Workshop

USING AUTHENTIC SOURCES IN TEACHING LOGISTIC GROWTH: A NARRATIVE DESIGN PERSPECTIVE

Kristian Danielsen[a] & Henrik Kragh Sørensen[b]

[a]Randers Statsskole, [a,b]Aarhus University

In this paper, we present and discuss newly designed materials intended to teach logistic growth in Danish upper-secondary mathematics classes using an authentic source. The material was developed in conjunction with teaching a university-level course for future mathematics teachers and published in the form of a small booklet that introduces, contextualises, explains and discusses an original source by the Belgian mathematician Pierre François Verhulst (published 1838). The material offers multiple strategies for classroom implementation, ranging from a basic introduction to the topic to a full-scale teaching unit, as well as detailed suggestions for its cross-disciplinary integration with other subjects such as history, languages, art and social science. Our approach has shown potential for engaging students in enquiry-driven learning of both mathematics and its history.

HISTORY OF MATHEMATICS IN DANISH MATHEMATICS EDUCATION

Since the 1980s, history of mathematics has been required as an integrated part of the mathematics curriculum in the Danish upper-secondary schools. The arguments for teaching history of mathematics include its potential for showing mathematics as a human and creative activity, embedded in broader culture and responding to relevant cultural questions, rather than a fossilised or sterile discipline of formal reasoning and rote learning (Jankvist, 2009; Kjeldsen & Carter, 2014). Since a reform in 2004, cross-disciplinary collaboration between subjects have also been mandatory, and history of mathematics is a vehicle for integrating mathematics in its cultural and historical contexts.

With such aims in mind, the university courses in history (and philosophy) of mathematics offer opportunities for both becoming acquainted with the context and contents of important mathematical concepts and episodes, for initiating methodological reflections, and for stimulating identity-formation in students, whether they are future teachers or researchers.

For the history of mathematics courses at Aarhus University aimed at future teachers, we have identified four focal points (in addition to the learning objectives) that will aid the teacher in using history relevantly in mathematics teaching: tools for critical reading of i) primary and ii) secondary sources; and iii) historiographical reflections, in particular concerning iv) biographies of mathematicians.

In order to implement these goals, two challenges therefore arise for us: How do we (best) educate the teachers and how do we devise suitable material for classroom usage? Historians and educators of mathematics have addressed these two questions for decades (see e.g. Fried, 2001; Jahnke et al., 2002; Pengelley, 2011; for a discussion of the use of primary sources in mathematics education, see also Jankvist 2014). We have combined these mathematics-bound discussions with inspiration drawn from teaching Nature-of-Science (NOS) through a narrative-based approach (see, for example, Alchin 2012) as this approach proposes to lead students to independent enquiry of complex questions while providing relevance and identification through the narrative scaffolding. In this paper, we present our perspective on the design of teaching material using authentic sources and based on a situating narrative and a historical contextualization. Although the material has been tried and is being used, this paper does not purport to systematically analyse results from its actual class-room usage.

In the Danish upper-secondary mathematics curriculum, differential equations and the logistic model are core topics. We think that, ideally, successful historical material should be chosen among the core topics or in close relation to them. Such choices make it possible to simultaneously teach mathematics and history of mathematics. And although many topics are not as prone to historical contextualization, a fair number of good historical sources exist that can be used in teaching elementary geometry, trigonometry, function theory, proofs, etc.

LOGISTIC GROWTH THROUGH VERHULST' AUTHENTIC SOURCE

Based on our teaching of a university course in the history of mathematics aimed at future teachers, we have developed teaching material about logistic growth. Central to the material is a short authentic mathematical source, namely Pierre-François Verhulst's article from 1838 in which he introduced what is now called the "logistic curve" (Verhulst, 1838; translated into English in Vogel et al., 1975). Although quite short, this source invites discussions of four important aspects of the use of history in upper-secondary mathematics education: teaching a central mathematical topic, illustrating and discussing the mathematical modelling process, addressing issues of philosophy of mathematics, and informing the students about the historical complexity of mathematics.

