Oral Presentation

FORMAL PROOF AND EXPLORATORY EXPERIMENTATION: A LAKATOSIAN VIEW ON THE INTERPLAY BETWEEN EXAMPLES AND DEDUCTIVE PROOF PRACTICES IN UPPER-SECONDARY SCHOOL

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This paper investigates conceptions of mathematical investigation and proof in upper-secondary students. The focus of the paper is an intervention that scaffolds the interaction between open explorative activities and the development of proof sketches through explorations of lattice polygons, aiming at proving Pick's theorem. In the process we investigate whether and how the conceptions of proofs and explanations in mathematics change. We work with the hypothesis that the problem of supporting the transition to deductive proofs in upper-secondary school students can at least partly be explained as a problem of bringing their empirical investigations into the deductive proof process in relevant and productive ways. Through our analyses of the portfolios and deliberations of the students, we are able to assess their performance of proofs and the conceptions of mathematical methodology before and after the intervention.

EXPERIMENTS AND PROOFS IN MATHEMATICS AND MATHEMATICS EDUCATION

Empirical and deductive proof schemes

The tendency among upper secondary students to "prove" mathematical statements by examples rather than by universal deductive reasoning has been established as a robust research result in mathematics education research (Arzarello et al., 2011). This educational problem is described as students possessing "empirical proof schemes" opposed to "deductive proof schemes". Phrased in these terms, a large amount of empirical studies have shown that students have difficulties performing and internalizing the movement towards deductive proof schemes, and that empirical proof schemes, and more broadly work with examples gives rise to difficulties (such as misunderstanding, difficulties and confusions) with the acquisition and performance of deductive proofs.

Such conflict results from "*the concept of formal proof is completely outside mainstream thinking*" (Arzarello et al., 2011, p. 51) suggesting an irreducible gap between everyday empirical thinking and formal mathematical thinking. The existence of such a gap is well supported by results from cognitive science (Kahneman, 2011), but little is known about specific approaches to overcome this gap and especially: "*the evidence about the transition from empirical to general proof schemes is based on* *limited evidence collected in suitable environments"* (Arzarello et al., 2011, p. 53).However a few distinctions can already be made: (1) empirical proof-schemes can be seen either as a necessity or as a problem in the transition to deductive proofs, and (2) the transition to deductive proofs can be seen either as a radical change in the mode of reasoning or as a natural continuation and refinement of empirical proof schemes.

In this project we suggest a continuous approach, activating rather than suppressing example work and the empirical proof schemes inherent in the students. These choices are informed by new "maverick" trends in the philosophy of mathematics, suggesting that investigations and heuristics are closely connected to more formal justificatory practices in mathematics.

Exploratory experimentation as a maverick approach to mathematical justification – Lakatos on the mathematical proof

Much traditional philosophy of mathematics has focused on providing accounts of the certainty of mathematical results. However, over the past decades, a new 'maverick' trend has been focusing on a broader and practice-informed philosophy of mathematics (Davis & Hersh, 1981; Lakatos, 1976; Mancosu, 2008). Among the insights thus produced is that the sharp context-distinction between a *context of discovery* and a *context of justification* does not square well with actual practice. In particular, Imre Lakatos' (1922-1974) book *Proofs and Refutations* put great focus on the informal aspects of mathematical knowledge production and on the epistemic roles played by examples and counter examples (Lakatos, 1976). Lakatos argued by a rational reconstruction of the history of Euler's polyhedral formula that counter examples and proof analysis play crucial roles in shaping mathematical concepts and developing increasingly refined proofs.

On Lakatos' account, the dialectic process of proofs and refutations (counter examples) can be used to develop mathematical knowledge about initially naively defined or partially understood concepts. Thus, if the classic context-distinction was to be imposed, Lakatos' dialectic belongs partly to the realm of heuristics in gaining insights about those concepts and partly to the realm of justification in providing and shaping the proofs of the theorem as they develop.

