Workshop

MAKING (MORE) SENSE OF THE DERIVATIVE BY COMBINING HISTORICAL SOURCES AND ICT

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To complement a teaching module on the introduction of the derivative, we designed three worksheets based on historical sources. The three worksheets illustrate different aspects of the derivative; select and use historical sources in different ways; and make different (but systematic) use of ICT (information & Communication Technology): dynamic geometry, programming languages, computer algebra system, and spreadsheet.

CONTEXT AND CONTENTS OF THE PAPER

We would like to present three sets of historical texts, and three classroom activities based on them. Our goal is to help high school students make sense of the new and thorny concept of the derivative. Needless to say, the literature on this topic is huge: it shows quite clearly where the main difficulties lie, and offers many fruitful leads [1]. We retained an approach emphasizing the importance of task solving, and designed a learning path which gradually expands the concept by unveiling new and *efficient* aspects: local straightness, limit-position of secants, affine approximation, and iterated affine approximation. The use of original sources *complements* these key stages.

More specifically, our target population is that of students in the French "Première" class who choose to major in maths and the sciences (the so-called scientific stream). As far as the teaching of mathematical analysis is concerned, the curricular context is the following:

Seconde Age 15-16	Basic notions on functions
Première Age 16-17 Scientific stream	The derivative • As an object: definition, geometrical interpretation, standard formulae • As a tool to study the qualitative behaviour of a function (variations, extrema...) Transversal methodological goals • Emphasis on algorithms (in the natural language or in a programming language) • Emphasis on proof and reasoning
Terminale Age 17-18	Integral calculus Transcendental functions: ln, exp, sin, cos

With high-school teacher and former teacher-trainer Sylvie Alory we designed a teaching module for the "derivative as an object" chapter. Among the various well-known ways to introduce the derivative we chose the "local-straightness" approach, which we feel provides the necessary feedback when students are to engage in rather open-ended tasks. To make a long story short, the two-week module starts and builds upon a new mathematical "experimental fact": if you use Geogebra to zoom in hard enough onto a point on a functional graph, you quickly get something that you cannot visually distinguish from a line-segment.

Two weeks later, we aim at reaching the following:

- Some functions have a derivative, some do not (at least not everywhere, for various reasons which can be illustrated graphically)
- A definition for the derivative $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, based on a loose and intuitive notion of what a limit is.
- The generic equation of a tangent to the *f*-graph is $y = f'(a)(x-a) + f(a)$
- Some basic formulae, such as $[x^n]' = n\,x^{n-1}$, $[uv]' = u'v + uv'$ …

This teaching module is not based on the history of mathematics [2]: it does not explicitly use historical sources; its design was not based on any form of rediscovery/genetic approach; when Sylvie and I discussed what we would consider to be a satisfactory way to teach this thorny chapter, the history of mathematical analysis was never mentioned. This is the reason why the present paper is not about this teaching module *per se*, but about how we chose to *complement* it.

Our background knowledge of the history of mathematics suggested that several topics could be investigated at various points of the "Première" year, in order to *flesh out* the derivative concept. In the main teaching module, we felt that two much context would stand in the way from a cognitive viewpoint; in contrast to that, we know that to make real sense of a new, tricky and rich concept such as that of the derivative, it is useful to study its role in several different contexts, to see it from a variety of angles and in various semiotic environment. We do not claim to cover all – not even all the important – contexts in which the derivative plays a part: we did not include kinematic aspects, or the reflection of light-rays, or the notion of visual horizon for an observer sitting on a curve etc... It so happens that for the three topics which we retained, we felt we could benefit from an ICT-rich environment (ICT = Information & Communication Tehcnology); in fact, the final worksheets cannot be implemented without a dynamic geometry software, a programming language, a computer algebra system and a spreadsheet. The rich environment provided by the original sources suggested that we could also address methodological teaching goals pertaining to proof and reasoning, and algorithmic thinking.