Our material (Danielsen & Sørensen, 2014) includes a translation of the main source into Danish as well as a structured narrative providing the students with a window into the historical context and an understanding of Verhulst's thought process. The approach of working directly with the primary, authentic source forced us to situate it in a historical narrative that briefly introduces the main protagonists, their social and political contexts, and the mathematics of the early nineteenth century to the teachers and, eventually, to their students. Such a narrative approach is employed also to aid student identification with the problem at hand and make accessible the methods of solving it (see also Allchin, 2012).

After the translated source is reproduced *in extenso*, our material also contains a section-by-section elaboration of it intended to aid the teacher in acquiring the relevant information and perspectives. Along the way, numerous exercises interspersed with the sections of the source point to tasks to guide students in their engagement with the source. Finally, a number of broader perspectives are presented which lend themselves to either detailed mathematical treatment or to cross-disciplinary collaborations with other subjects in the upper-secondary school. While parts of the material are directly useful for students, the booklet as a whole is intended for teachers who should appropriate its content to their specific usage.

Verhulst's short text can be structured into four sections with different content and different perspectives on the source.

Part 1: Background of the problem (§1-4)

In the first paragraphs of the article, Verhulst introduced the background of his model by referring to Malthus' idea that populations grow exponentially. While this could be maintained if no restrictions were put on the growth, this was not realistic in comparison to most European societies where cultivation of the land had already been cultivated to near its capacity. Thus, a refined model would be required.

Concerning these passages, we can pose enquiry-enabling questions in the form of an exercise based on reading the source and using the internet to find relevant information:

1. The name of the author is Pierre-François Verhulst (1804–1849). What can you find out about him on the internet?
2. How is Thomas Robert Malthus (1766–1834) described in the source? What can you find out about him online?
3. What can you find out about the context in which the source was produced?
4. How is the influence of agriculture and cultivation on populations described?

Part 2: Outlining the model (§5-9)

In light of the insufficiency of exponential growth to model realistic populations, Verhulst in the following paragraphs set out to formalise a more refined model. He wanted to conceptualise the "hindrances" to the exponential growth, and he drew on an analogy with Quetelet's "social physics" which likened the hindrances to the resistance on the motion of a body falling through a liquid. Thus, Verhulst obtained assumptions that would ultimately underpin his mathematical model such as his realisation that populations have a capacity:

> All the formulas by which one may attempt to represent the law of population must therefore satisfy the condition of allowing for a maximum which will only be reached in an extremely remote era. This maximum will be the total of the population which tends to become stationary. (Translated from Verhulst 1838, 114-115)

He then went on to describe how he had not been completely successful in this endeavour since his assumptions and the available data were not sufficient to completely determine the model:

> I have tried for a long time to determine by analysis a probable law of population; but I have abandoned this type of research because the available data is too limited to allow the verification of my formula, so as to have no question about its accuracy. However, the course which I followed seemed to lead me to the understanding of the actual law, which when sufficient data becomes available, will support my speculations. (Translated from Verhulst 1838, 115).

As seen from these quotes, Verhulst's paper provides a rare snapshot of the modelling process in mid-action. In order to assist the students in engaging with this piece of text, we pose questions such as:

5. How is exponential growth described?
6. Population growth is compared to an example from physics. How?
7. Why do models of population growth require the inclusion of an upper limit?
8. Which problems have Verhulst had to overcome to build his model?

Part 3: The mathematical model (§10-11)

Verhulst proceeded to formulate a mathematical description of population growth which is equally interesting in its details. He set out from the case of exponential growth, $dp/dt=mp$, where m is a constant, and sought to "slow" it by an unknown function, thus yielding $dp/dt=mp-\varphi(p)$. He then proceeded:

> The simplest hypothesis that can be made on the form of the function φ is to suppose that $\varphi(p) = np^2$. (Translated from Verhulst 1838, 115)

And for this case, he could solve the equation by separation of variables and eventually deduce a formula for the population p as a function of time, t, from which also followed an expression for the upper limit of the population.

This is the most technical of the sections in Verhulst's paper, and depending on the mathematical level of the students, various approaches can be taken. Among the questions to be posed are:

9. Which equation is eventually used to describe population growth?
10. Can you rework the equation into a form that is recognizable and understandable from a modern, school perspective?
11. How is the solution formulated in the source?

12. Use a CAS tool to find a solution. What do you find?
13. Can you find a relevant formula in your textbook? What is the textbook's solution?