Thus, Lakatos implored us, mathematical statements are not static and do not epistemologically predate their warrants; and conversely a mathematical proof is not an analytical afterthought warranting a previously existing mathematical insight. Rather, proof-practices are active in creating the mathematical landscape of theorems and claims.

Recently, new practice-oriented trends in the philosophy of mathematics have investigated how the present availability of desktop computers with flexible mathematical software systems increases the interplay between proving, investigating examples, and suggesting new theoretical concepts. Using computers not only to verify proofs or generate data for heuristic conjecture formation, it is possible to undertake what has been described as "exploratory experimentation" in mathematics in which concepts are formed through experimentation and in which experiments critically inform (if not warrant) proof (Sørensen, 2010 see also e.g. Borwein 2012).

In this paper we bring these two conceptions from the recent philosophy of mathematics – the continuous overlap between empirical and deductive proof schemes involved in exploratory approaches to mathematical research and the specific role of computer-assisted experimentation to bear on the didactical situation where the two proof schemes are often (misguidedly, we claim) separated. We do so by first detailing the discussion of different proof schemes and their potential overlap, before we discuss the role of computers in exploratory experimentation. We then describe the context and content of our intervention and the data produced, which is subsequently analyzed, bringing to the fore both some of the successes in integrating exploratory experimentation in mathematics education and some problems which students experienced in completing the transition to deductive proof schemes.

DISCOVERY AND JUSTIFICATION: TWO DISTINCT CONTEXTS OR BLENDED DOMAINS?

The transition from experimentation with specific examples to formal proof can be studied as a change from a heuristic context of discovery to a justificatory context of proof or as a matter of drawing upon both empirical and deductive proof schemes (Arzarello et al., 2011). Lakatos suggests that even though we can talk about a transition *to proving,* there is no such thing as a *transition away from working with examples.*

What Lakatos thus points out is that, in mathematics, the contexts of discovery and justification are not to be too sharply distinguished, neither temporally nor methodologically. Initially, Lakatos' analyses were aimed at research-level mathematics and the production of new mathematical knowledge, but they also have important implications for mathematics education, such as have been pursued by contemporary mathematics educators (see e.g. Ernest, 1991).

Therefore we suggest using the notion of contemporaneous empirical and deductive proof schemes to conceptualize not different contexts that students move in and out of in a binary fashion, but as relating to different domains influencing their experiences of working with mathematics. In our analysis we will discuss whether or not students *frame their activities towards the domains of examples or towards the domain of formal proofs.* This distinction is an analytical one inspired by (Hanghøj, Misfeldt, Bundsgaard, Dohn, & Fougt, n.d.), and we expect to see that students express references to both domains in their work with constructing proofs..

The domain of tasks and mathematical examples

When students in Danish upper-secondary mathematics classes work with word problems and similar tasks, they work almost exclusively with examples. Hence, such examples represent tasks and situations where the student is expected to apply mathematical theory. Formulas are *tools for working with examples* and proofs are hardly relevant when considering examples. The type of reasoning applied when working with word-problems is deductive, but specific: students need to use rules, theorems and formulas to calculate the solution to a certain problem.

Moreover, in their textbooks examples are often used to show how a certain type of task is performed or how a mathematical result is activated. And finally, examples can be used in a theory-generating fashion – typically as motivational devices preceding a theoretical construct. Hence in the domain of examples *the objects* are *specific rather than general*, *formulas* are *tools*, and *proofs* are of *little relevance. Such a view* promotes a process going from *problem situation* to *solution* by *using mathematical theory*, as well as a tendency to describe the involved objects in *specific* rather than general *terms.*

The generative uses of examples described above come close to the way examples will be used in our material: Focusing on the transition from examples to proofs, we will use examples (of lattice polygons) both in order to motivate, as specific stand-ins for general objects, and as objects unto which the general theory is to be applied Yet, our intervention is designed so as to facilitate a *continuous* transition in which knowledge acquired in the empirical investigation of examples is to feed constructively into the shaping of deductive proofs.