Let us give the outline of the three topics:

- Lines tangent to a circle / lines tangent to other curves
  Historical sources: Euclid, Clairaut (1713-1765).
  ICT: Dynamic geometry
  Maths contents: Tangents as limit positions of secants (chords). Proof and reasoning (reading, analysing, and assessing proofs).
- A Babylonian method to approximate square roots
  Historical source (indirect use): Cuneiform tablets BM 96957 and VAT 6598
  ICT: Programming. Dynamic geometry.
  Maths contents: Algebra and inequalities. Derivative as best local affine approximation. Proof and reasoning.
- An iterative method to approximate the roots of a polynomial
  Historical source: Euler's Elements of Algebra
  ICT: Computer algebra system. Spreadsheet.
  Maths contents: Linear approximation. Iterative algorithms and recursively defined sequences.

Needless to say, our selection of original sources depends heavily on the teaching goals. Here, historical documents are used as means to teach the derivative from a variety of angles, and tackle general methodological goals as well. In fact, we did not use any of the sources which represent milestones in the history of the calculus (tangents in Descartes, Fermat, Roberval; the calculus according to Newton and Leibniz); in contrast, these landmark texts are those we cover in our history of maths courses for pre-service teachers.

For all three topics, we designed a thematic worksheet, with a student version and a teacher version; only the student versions are presented here, with a few introductory remarks (italicized). The main teaching module has been implemented several times in the classroom, and feedback is being analyzed; the thematic-worksheets, however, have not yet been tried out. The package will be made available to teachers and teacher-trainers in France in the fall of 2014.

THEMATIC WORKSHEET #1: LINES TANGENT TO THE CIRCLE.

Students learn about the tangents to the circle in middle-school, but this first encounter with the notion of tangent is not usually very helpful when it comes to introducing a new (or, rather, more general) notion of tangent in high school: students generally remember one fact only, namely that the tangent is the perpendicular to the radius drawn from a point on the circle; a property which cannot be generalized. In this worksheet, we provide material on the basis of which the notion of tangent can be studied from a variety of angles; in particular, in Clairaut's text, a deep theorem is proved in a context where the tangent is seen as the limit position of a one-parameter family of chords with one fixed endpoint.

Dynamic geometry is used to help students visualize invariants and re-enact dynamic arguments such as Clairaut's. In this worksheet, a strong emphasis lies on methodological goals, in particular proof and reasoning. In the first part, excerpts from Euclid's Elements are analyzed, in order to exemplify the notions of "existence theorem", "uniqueness theorem" and "proof by contradiction". In the second part, students (and teachers) are confronted with a text which most of us would not consider a bona fide proof. It is interesting, and to some extent convincing, but several of its features are clearly non conventional: it relies on some form of dynamic geometrical intuition, on an implicit continuity principle; its style is very rhetorical.

Session 1: Definition and characteristic property of tangents to a circle, in Euclid's Elements (Heath, 1908).

1. In middle-school you studied the notion of a tangent to a circle. Can you recall its definition?

At the beginning of Book III of the Elements (written circa 300 BC), Euclid gave the following definition: *A straight line is said to touch a circle which, meeting the circle and being produced, does not cut the circle*.

2. Can you reformulate this definition with your own words or with diagrams; in particular, explain the difference between "touch" and "cut".

In Book III, proposition 16 reads: *The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed*.

Here is the proof of the first part of the proposition:

Let ABC be a circle about D as centre and AB as diameter;

I say that the straight line drawn from A at right angles to AB from its extremity will fall outside the circle.

For suppose it does not, but, if possible, let it fall within as CA, and let DC be joined.

Since DA is equal to DC,

the angle DAC is also equal to the angle ACD. [I. 5]

But the angle DAC is right;

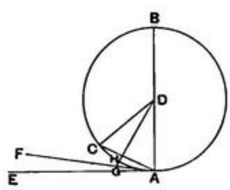
therefore the angle ACD is also right:

thus, in the triangle ACD, the two angles DAC, ACD are equal to two right angles; which is impossible. [I. 17]

Therefore the straight line drawn from the point A at right angles to BA will not fall within the circle.

Similarly we can prove that neither will it fall on the circumference;

therefore it will fall outside.



3. a. Draw a diagram showing only those elements which are relevant for this part of the proof.

3. b. The proof refers to propositions 5 and 17 from book I. Can you make a conjecture as to what these propositions state?

