' high level competencies without losing their volition to learn.

I would like to compare the situation regarding the document in Fig. 3 with the situation in which students were requested to discover the operation Bombelli used in the calculation shown in materials “For participants’ working groups-2.a” (see next paragraph). Also in this circumstance, students had to examine some possible solutions, specifically different operations: addition, subtraction and multiplication, without others, considering what they had previously learnt. Why did not they conjecture which one would work? Beyond the answer regarding the teacher's role, I suppose that this lack of disposability to conjecture could have a social origin, considering that in everyday life a false statement could produce damage to personal prestige or standing (in fact I believe that most of us would usually not knowingly risk to say something which could be wrong).

From Ghaligai’s *Pratica d’Arithmetica.* Ghaligai's documents was introduced in the workshop as proposals to analyse in order to plan activities for the class. The whole *Pratica d'Arithmetica* was at disposal in a .pdf file (in Italian).

The first part “LE FIGVRE” [the figures] was considered self explanatory, so no translation was given, apart from “di” means “of”. The main idea about possible use of this part in class was that Ghaligai's symbols show an alternative way to represent powers and to work with their properties. I believe that also the fact that there are specific symbols for the powers having a prime number as exponent (“censo, cubo, relato, pronico, tromico, dromico”) could be profitable for class. The other short Chapters introduce a brief itinerary about operation with radicals (fig. 9 and fig. 10).


LE FIGVRE.

Numero.
Cosa.
Censo.
Cubo.
Relato.
Pronico.
Tromico.
Dromico.

Numero — 1
Cosa — 2
Censo — 4
Cubo — 8
□ di □ — □ di □ — 16
Relato — 32
di □ — di □ — 64
Pronico — 128
□ di □ di □ — □ di □ di □ — 256
di — di — 512
di □ — di □ — 1024
Tromico — 2048
di □ di □ — di □ di □ — 4096
Dromico — 8192
di □ — di □ — 16384
. — . — 32768

Fig. 8: Francesco Ghaligai’s *Pratica d’Arithmetica,* Book tenth, folio 72.

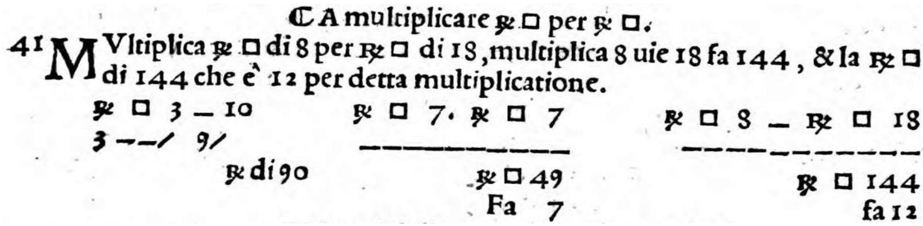

℃ A multiplicare ℞ □ per ℞ □.
41 MVltiplica ℞ □ di 8 per ℞ □ di 18, multiplica 8 uie 18 fa 144, & la ℞ □ di 144 che è 12 per detta multiplicatione.

Fig. 9: ***Pratica d'Arithmetica,*** **Book tenth, folio 75. "**C About multiplying square roots by square roots. Multiply the square root of 8 by the square root of 18; multiply 8 by 18: it is 144, and the square root of 144 is 12 for that multiplication".