The domain of formal proof

The domain of formal proof differs from the domain of examples in a number of ways. On the object-level *proofs* and *theorems* are at the *center of the activity*, and correspondingly, on the meta-level, the involved objects are described *as generally as possible* and the argumentative schema goes *from theorem* (stating a result) *to proof* (warranting the result). When students in Danish upper-secondary schools work with formal proofs they are usually expected to read and understand these proofs and in some cases also memorize and perform them. In this context, formal proofs usually have to them the flavor of “divinely informed calculations” with little explanatory motivation given. Understanding the proof largely consists in remembering a few main ideas, typically developed over generations of mathematicians to a very elegant and condensed form. It is much rarer for these students to develop their own mathematical proofs. Hence in the domain of formal proof, the *official mathematical text is at the center;* whereas in the domain of examples the *student’s own voice is acknowledged*.

EXPLORATORY EXPERIMENTATION IN A LAKATOSIAN FRAMEWORK

As mentioned, Lakatos' original description of the proof process saw it as a perpetual dialectic between what we call domains of examples and domains of formal proof. Building on this, we suggest to add a process of *presenting codified proofs* such as they typically come to appear in accepted mathematical communication, including textbooks. Obviously, as is one of Lakatos' main points, such a codified proof could still be subjected to further dialectic treatment, but it seems to us an important part of the process of teaching mathematics to reach a recognizable, relatively stable notion of a (written, codified) proof. Furthermore, still building on the Lakatosian approach, we wish to emphasize the wide applications for exploratory experimentation, some of which (examples and counter-examples) are to be found also in works by Polya (Pólya, 1945) and Lakatos. Such experimentation is readily available through the use of software, the use of which is, itself, a goal of Danish upper-secondary mathematics education and, of course, a topic of educational research (see also Conner et al, Guven et al., Guin, Ruthven, & Trouche, 2005) However, as we aim to show, such experimentation can also be important (beyond the roles of mere motivations or illustrations) for shaping sub-arguments of larger mathematical proofs, thereby also giving rise to proof-generated concepts as emphasized by Lakatos. The result of these considerations is an envisioned process of moving from idea generation to experiment to proof. This process is shown in figure 1 and represents our envisioned learning trajectory (Cobb & Gravemeijer, 2008) for the movement from example to proof in the students of our intervention.

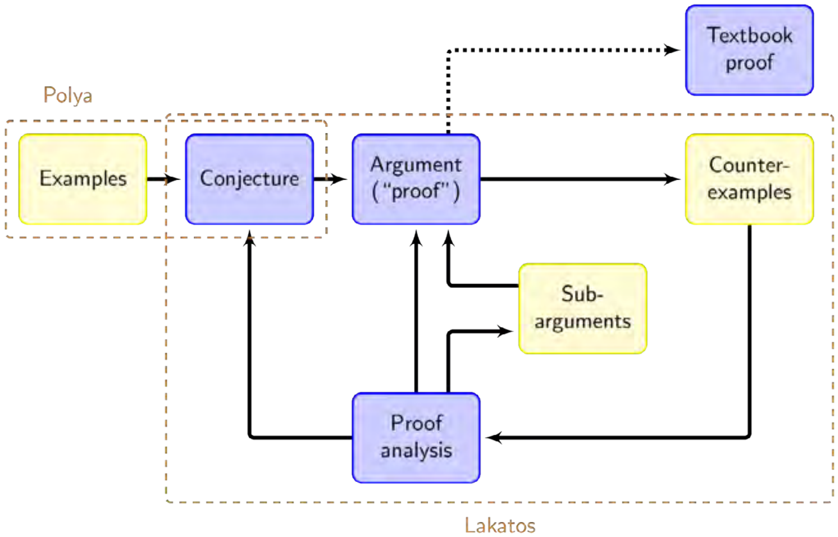

Figure 1: The envisioned process of experiments and proofs (and our intended learning trajectory) based on inspiration from Polya and Lakatos. This scheme shows

the Lakatosian dialectic of conjectures, refutations, and proof analysis leading to refined conjectures and proofs. It also suggests how examples (yellow boxes) have multiple functions, both in forming conjectures (Polya), as refutations that prompt proof analysis (Lakatos) and as means to calibrate proof analysis and lemmas (Lakatos). Added to the Lakatosian framework is our suggestion of a process of presenting codified (textbook-like) proofs that transcend the dialectic of proofs and refutations.