3. c. What makes this proof a proof by contradiction (also called *reductio ad absurdum*)?

Here is the proof of the second part of the proposition:

> Let it fall as *AE*;
>
> I say next that into the space between the straight line *AE* and the circumference *CHA* another straight line cannot be interposed.
>
> For, if possible, let another straight line be so interposed, as *FA*, and let *DG* be drawn from the point *D* perpendicular to *FA*.
>
> Then, since the angle *AGD* is right,
> and the angle *DAG* is less than a right angle,
> *AD* is greater than *DG*. [I. 19]
>
> But *DA* is equal to *DH*;
> therefore *DH* is greater than *DG*, the less than the greater: which is impossible.
>
> Therefore another straight line cannot be interposed into the space between the straight line and the circumference.

4. a. In the diagram, the position of line AF doesn't seem to be quite right. Is it a mistake, ascribable either to the author or to the publisher?

4. b. A key argument in the proof comes from proposition 19 of Book I, the content of which may not be familiar to you. This proposition states an intuitive relationship between the longest of the three sides and the greatest of the three angles, in any triangle. Can you suggest a statement of this proposition?

4. c. In what respect is proposition 16 of Book III an existence theorem? In what respect is it a uniqueness theorem?

Session 2: The alternate segment theorem, according to Clairaut (1713-1765).

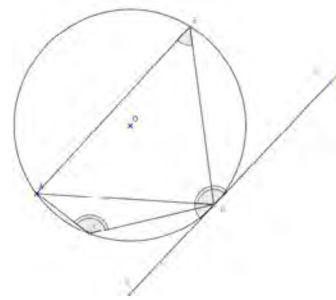
Let us consider a circle **C** with centre O; let AB be a chord (but not a diameter), and E another point on the circle. Draw triangle ABE.

1. a. You studied in middle-school a theorem about two angles subtended by the same arc AB: an angle at the centre of the circle, and an angle at the perimeter (or circumference) of the circle. Can you state this theorem?

[In Euclid's Elements, this theorem is proposition 20 of Book I]

1. b. Create a *Geogebra* file in order to illustrate this property.

1. c. In the particular case when AB is a diameter, which well-known property does this general theorem boil down to?



After proving this theorem, Euclid stated a new result on tangents, which we now call the alternate-segment theorem. Proposition 32 of Book III reads: *If a straight line touches a circle, and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle*.

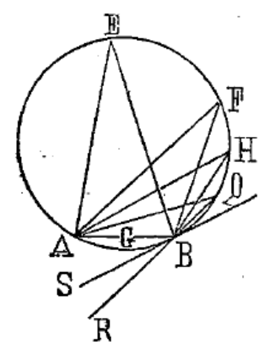
In this diagram, line SB touches the circle at B, and AB is the line "cutting the circle".

In Euclid's Elements, the proof of this proposition did not rely on proposition III.20 (studied above). However, in his Eléments de Géométrie, Alexis Clairaut (1713-1765) derived the alternate-segment theorem from proposition 20.

Here are Clairaut's proposition and his justification (Clairaut, 1853) [3]:

The tangent to a circle is the line which touches it at only one point. The angle to the segment is that between the chord and the tangent. Its measure is half that of the arc of the segment.

Fig. **87.**



Since we saw that the angles on the perimeter AEB, AFB, AHB (fig. 87) are all equal, one wonders what becomes of angle AQB as its vertex Q coincides with point B, the extremity of its base. Would this angle then vanish? One cannot see after which point this angle would cease to exist; how, then, could we measure this angle? The only way out of this conundrum is to resort to the geometry of the infinite; a geometry of which all men have some (maybe imperfect) grasp, and which we aim at improving.

Let us first observe that, as point E approaches point B, thus becoming F, H, Q etc., line EB gradually decreases, as the angle EBA which it makes with line AB increases ever more. But, however short line QB may become, the angle QBA will not cease to be an angle, since, to make it perceptible, we only need to extend line QB to point R. Will the same hold for line QB once it has decreased to the point of vanishing? What has then become of its position? What about its extension QR?