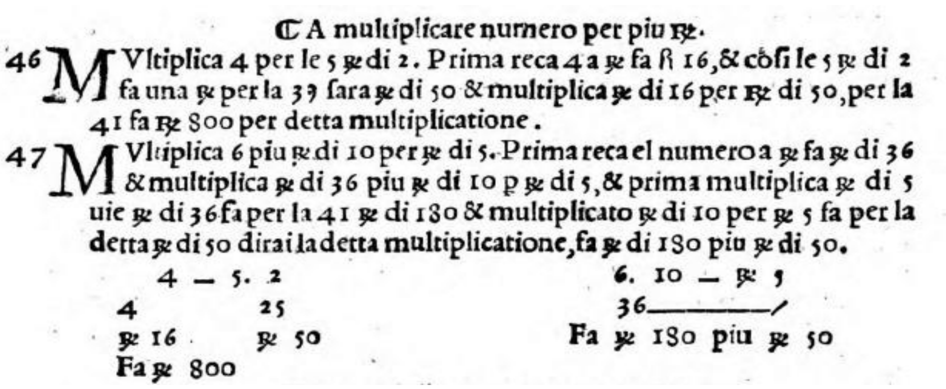

℃ A multiplicare numero per piu ℞.
46 MVltiplica 4 per le 5 ℞ di 2. Prima reca 4 a ℞ fa ℞ 16, & cosi le 5 ℞ di 2 fa una ℞ per la 39 sara ℞ di 50 & multiplica ℞ di 16 per ℞ di 50, per la 41 fa ℞ 800 per detta multiplicatione.
47 MVltiplica 6 piu ℞ di 10 per ℞ di 5. Prima reca el numero a ℞ fa ℞ di 36 & multiplica ℞ di 36 piu ℞ di 10 p ℞ di 5, & prima multiplica ℞ di 5 uie ℞ di 36 fa per la 41 ℞ di 180 & multiplicato ℞ di 10 per ℞ 5 fa per la detta ℞ di 50 dirai la detta multiplicatione, fa ℞ di 180 piu ℞ di 50.

Fig. 10: ***Pratica d'Arithmetica,*** **Book tenth, folio 75.** "C About multiplying a number by several square roots. Multiply 4 by the 5 square roots of 2. First of all, bring 4 to square root: it is square root of 16, such as the 5 square roots of 2 is one square root for the [proposition] 39 it will be square root of 50, and multiply square root of 16 by square root of 50, for the [proposition] 41 it is square root of 800 […]".

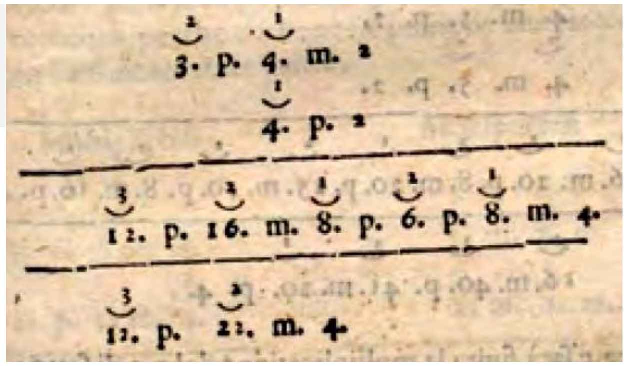
Worksheet for participants' working groups

Among the following questions/activities, feel free to choose and discuss those you prefer.

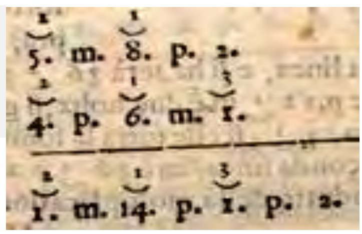
1. Compare Ghaligai's and Bombelli's notation. Why would students have to analyze it, that is: why could it be useful for students? What prerequisites do students need to interpret Ghaligai's and Bombelli's originals?

2. Two exercises from Bombelli's original (hermeneutic approach):

a) Write the following multiplication using modern symbols.



b) Which operation does Bombelli use in this calculation? Explain your answer.



Write other exercises.

3. I guess that Ghaligai's symbols could help students to tackle some difficulties. For example: using symbols with reference to formal properties and being able to control the meaning of symbols. Agreement/disagreement and why.

4. Working with Ghaligai's or Bombelli's documents, what kind of students' capabilities could the hermeneutic approach improve?