    Compare the various solutions obtained.
14. Which limit is contained in the formulas?

Part 4: Verhulst's discussion of the model (§12-16)

Finally, Verhulst provided some discussion of his own model. For instance, he noticed that he had tried various alternatives for the function φ but that these were all indistinguishable from the simplest one given the limited amount of data at his disposal. He appended to his articles four tables of populations in Belgium, France, Essex and Russia, and these all show demographic statistics was still in its infancy and data was scarce.

Concerning this part of the article, we pose questions such as:

15. Is the description of population growth unique?
16. In Verhulst' paper, he introduced four sets of data for populations in France, Belgium, Essex and Russia. These data can be downloaded from www.matematikhistorie.dk/logistisk-vaekst. How would you treat the data that Verhulst presented?

Working with authentic data is another requirement for upper-secondary mathematics education and this can be achieved either by using e.g. Verhulst's historic data or by trying to apply logistic growth to e.g. the population census of Denmark during the twentieth century.

META-PERSPECTIVES GAINED FROM THE AUTHENTIC SOURCE

The material has been tested in different classes in different modules. In the design, it was important to allow for flexibility in the scope: the historical perspective could be used to briefly introduce logistic growth, or used throughout as a means of teaching logistic growth, or as the nexus for various collaborations with other subjects. These experiences have all been positive, and we are still receiving feedback, which we look forward to analysing, in particular concerning the full-scale student project in history and mathematics.

Discussions about mathematical models

Mathematical models and the ability to critically assess them are key topics in mathematics classrooms. By using historical sources such as Verhulst's paper, the authentic modelling process with its incompleteness and contingent choices become visible to the students. In particular, Verhulst's aesthetic and pragmatic choices in building the model are otherwise difficult to illustrate. Thus, students get to "look into the workshop" of the mathematical modelling process, and what they see may nuance the picture often presented in textbooks of a finished, fossilised model. The fact that historical sources can help teachers identify what Anna Sfard denotes *commognitive conflicts* (Sfard 2010) has also been remarked in (Kjeldsen & Petersen, 2014) where it is shown how exposure to an authentic historical source brought to light some key problems about mathematical symbols that were otherwise invisible to the teacher. Thus, when students are presented with Verhulst's original notation and wording, they will have to come to accept, for instance, that his p' is *not* the derivative of p but rather the population at $t=0$. In performing such translations – which are not often present in stream-lined textbooks – their understanding of the conceptual level might well be revealed. And when they confront the contingencies present in the authentic source with the often sterilized product found in textbooks, they can be led to a deeper understanding of the processes and practices involved in mathematical modelling and, more generally, in mathematical research (for more on the role of history in teaching mathematical modelling, see also Kjeldsen & Blomhøj, 2013).

In the source, Verhulst provided remarks on the process of building his model and on the factors that influenced it. In an exercise, we put these into perspective and discuss the modelling process and the status of the model in relation to data:

1. First outline the model of mathematical modelling as presented in figure 1. Be sure to explain how the mathematical model is related to reality by focusing on how the model is built and what kinds of statements about reality that it warrants.
2. Now analyse Verhulst's source to find those places in it where he explains or comments upon his model. What does he, for instance, say about the relation between the model and the actual population sizes? Again, you should focus on the construction of the model and the types of warrant that it provides.
3. Discuss the perspective on the process of building mathematical models which you have achieved in the first two tasks by discussing the types of knowledge that mathematical models can provide of the real world.