It is obvious that it becomes no other than the line BS which touches the circle only at B, without meeting it at any other points; for this reason, this line is called the tangent.

Moreover, it is clear that as line EB continuously decreases and eventually vanishes, the line AE, which successively becomes AF, AH and AQ etc., comes ever closer to AB, and eventually coincides with it: hence the angle AEB subtended at the *perimeter, after becoming AFB, AHB and AQB, eventually becomes the angle ABS between chord AB and tangent BS; and this angle, which is called the alternate-segment angle, must retain the property of being half of the measure of arc AGB.*

In spite of the fact that this proof may be a little abstract for the beginner, I thought fit to include it, since it will be very useful for those who will further their study into the geometry of the infinite to become accustomed to these considerations fairly early on.

2. a. Illustrate Clairaut's reasoning on your *Geogebra* file, using E as a moving point.

2. b. Would you call Clairaut's reasoning a *proof*?

2. c. In his reasoning, Clairaut never mentioned the fact that a tangent to the circle is perpendicular to a radius. What are the two features of the tangent to a circle that he mentioned or used?

2. d. Compare Clairaut's notion of tangent to the notion used in your lesson on the derivative of a function.

2. e. Clairaut wrote that his reasoning could help accustom beginners to the *geometry of the infinite*. In your opinion, did he mean "infinitely small" or "infinitely large"?

THEMATIC WORKSHEET #2: A BABYLONIAN PROCEDURE TO APPROXIMATE SQUARE ROOTS

In this worksheet, an approximation method is first studied from a mathematical viewpoint independently from the derivative context (only basic algebra and the algebra of inequalities are required); the method is also implemented in a programming language (ALGOBOX being the one most commonly used in French schools). The connection with the derivative is made in the third part, where the intuitive notion of "best linear approximation" is brought into the picture.

Here, the use of historical sources is quite unusual. We felt the Babylonian tablet was too difficult to study, which is why, in the second part of the worksheet, we decided a secondary source could be studied instead. We chose an excerpt from Fowler and Robson's paper on tablet YBC 7289 to discuss a possible geometric argument accounting for the approximation method. Of course, we would not object to anyone using the original source in the classroom! This is why it is included in an Appendix, along with reading tips.

Part 1

Some Babylonian clay tablets from the 2nd millennium BC display a procedure to approximate the square root of a number. This procedure can be summed up as follows:

To find the square root of N, look for the largest integer B whose square is less than (or equal to) N. We have $N = B^2 + A$. An estimate for $\sqrt{N}$ is given by $\sqrt{N} = \sqrt{B^2 + A} \approx B + \frac{A}{2B}$.

1. What output values does this procedure yield for the following $\sqrt{N}$:

$$\sqrt{104} \qquad \sqrt{4{,}5} \qquad \sqrt{10} \qquad \sqrt{81} \qquad ?$$

2. As we know, an approximate value can be either above (an overestimate) or below (an underestimate) the target exact value.

2. a. In the examples from question 1, does this procedure provide underestimates or overestimates? Can you answer this question without using the "square root" button of your calculator?

2. b. Work out the square of $B + \frac{A}{2B}$. How can you generalize your answer to question 2.a?

3. We would like to write an algorithm implementing this procedure for any input integer chosen by the user.

3. a. In the first part of the algorithm, when looking for the value of B, will we need a FOR-loop or a WHILE-loop?

3. b. Write the complete algorithm. Check it with the values studied in question 1, and for perfect squares.

4. The Babylonian method is deeply connected to the following approximation formula:

T formula: if a is close to zero, then $\sqrt{1 + a} \approx 1 + \frac{a}{2}$

4. a. Interpreting "close to zero" as "lying between 0 and 1", check that formula T is a special case of the what the Babylonian procedure yields.

4. b. With the same interpretation, show that, in the Babylonian method, $\frac{A}{B^2}$ is close to zero when B is greater than 2.

4. c. Under this condition, factorize B^2 in $\sqrt{N} = \sqrt{B^2 + A}$, and show that the Babylonian formula is a special case of formula T.