Hence we view the classical context-distinction not as transition, nor as just two complementary views on the subject, but as a process with repeated feedback loops where activities are framed towards different domains at various stages (see figure 1).

THE EDUCATIONAL SCENARIO

Our intervention is centered on a beautiful, yet somewhat atypical and slightly complicated theorem about areas of polygons in a lattice. In this section we describe this result – Picks theorem – and suggest why it is an interesting case to support the development of formal proof strategies. In the next section we then proceed to describe the teaching material and the classroom intervention we have conducted. For further documentation, we refer to (Danielsen, Misfeldt and Sørensen, 2014).

Pick's Theorem

The theorem at hand is known as Pick's Theorem named after the Austrian mathematician Georg Alexander Pick (1859-1942), who first described the result in 1899. Published under the title "Geometrisches zur Zahlenlehre", Pick's theorem is located on the intersection of geometry and arithmetic that was cultivated around 1900, in particular by the German mathematician Hermann Minkowski (1864-1909). Educated in Vienna, Pick spent his entire career in Prague, where he also published his result in a relatively obscure journal of the German-language scientific and medical association (Pick, 1899). During the 1930s, Pick became a victim of Nazi persecution, and he perished in Theresienstadt in 1942. Over the years, the theorem has been proved repeatedly and in various ways; it has also been used to train mathematics teachers at various levels, but it is (we believe) relatively rarely taught to students. As a theorem, it is remarkable for a number of reasons that include the following:

a) It can be inductively approached using either physical lattices (in Danish: "sømbræt") or computer-based experimentation (see figure 2).
b) It links two domains of mathematics by showing that in some cases, you can actually *count* an area which is normally something to be *measured*.
c) It involves a number of basic geometrical ideas such as triangulation and knowledge about basic geometrical concepts such as polygons, areas of triangles etc.
d) Its proof is slightly more complicated and intricate than proofs by traditional derivations; yet, it is at a level of complexity where it can be taught to students.

The theorem provides a way of computing the area of a lattice polygon, i.e. a polygon whose vertices are located in the grid (lattice) Z × Z. If P designates such a lattice polygon, i(P) counts the number of interior lattice points in P, and b(P) counts the number of lattice points located on the boundary of P, then Pick's theorem states the relation $A(P) = i(P) + b(P)/2 - 1$, where A(P) is the area of the polygon P. The proof traditionally operates by three important steps:

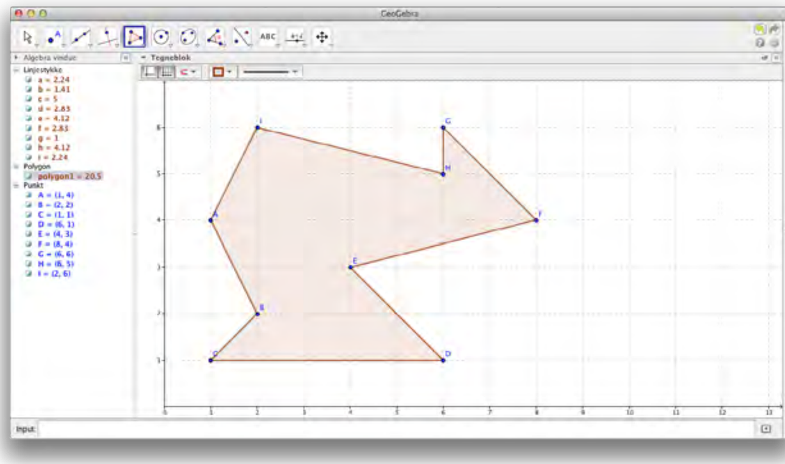

Figure 2: Example of a lattice polygon drawn in GeoGebra.