5. Write proposals of laboratories using Ghaligai's, Bombelli's or other authors' algebraic works (main points, keywords, activities, resources etc.).

Transcription of the previous documents in modern symbols

From Bombelli's *L'Algebra* (fig. 3, fig. 4, fig. 6, fig. 5, fig. 7):

205 $x \cdot x = x^2$ $x^2 \cdot x^2 = x^4$

$x \cdot x^2 = x^3$ $x^2 \cdot x^3 = x^5$

$x \cdot x^3 = x^4$ $x^2 \cdot x^4 = x^6$

$x \cdot x^4 = x^5$ $x^2 \cdot x^5 = x^7$

[...] [...]

206 $4 \cdot 3x^2 = 12x^2$ $3x^4 \cdot 5x^5 = 15x^9$

$7x^2 \cdot 18x^2 = 126x^4$ $56x \cdot 12x^4 = 671x^5$

$5x \cdot 8x^2 = 40x^3$ $7x^2 \cdot 84 = 588x^2$

$4x^2 \cdot 6x^2 = 24x^4$

$5x \cdot 7x^3 = 35x^4$

$3x \cdot 8x^2 = 24x^3$

212 $(6x + 4) + (5x + 6) = 11x + 10$ $(6x + 4) + (5x - 3) = 11x + 1$

$(6x + 4) + (8 - 2x) = 4x + 12$ $(6x - 2) + (5x^2 - 2x) = 5x^2 + 4x - 2$

$(6x + 8) + (-15x) = 8 - 9x$ $(12x^2-6x+4)+(5x^2+9x-5) = 17x^2+3x-1$

213 $(4x + 6) - (2x + 5) = 2x + 1$ $(4x + 6) - (5x - 8) = -x - 2$

214 $(6x + 2) \cdot (6x - 2) = 36x^2 + 12x - 12x - 4 = 36x^2 - 4$

$(6x + 2) \cdot (6x + 2) = 36x^2 + 12x + 12x + 4 = 36x^2 + 24x + 4$

From the material "For participants' working groups, 2 a) and 2 b)":

215 $(3x^2 + 4x - 2) \cdot (4x+2) = 12x^3 + 16x^2 - 8x + 6x^2 + 8x - 4 = 12x^3 + 22x^2 - 4$

213 Subtraction: $(5x^2 - 8x + 2) - (4x^2 + 6x - x^3) = x^2 - 14x + x^3 + 2$

From Ghaligai's *Pratica d'Arithmetica* (fig. 9, fig. 10):

f.75 **41** $3\sqrt{10} = \sqrt{9}\,\sqrt{10} = \sqrt{90}$ $\sqrt{7}\,\sqrt{7} = \sqrt{49} = 7$ $\sqrt{8}\,\sqrt{18} = \sqrt{144} = 12$

46 $4 \cdot 5\,\sqrt{2} = \sqrt{16}\,\sqrt{25}\,\sqrt{2} = \sqrt{16}\,\sqrt{50} = \sqrt{800}$

47 $(6 + \sqrt{10})\sqrt{5} = (\sqrt{36} + \sqrt{10})\sqrt{5} = \sqrt{180} + \sqrt{50}$

DISCUSSION DURING THE WORKSHOP

Participants worked above all on the interpretation of the documents. I collected their opinions which were homogeneous about the fact that Ghaligai's documents would not be easy for students. Some participants were oriented to sketch a proposal of activities for the class after ESU7 Conference. They also found a good opportunity for students to reflect on a calculation in a historical document and to discover the operation used. This task was considered quite meaningful because 15th and 16th century originals do not yet use our modern symbols for arithmetical operations. On the contrary, somebody else believed that it is more profitable to use historical problems instead of introducing different symbolisms in class.

During the presentation, a participant posed the following question: Why did Bombelli write in Italian, unlike Cardano, for instance, who used Latin, the language of science at that time? We shared the hypothesis that the answer has to be looked for in their biographies: the former lived in Italy (was born and dead in the Papal States), the latter travelled to Germany, France, Scotland and England. Despite this, we observed similarities between Bombelli's and Stevin's symbolism about the manner to write powers.