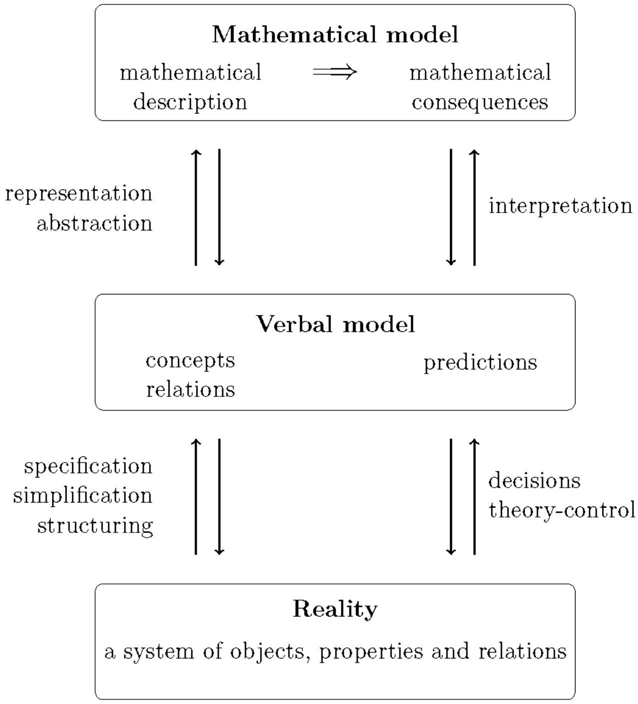

Figure 1: A simple model of the mathematical modelling process. The process starts by delineating a section of "reality" to be modelled. The arrows then indicate dialectic processes, and the entire modelling process is also to be considered iterative (see also Johansen & Sørensen 2014, chap. 10).

Illustrating historical complexity and context

The material is designed to include sufficient historical context to aid the teacher in situating the source. And the source is given in translation quite close to the original in order to allow the teacher to address historical complexities such as differences in notation or conceptions about mathematical notions and objects. These are, we believe, very important aspects that history can bring to the mathematics classroom. Students will thus be challenged – and eventually benefit – from having to appropriate knowledge from the source into their own mathematical frameworks; Elaborating on the commognitive conflicts alluded to above, we hope that in reappropriating the source into their modern framework, students are led to both an appreciation of the historical development and a richer understanding of the modern theory. Obviously, since logistic growth is a core component of the curriculum and can be included in the written exams, students must also be comfortable with such translations to the extent that they can solve relevant exam questions about the topic. If the historical source is indeed used as the focal point of teaching logistic growth, these translations will have to be explicitly stressed in order to achieve the abilities at problem-solving required for the exam.

In order to address the contextual aspects of the authentic mathematical source, we also pose exercises that can either be discussed directly in the mathematics class or can lead to integration with history, for instance in the form of individual reports which the students are required to write in the final, third year.

This is a good opportunity to explain that any mathematical creation is formed in a specific context, and it is often very important to know about this context in order to understand and explain the mathematics created, its purpose and its relevance. In the case of Verhulst's modelling, some of the relevant factors can be probed based on the following guiding tasks and questions:

1. Start by comparing two maps of Europe from, say, 1800 and 1900 and describe the changes in nations and their borders that you see.
2. Search the internet for information on the so-called "Congress of Vienna" of 1814-1815 and describe in more detail the changes to the nations of Denmark and The Netherlands.
3. Several European nations introduced new constitutions in the decades between 1830 and 1850: Belgium in 1831 and Denmark in 1848. The Danish constitution was partly modelled on the Belgium one. Analyse the constitutional rights of individual citizens granted by the Danish constitution of 1848 (chapter IV). Who were granted such rights? What shift of democratic power did this entail?
4. The new states were to a larger extent formed by (emerging) criteria of national belonging. Describe differences between the absolutist state and the nation state by focusing on the demarcation of the state.

The many new European states had a need for forming their own identity. This was partly a conscious process and partly the background for the emergence of new states in the first place.

5. Analyse some Danish national romantic works (poems and paintings) from the period 1830-1850, focusing on their role in forming a national identity. You should include knowledge from other subjects (Danish and history) in analysing the context and interpretation of these works.

An important part of the formation of new states consisted in the institutional development of a bureaucratic system for supporting parliamentarian and democratic decision processes.

6. The Danish Statistical Bureau was formed in 1849 as the precursor of the modern *Danmarks Statistik*. What can you find out about this bureau and the background for its establishment?

7. The geographic survey of Denmark was begun under the auspices of the Royal Danish Academy of Sciences and Letters towards the end of the eighteenth century. What can you find out about this project and the mathematicians who were involved in it?

Integration with other subjects

In the Danish upper-secondary school system, integration of different subjects is very important, both in the day-to-day teaching and in special projects. Thus, it becomes a special challenge to provide material suitable for the itegration of mathematics with , for instance, history. The above exercise on the context of Verhulst' model provides one example of how this can be accomplished, but other possibilities also exist based on different views of how to integrate the two topics.