1. Proof that the Pick function defined by $\Pi(P) = i(P) + b(P)/2 - 1$ is additive when two adjacent lattice polygons are merged into one.
2. Proof that any lattice polygon can be triangulated into lattice triangles.
3. Proof that for any lattice triangle T, $\Pi(T) = A(T)$.

This three-step proof scheme might appear complicated or foreign to students, since it does not reduce to either a calculation or a traditional Euclidean proof scheme. It purports to show a complicated identity by showing that the identity holds for atomic configurations and that it is preserved when complex configurations are built up from such atomic building blocks. Although such proofs are relatively rare in teaching on the upper-secondary level, similar proofs are actually abundant in mathematics, and students will also encounter them, for instance when it is shown that any (sufficiently simple) function is differentiable.

Teaching material and educational intervention

The educational intervention was situated in one upper-secondary class (senior year, 3.g STX MAT-A) taught by the second author of this paper. It consisted of 10 onehour lessons and was planned to consist of 5 modules. As an answer to the students' needs, the teacher added two more modules. Six of the seven modules were built on the same template (described below) and the last module was a blackboard-based proof of Picks theorem serving an institutionalization purpose showing the students how the knowledge that they had explored and the propositions that they had justified fit into a larger landscape of official, codified mathematical knowledge (Brousseau, 1997). Each module contained the following elements:

- Introductory activity: a simple activity introducing one of the ideas in the module in a simple way
- Closed task introducing an important tool or concept
- Investigation prompted by an open task/invitation
- Buffer activity to make sure that everyone had something relevant to do
- A collaborative reflection activity

The rationale behind this template was to scaffold (1) individual or small group investigations of a specific aspect of mathematics, and (2) collective reflection and formulation of results of that activity. For instance, in one module (module 5) on triangulation of lattice polygons, the work sheet involved the following activities:

- Activity 5.1: What happens to points and areas when a polygon is divided into two (or more) polygons?
- Activity 5.2: What happens to points and areas when two (or more) polygons are put together?
- Activity 5.3: Formulate some rules for the number of points in a lattice polygon when you divide a polygon or put polygons together. You should introduce some suitable names and notation for the elements you use. Make the rules as simple as possible and save all your suggestions for later (also the ones that turn out to be wrong).
- Activity 5.4 (buffer activity, intended to make sure that students are at same pace, when starting activity 5.5); Try your best rules on a lot of different cases, using the computer to produce the cases. Correct the rules if necessary. Save all your suggestions (also the one that turn out to be wrong).
- Activity 5.5; Do your rules hold in all cases? Do you think you have made a theorem? Do you think the theorem is proved?

The teacher was mainly acting as guide and supervisor with respect to the mathematical aspects and as a process facilitator with respect to the progress of the modules.

The topics dealt with in the individual modules were:

1. Module 1: Areas of polygons
2. Module 2: Lattice polygons

3. Module 3: Areas of simple lattice polygons
4. Module 4: Generals aspects of the area of lattice polygons
5. Extra Module: Formulas
6. Module 5: Triangulation of lattice polygons
7. Extra Module: proof conducted on the blackboard as a combination of lecture and plenary discussion in the class.

Upon completion of the work originally intended to prove Picks theorem (module 5 and the extra module), the teacher decided to change the work method towards whole-class discussion and lecture. The rational for doing that was that the students were increasingly working without direction and with the need of so much teacher guidance that the idea of working individually and in small groups collapsed and a whole-class discussion seemed like the healthy pedagogical choice. This “collapse” happened in the transition from experiments generating and verifying formulas for the area of lattice polygons to establishing a mathematical theorem with a proof and as such is very interesting for our design; therefore this “collapse” is further described and discussed in the data and results sections that follow.