Eglie una torre che e alta 40 braccia e dap pie u passa uno fiume che e largho 30 brac cia. uo sapere quanto sara lungha una fune che sia appicata alla ri ua del fiume e alla ci ma della torre

40 — 30
40 30

1600
900
la radice di 2500
sara lunga 50 brac cia

Fig. 11: From Calandri's *Arithmetic*

HISTORICAL AND PEDAGOGICAL REMARKS

I would like to focus on the schemes that appear at the end of each text in Ghaligai's documents, comparing them with that in the document (Calandri, 1491/2, figure 11): "A tower is 40 braccia high and at its base runs a river which is 30 braccia wide. I want to know how long a rope which runs from the top of the tower to the other side of the river will be?" (Katz 2000, p.65). Ghaligai uses the schemes as a summary of the text. In Calandri's document the scheme encompasses the whole solution which is not accompanied by any words, apart from "la radice di" i.e. "the root of". In Ghaligai, the symbol of square root appears in the text and in the scheme as well. On the contrary, Calandri does not use a specific symbol. At the beginning in paragraph 41 (fig. 9), Ghaligai makes explicit that he is using the square root. Note that a preceding part of his *Pratica d'Arithmetica* was devoted to cubic root.

There is a similar use of lines connecting numbers which have to be combined in calculation. Note that in Ghaligai they are sometimes replaced by dots. In Calandri as well as in Ghaligai, numbers that are used in the same calculation are not always connected. Horizontal lines are also used for another goal: in Calandri, just before the result of an addition in which the two addends are written one above the other; in Ghaligai, before the product of roots. Some questions arise and regard the role of publishers and printers. How could they take into account the fact that the author was moving toward symbolism (if this would be the case)? How could they coherently interpret the meaning of each sign contained in the author's manuscript? What kind of typographic solutions could their possible misunderstandings produce? With respect to our analysis, the previous explanation regarding the role of a horizontal line might eventually be changed: like in the last scheme of paragraph 41, is the dashed line to be interpreted as implication, considering the application of a general rule, in this case regarding multiplication of roots with the same index? We cannot hide that these questions implicitly depends on the assumption that in the European Renaissance conditions for the new, efficient symbolism emerged (Radford, 2006).

CONCLUSION

The most common feature of the participants' answers is that the history of mathematics can be a way to achieve high level educational aims. More precisely, it is believed to be a resource for students to improve their way to work in mathematics. Narratives, problems, suggestions for laboratory activities can be only some of the many possible resources.

The image of mathematics which appears in the answers is that of an 'open science' either from the point of view of method or content. This shows the innovative role which the history of mathematics can play. In my opinion, it can address effectively some unsatisfactory aspects in our classrooms: those regarding, in short, the difficulties students have in managing mathematics when it is presented only in a formalized manner.

The "unfavorable factors" can be also part of students' opinions. This fact shows how the use of history in a profitable manner requires considering the different school actors. The perspective to achieve high-level aims is motivating but involves a wide investment of resources.

Considering the lack of a coherent symbolism, suggestions in favour of students who are requested to interpret schemes like those in paragraph 41 could be appropriate. A suggestion is to highlight the numbers, to search how to combine them and what operation to choose to get the result (see, for example, the first and the second one in paragraph 41 – Fig. 9 that are not accompanied by explanations). In my opinion, this way, students get a basic key to get a grasp of the hermeneutic approach. In fact, interpretation is achieved starting from an incomplete set of data. It is incomplete because the interpreters have to reconstruct other parts of reasoning in order to obtain a frame that has to be satisfactory with respect to their personal previous knowledge. Supplementary historical endeavour is necessary to interpret the document with respect to the author's viewpoint and knowledge about the topic to which document refers.

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