Thus, whereas the content of the above exercise was essentially to situate the mathematical source in its relevant context, it is also possible to use the mathematical source – and the ideas that it entail – to situate discussions about demographics, statistics, and the governing of states and people through quantification. Although this is perhaps further from the core mathematics curriculum, it integrates well with a variety of other subjects in the upper-secondary school, including history and Danish, of course, but also spanning classical culture, social science, art, and languages. Among the topics suggested, we mention:

A central element of Verhulst's modelling is the assumption that social and human relations are regulated by laws and can be studied in the same way as (natural) science studies Nature.

1. The conception of a "social physics" as the study of social relations subjected to laws analogous to the laws of nature goes back to the scientists and philosophers Adolphe Quetelet (1796-1874) and Auguste Comte (1798-1857). What does Quetetelet mean when he in *Sur l'homme et le développement de ses facultes, ou essai de physique sociale* (1835) talks about "the average man"? What else can you find out about the social physics of Quetelet and Comte?

Human beings and their properties have been quantified in many different ways over time. One of the purposes of associating numbers to individuals has been to define what it means to be 'normal' – and sometimes 'ideal'.

2. In the arts (paintings and in particular sculptures), human proportions have been associated with numbers in order to describe 'the ideal body'. Analyse selected works by e.g. Vitruvius (ca.75BC-ca.25BC), Leonardo Da Vinci (1452–1519) or Le Corbusier (1887-1965), focusing on their description and use of ideal human proportions.
3. Attempts have also been made to quantify non-physical aspects of human beings like personality, intelligence or crime. This became most explicit in the

pseudo-science phrenology which was taken quite seriously in the nineteenth century. What can you find out about phrenology?

4. In order to illustrate how quantification can be used to define normality, do some research on the notion of ‘body mass index’ which was first suggested by Quetelet.

Thus far, students have been asked to explore some applications of quantification in social and human domains. These are not value-neutral as the following questions explore:

5. Present Malthus’ political and philosophical position which is the foundation for his analysis of population growth. Which views of revolution and democracy are behind his position?
6. The quantification of individuals is of course an important part of democratic government in which all votes are equal (with some provision for representative democracy). But it is not (and has not been) unproblematic to decide which votes and voices are to be counted. Analyse selected democracy-critical thinkers with special attention given to their presentation of the quantification of individuals.
7. The quantification of individuals can also lead to alienation and the loss of individuality and identity. Analyse selected works (fiction and movies) for their presentation of this form of alienation through quantification. Examples could include Dickens’ novel *Hard Times* (1854) or episode of the TV-series “The Prisoner” (1967). Sometimes, such alienations are related to certain political doctrines, so discussions should also include the previous task.

CONCLUSIONS AND PERSPECTIVES

In this paper, we have outlined the contents and design of our material for teaching logistic growth in Danish upper-secondary mathematics classes based on the authentic source of Verhulst’s 1838-paper. Although short, the paper allows for a variety of perspectives and avenues for students to follow in their engagement with the source: Sometimes, they will have to search for factual information about the context of the source; sometimes, they will have to work with the mathematical description to bridge the gap between the historical presentation and contemporary school mathematics; sometimes, they will be hard-pressed to understand Verhulst’s ideas about the modelling process and the relation between his “laws” governing population growth and the actual statistical data; and sometimes, they will be asked to consider much broader questions about the interactions and relations between mathematics and the social and cultural context in which it is produced and used. All these perspectives (and more) belong to what we consider relevant mathematics education in the upper-secondary level: they all contribute to understanding the processes that led to technical advances in mathematics as well as its applications to relevant problems, and if historically contextualised, they also contribute to showing how mathematics (and quantification) came to be such a dominant approach to understanding modern society. All this can be approached through unwrapping of a single historical source, as we show with our material.

We conclude with several remarks concerning the design of our materials. First, because of the open-endedness of the student-driven, enquiry-based approach and the vast variety of issues that can be taken up, the material had to be designed in an open way that invites and helps teachers to appropriate it for their specific needs. This was achieved through the use of narrative contextualisation and explanatory sections that provide pieces of information, pointers, and perspectives for the teacher to use. Second, the historiographical approach has been to contextualise and show the complexity of a *single* historical source, rather than teaching history through episodic sources connected by some theme. This is deliberate, as we both found that a well-chosen single source can allow for a richer contextualisation and that longitudinal connections are often harder to teach without resorting to simplified connections.