DATA AND RESULTS

As data from the invention, we can draw on the teacher’s impressions and experiences teaching the students combined with various products of students and their answers to a questionnaire with six qualitative questions:

1. How did the activities in the modules work? How can they be improved?
2. Did you learn anything about mathematical objects (triangles, polygons, functions, etc.)?
3. Did you learn anything about formulas?
4. Did you learn anything about proofs?
5. Did anything surprise you?
6. Do you have other comments?

The students generally enjoyed working with the material and considered it a nice variation away from the typical classroom work. They found it nice to work in a different way with a subject, and they liked the structure with more independent work enjoying the opportunity to take active part in developing mathematical theory; Representative of their evaluations, they expressed that:

"It was a good and different way of working with polygons" and similarly *"It was good with more independent work, which was followed up afterwards."*

Although the students liked to work independently, some of them also lacked the overview gained from a lecture structured by a teacher. In several cases, the students were able to express rather elaborately what they had learned. On the topic of creating mathematical results as consequences of considering, combining and dividing into simple examples, several different students expressed view such as:

"I have learned how to come up with results in different ways. For example by looking at a triangle, it is possible to say something about a square. I have also learned that there is a relationship between the dots in a lattice polygon and its area ... "

"We have been working on how a complex object can be simplified by dividing it into simpler shapes."

"We have learned something about how to calculate the area of polygons by dividing them into triangles. We also learned something about the connection between the formulas you can use to calculate the area of lattice polygons, and how to derive such formulas."

Several students highlighted the fact that they were able to develop their own formula to describe a rather surprising and strong mathematical result. Making your own formula was considered fun:

"It was fun to try to come up with your own formula to solve a particular problem. It is probably this activity, which I liked best."

It was also considered less complicated than the students would have thought, and directly connected to inductive reasoning:

"I've learned that by sitting and trying various possibilities, you can relatively easily come up with your own formula, and it need not be very complicated."

Furthermore the students expressed that they had leaned something about mathematical reasoning and proving. Some noticed that proofs can consist of many independent parts: "*Yes, [we have learned] that a proof can easily contain smaller elements of proofs that together form the basis for proving the same theorem."* And similarly "*The proof we did was different from the ones we normally do, because the proof was divided into several parts (various geometric shapes) to eventually cover all polygons."* The open nature of the proof-process was also explicitly noticed by some students as being different: *"[We have learned] that there are many ways to prove a claim; and this is different from the classical proofs we have done in the past."*

ANALYSIS

Two positive results in terms of the students' mathematics learning are suggested by the data:

1. The students are able to come up with a formula based on experimenting with various cases. The students expressed that it is new to them that formulas in that sense can be created.
2. The students participate in warranting their formula by considering how the formula is true for various examples and classes of examples.

The aim of this project was to create a situation where the inductive reasoning would suggest structure and propositions strongly enough that some students would eventually start proving without strong teacher support. This did not happen to the degree we had envisioned. The students experienced a lot of difficulties with the transition from considering (classes of) examples to a formal proof. However finding the mathematical result and starting the line of argumentation warranting this result in simple examples was possible for the students.

There are several aspects of the two results mentioned above that suggests that our intervention has activated and blended the two domains of proofs and examples. The students are surprised that they are able to come up with an official mathematical result themselves; they expected this to be "*much harder*". This can be described by observing that official mathematical theory resides in the domain of formal mathematics where students, by the prototypical conception, are not able to contribute. However examining a large number of examples is relatively easy for the students using suitable software, and when they do that using our material, it is easy to propose the general formula.