Part 2

5. Historians of mathematics David Fowler and Eleanor Robson (Fowler & Robson, 1998, pp.370-372) reconstructed a geometrical argument which they think could have led Babylonian mathematicians to their procedure.

5. a. Up to now, we focused on the following numerical problem: "to find the square root of a given number". Can you think of a geometrical problem which would – to a large extent – be equivalent to this numerical problem?

Here is an excerpt from the Fowler and Robson paper:

(...) to help the reader, we shall use lower-case names such as "approx", "new approx" for lengths, and capitalized names such as "Number" and "Bit" for areas.

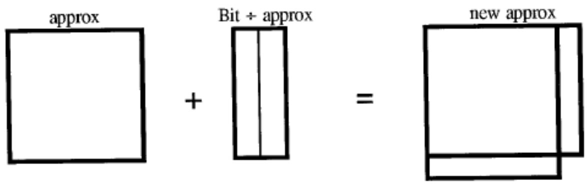


Figure 2

So suppose we want to evaluate the "side of a Number" (our square root). We start from some approximation, and let us first examine the case where this is an underestimate, so

Number = Square of approx + Bit

which, geometrically, can be represented by the sum of a square with sides approx and the leftover Bit. Now express this Bit as a rectangle with sides approx, and therefore, Bit ÷ approx, or, Old Babylonian style, Bit × IGI approx [IGI means *reciprocal*]*; cut this in two lengthwise, and put the halves on two adjacent sides of the square root of approx, as shown in Fig. 2. Hence*

new approx = approx + half of Bit x IGI approx,

and it will clearly be an overestimate because of the bite out of the corner.

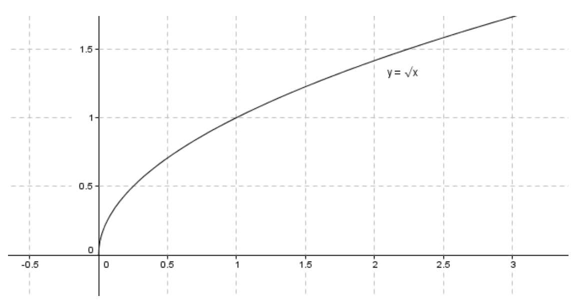
5. b. Draw the diagrams which illustrate Fowler and Robson's argument for the case $N = 104 = 10^2 + 4$

5. c. Use the letters N, B and A in the diagrams to recover what we called the Babylonian formula.

5. d. Explain "it will clearly be an overestimate because of the bite out of the corner", and compare with question 2.b.

6. We saw that the Babylonian procedure was, to a large extent, equivalent to formula T. Leaving aside Babylonian-style arguments, we will see that the study of the square root function (and its graph) can lead to formula T.

Let us define f by the formula $f(x) = \sqrt{x}$, where x denotes a non-negative real number. Let C denote the graph of f.



6. a. Work out $f(1)$ and $f'(1)$.

6. b. Show that the tangent line to curve C at point (1,1) has equation

$$y = \frac{1}{2}(x-1)+1$$

6. c. For which values of x do you think the following formula would be relevant, and why?

$$\sqrt{x} \approx \frac{1}{2}(x-1)+1$$

6. d. Substitute 1 + *a* for *x* in the formula. What formula do you get, and for which values of *a* would it be relevant?

6. e. Both formula T and the Babylonian procedure yield overestimates. Can you make sense of it geometrically?

6. f. One could come up with a wealth of linear approximation formulae similar to formula T, such as

if a is close to zero, then $\sqrt{1+a} \approx 1 + \frac{a}{3}$

if a is close to zero, then $\sqrt{1+a} \approx 1 + 2a$

if a is close to zero, then $\sqrt{1+a} \approx 1 - \frac{a}{2}$

Could a quick look at the graphs suggest that they yield poorer estimates than formula T?