Finally, we have reasons to believe that this approach can be very successful in stimulating mathematical as well as historical competences in the students. When set to work with authentic mathematical texts, students are required to use both linguistic and mathematical translations that activate their representational competences. And, although the case of Verhulst's paper is ideal in many respects, because it is short, addresses a core topic in the curriculum, and opens for such vast philosophical and historical perspectives that are also on the curriculum, we believe it is not unique. We are currently working with other projects to produce other materials by the same method; and we have come to believe that although it is challenging to find the relevant sources and prepare them for teachers to use, it is also worthwhile and certainly in demand in the Danish system, where teaching history of mathematics is mandatory but where teachers complain about the lack of suited materials and about their own insecurity in teaching what is still often considered a different and difficult subject.

REFERENCES

Allchin, D. (2012). The Minnesota Case Study Collection: New Historical Inquiry Case Studies for Nature of Science Education. *Science & Education, 21*, 1263-1281.

Danielsen, K. & Sørensen, H.K. (2014). *Vækst i nationens tjeneste. Hvordan Verhulst fik beskrevet logistisk vækst*. København: Matematiklærerforeningen.

Fried, M.N. (2001). Can Mathematics Education and History of Mathematics Coexist? *Science & Education*, *10*, 391-408.

Jahnke, H.N. et al. (2002). The use of original sources in the mathematics classroom. In: J. Fauvel and J.V. Maanen (Eds.) *History in Mathematics Education. The ICMI Study*. New ICMI Study Series 6, pp. 291-328. Springer.

Jankvist, U.T. (2009). A categorization of the “whys” and “hows” of using history in mathematics education. *Educational Studies in Mathematics*, *71*, 235-261.

Jankvist, U.T. (2014). On the Use of Primary Sources in the Teaching and Learning of Mathematics. In: M.R. Matthews (Ed.) *International Handbook of Research in History, Philosophy and Science Teaching*, pp. 873-908. Dordrecht: Springer.

Johansen, M.W. & Sørensen, H.K. (2014). *Invitation til matematikkens videnskabsteori*. København: Forlaget Samfundslitteratur.

Kjeldsen, T.H. & Blomhøj, M. (2013). Developing Students’ Reflections on the Function and Status of Mathematical Modeling in Different Scientific Practices: History as a Provider of Cases. *Science & Education*, *22*, 2157-2171.

Kjeldsen, T.H. & Carter, J. (2014). The Role of History and Philosophy in University Mathematics. In: M.R. Matthews (Ed.) *International Handbook of Research in History, Philosophy and Science Teaching*, pp. 837-871. Dordrecht: Springer.

Kjeldsen, T.H. & Petersen, P.H. (2014). Bridging History of the Concept of Function with Learning of Mathematics: Students’ Meta-Discursive Rules, Concept Formation and Historical Awareness. *Science & Education*, *23*, 29-45.

Pengelley, D. (2011). Teaching with Primary Historical Sources: Should It Go Mainstream? Can It? In: C. Katz and C. Tzanakis (Eds) *Recent Developments on Introducing a Historical Dimension in Mathematics Education*, pp. 1-8. MAA Notes 78. Washington DC: Mathematical Association of America.

Sfard, A. (2010). *Thinking as Communicating. Human Development, the Growth of Discourses, and Mathematizing*. Learning in Doing: Social, Cognitive and Computational Perspectives. Cambridge University Press.

Verhulst, P.-F. (1838). Notice sur la loi que la population suit dans son accroissement. *Correspondance mathématique et physique de l’Observatoire de Bruxelles*, *10*, 113-121.

Verhulst, P.-F. (1845). Recherches mathématiques sur la loi d’accroissement de la population. *Nouveaux mémoires de l’Académie Royale des Sciences et Belles-Lettres de Bruxelles*, *18*, 1-38.

Vogel, M. et al. (1975). P.F. Verhulst’s ‘Notice sur la loi que la populations suit dans son accroissement’ from Correspondence Mathematique et Physique. Ghent, Vol. X, 1838. *Physical Biological Sciences*, *3*, 183-192.