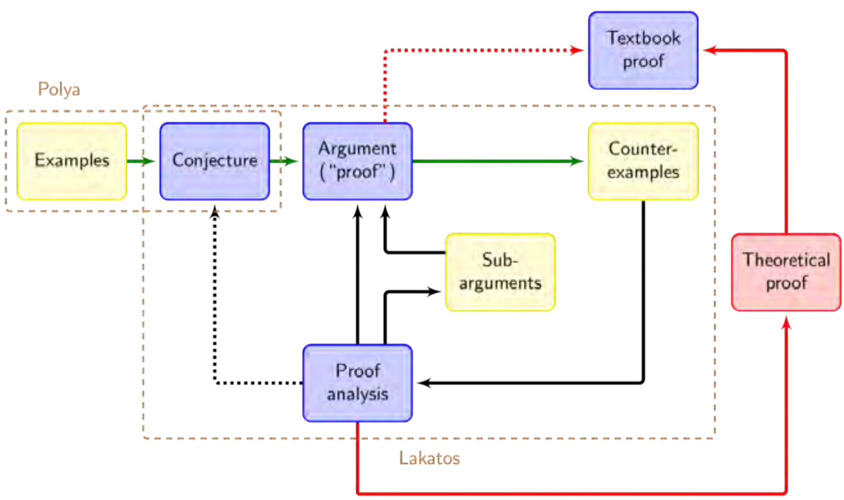

Figure 3: Refined schema of the learning trajectory, (figure 1) under the influence of our intervention. The arrows in green represent paths that the students were able to follow and perform with only minimal guidance. The red arrows point to the conflict that emerged when students were asked whether the proof was complete. That conflict

points to a difficulty in bringing isolated exploratory experiences in the domain of examples in the form of a codified mathematical text in the domain of deductive proofs.

Verifying the formula in simple situations can be viewed as residing in the domain of examples. Checking that a formula holds for a specific geometrical shape is a typical task for students at this level. This part of the activity was performed successfully by most students. Considering the augmentation of one example to a whole class of examples is also a relatively doable activity for the students. However, understanding the reason for doing this, and especially seeing how one can be assured that all cases are covered by securing a number of simple cases and procedures of augmentation, requires the student to move into the domain of formal mathematics. In that sense it is not surprising that the students' independent work collapsed at this stage. This suggests that the difficulties that students faced when bringing their empirical findings into the realm of deductive proofs was not just a matter of reasoning mode or of moving from a small number of examples to a general statement and proof. Instead, what they found difficult was the structure of the theorem to be proposed: They did not internalize how the different experiments fit together to prove the generality of the theorem.

Thus, when the students were expected to change domain to formal mathematics and construct their own proof (rather than reading, understanding and performing official proofs) based on insights gained in the domain of examples and tasks, a number of potential difficulties became visible:

Can I use examples in a formal argument? In the domain of tasks and examples students are interested in the specificity of the example, not in the general class of objects dealt with. This change in view is rather radical and is likely to confuse students. When reporting on their learning, the students point to the fact that a proof can consist of smaller parts when describing what they learn; this seems to suggest that this particular use of multiple examples and cases in a complicated proof is new to these students. Thus, it may not be the examples as much as the structures of a proof consisting of elaborate sub-arguments that actually were in contrast with the expectations of the students.

Can I construct a proof? As simple as this might seem, students do express a big surprise that they can contribute in the formal domain, both with proposing a formula (which they actually found easy) and with a proof. This specific proof is not so easy and hence perhaps not the best starting point for creating formal mathematics.

Can a proof be non-algebraic? Apart from consisting of various cases and examples, the proof that the students were asked to contribute to is also very multimodal. We do not have strong empirical evidence here, but the explicit use of non-algebraic (but logical) reasoning makes the proof of Pick's theorem quite different from the textbook proofs these students have seen, perhaps also feeding into their conflict when asked to present a codified proof.

CONCLUSION

In this paper we have investigated a case where students were brought to use empirical strategies or example-driven reasoning in proposing, constructing and proving mathematical results. We suggest analyzing the difficulties that the students have as difficulties with combining a domain of mathematical tasks and examples and the domain of formal mathematics. Using this lens we came to see how the mutual activation of proofs and examples gave rise to certain conflicts that reside not with the empirical mode of investigation, neither with the deductive reasoning mode as such when applied to sub-arguments in the proof, but with the ability to gain a comprehension of the structure of more complex deductive proofs such as the proof of Pick's Theorem.

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