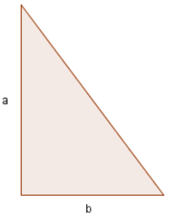
Appendix: A Babylonian worked exercise

Clay tablets BM 96957 and VAT 6598 display a series of worked exercises. Here is a translation of one of them:

A gate, of height ½ <rod> 2 cubits, and breadth 2 cubits. What is its diagonal? You: square 0;10, the breadth. You will see 0;01 40, the base. Take the reciprocal of 0;40 (cubits), the height ; multiply by 0;01 40, the base. You will see 0;02 30. Break in half 0;02 30. You will see 0 ;01 15. Add 0 ;01 15 to 0;40, the height. You will see 0;41 15. The diagonal is 0;41 15. The method.

Reading help:

To work out the length of the diagonal of a rectangle with sides *a* and *b* units of length, assuming *a* > *b*, the procedure corresponds to the following formula:



$$diagonal = \frac{(b^2 \times reciprocal(a))}{2} + a$$

Which can be interpreted as a combination of Pythagoras' rule, and the approximation method studied above:

$$diagonal = \sqrt{a^2 + b^2} \approx a + \frac{b^2}{2a}$$

Additional help:

- Units of length : 1 rod = 12 cubits (≈ 6m) (see Fowler & Robson op. cit., p.369 footnote 8)
- The numerical system used in this tablet has base 60 (sexagesimal system). In the transcription chosen by Fowler and Robson, the semicolon separates the whole part from the fractional part, and the 59 "digits" are separated by blank spaces.

  Example: 0 ; 01 40 stands for 0+ 01/60+40/3600
- Breadth = 2 cubits = 2/12 rod = 10/60 rod = 0 ; 10 rod.
- $(10/60)^2$ =100/3600=(60+40)/3600=60/3600+40/3600=1/60+40/3600 = 0 ; 01 40
- Height = ½ rod 2 cubits = 6 cubits + 2 cubits = 8 cubits = 8/12 rod = 40/60 rod = 0 ; 40
- The reciprocal of 40/60 is 60/40=(40+20)/40=1+20/40=1+1/2=1+30/60 = 1 ; 30

To know more about Babylonian mathematics, and learn how to use on-line sexagesimal calculators, you can Google *mesomath.*

THEMATIC WORKSHEET #3: AN ITERATIVE METHOD TO APPROXIMATE THE ROOTS OF A POLYNOMIAL

The French and English editions of Euler's Elements of Algebra include a very clear exposition of Lagrange's version of the Newton-Raphson approximation method (also known as the method of tangents). In addition to introducing a standard and powerful approximation method, it enables us to focus on two different mathematical topics.

The first one is that of iterative methods: these can be studied from an algorithmic viewpoint, or, if formulated in terms of formulae, through recursive sequences. Here, the algorithmic aspect is not studied with a programming language but with a computer algebra system, which is used step-by-step in an iterative way. The "sequence" point of view is studied both with a spreadsheet, and on pen and paper, to yield formulae such as $x_{n+1} = \frac{2(x_n)^3+2}{3(x_n)}$.

The other topic is, of course, that of the derivative. Here the notion of linear approximation is studied from the numerical angle, in a polynomial context. Euler does not mention the more general context in which the notion of derivative becomes necessary. We leave this to the teacher if he/she pleases, since the method of tangents is studied in most textbooks, usually from a graphical viewpoint.

Session 1: Discovering a new method

We shall use – and try to account for – a method for solving polynomial equations by approximation. This method can be found in many texts; we will use Leonard's Euler (1707-1783) *Elements of Algebra* (Euler, 1822, p.289).

> **786.** We shall illustrate this method first by an easy example, requiring by approximation the root of the equation $x^2 = 20$.
>
> Here we perceive, that x is greater than 4 and less than 5; making, therefore, $x = 4 + p$, we shall have $x^2 = 16 + 8p + p^2 = 20$; but as p^2 must be very small, we shall neglect it, in order that we may have only the equation $16 + 8p = 20$, or $8p = 4$. This gives $p = \frac{1}{2}$, and $x = 4\frac{1}{2}$, which already approaches nearer the true root. If, therefore, we now suppose $x = 4\frac{1}{2} + p'$; we are sure that p' expresses a fraction much smaller than before, and that we may neglect p'^2 with greater propriety. We have, therefore, $x^2 = 20\frac{1}{4} + 9p' = 20$, or $9p' = -\frac{1}{4}$; and consequently, $p' = -\frac{1}{36}$; therefore $x = 4\frac{1}{2} - \frac{1}{36} = 4\frac{17}{36}$.
>
> And if we wished to approximate still nearer to the true value, we must make $x = 4\frac{17}{36} + p''$, and should thus have $x^2 = 20\frac{1}{1296} + 8\frac{34}{36}p'' = 20$; so that $8\frac{34}{36}p'' = -\frac{1}{1296}$, $322p' = -\frac{16}{1296} = -\frac{1}{36}$, and
>
> $$p = -\frac{1}{36 \times 322} = -\frac{1}{11592}:$$
>
> therefore $x = 4\frac{17}{36} - \frac{1}{11592} = 4\frac{4473}{11592}$, a value which is so near the truth, that we may consider the error as of no importance.

1. Euler stated without justification that $\sqrt{20}$ "is greater than 4 and less than 5". Can you justify it?

2. Using pen and paper only, carry out Euler's calculations up to the $4\frac{17}{36}$ value. You should use different symbols (such as = and ≈) to distinguish between "equal" and "approximately equal".

3. Is $4\frac{17}{36}$ a better estimate of $\sqrt{20}$ than $4\frac{1}{2}$? You may use your calculator.

4. Euler repeatedly claimed that p^2 is "very small", hence can be "neglected". Does this sound reasonable to you?

5. To carry out Euler's procedure, all we need to do is to expand squares of sums, and solve linear equations; a computer algebra system can do this for us.

5. a. Carry out Euler's computations using *Geogebra's Algebra View*.

5. b. Euler stops at x=$4\frac{4473}{11592}$. Is this what you find with *Geogebra*? Is it the exact value of $\sqrt{20}$?

5. c. Carry out Euler's procedure one more time, to get an even better estimate of $\sqrt{20}$.

5. d. Work out $\sqrt{20}$ with your calculator. How many times do you need to carry out Euler's procedure to get the same number of decimal places?

Session 2: Applying the method to a variety of equations

After this detailed exposition of the case of equation $x^2 = 20$, Euler quickly explained how to adapt the same method to other equations.

6. For instance, for $x^3 = 2$ he wrote (Euler, 1822, p.291):

For example, let $x^3 = 2$; and let it be required to determine $\sqrt[3]{2}$. Here, if n is nearly the value of the number sought, the formula $\frac{2n^3+2}{3n^2}$ will express that number still more nearly; let us therefore make

1. $n = 1$, and we shall have $x = \frac{4}{3}$,
2. $n = \frac{4}{3}$, and we shall have $x = \frac{91}{72}$,
3. $n = \frac{91}{72}$, and we shall have $x = \frac{162130896}{128634294}$.

6. a. How can you be sure that there is a number between 1 and 2 whose cube is equal to 2? Is there only one?

6. b. Use the computer algebra system to check Euler's calculations.

6. c. Use algebra to justify Euler's claim: "if n is nearly the value of the number sought, the formula $\frac{2n^3+2}{3n^2}$ will express that number still more nearly". You may use either pen and paper, or *Geogebra*.

6. d Use the $\frac{2n^3+2}{3n^2}$ formula in a spreadsheet software to display a sequence of ever more accurate approximations of $\sqrt[3]{2}$.

7. Same questions as in (5), for the following excerpt (Euler, 1822, p.291):

790. In order to apply this operation to an example, let $x^3 + 2x^2 + 3x - 50 = 0$, in which $a = 2$, $b = 3$, and $c = -50$. If n is supposed to be nearly the value of one of the roots, $x = \frac{2n^3+2n^2+50}{3n^2+4n+3}$, will be a value still nearer the truth.

Now, the assumed value of $x = 3$ not being far from the true one, we shall suppose $n = 3$, which gives us $x = \frac{61}{21}$: and if we were to substitute this new value instead of n, we should find another still more exact.

NOTES

1. The list of references could go on forever. For a classical list of references, see (Artigue, 1991).

2. For a nuanced approach to the teaching of analysis, enriched on the basis of the historical knowledge of task designers, see (Hauchard & Schneider, 1996).

3. Free translation by R. Chorlay.

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