Oral Presentation

TEACHING THE AREA OF A CIRCLE FROM THE PERSPECTIVE OF HPM

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In China, importance is attached to three objectives in mathematics teaching: knowledge & skills, process & methods, affect & attitude, corresponding to which we have the following "whys" of integrating the history of mathematics into mathematical teaching: (1) The history of mathematics is helpful for deepening students' understanding of mathematics; (2) The history of mathematics provides a lot of problem-solving methods and can broaden students' thinking; (3) The history of mathematics increase students' interest and create their motivation. The above values had been really achieved in the mathematics classroom when the history of mathematics was integrated into the teaching of the area of a circle at the sixth grade in a junior high school, and the benefits were identified from the students' viewpoints.

INTRODUCTION

In the field of education in China, great importance has been attached in recent years to the Three-Dimension Instructional Objectives, i.e., knowledge & skills, process & methods, affect & beliefs. From the perspective of HPM, we have designed teaching projects to accomplish those objectives by integrating the history of mathematics into mathematical teaching. The reasons we use this approach are listed as follows:

(1) The history of mathematics is helpful for deepening students' understanding of mathematics;

(2) The history of mathematics provides a lot of problem-solving methods and can broaden students' thinking;

(3) The history of mathematics increases students' interest and creates their learning motivation.

Can those goals really be achieved? Next, we will introduce an experiment of using history of mathematics in the teaching of mathematics of one middle school in shanghai, and share our experience as well.

THEORIES RELATED TO HPM TEACHING

Principles of HPM teaching design

Italian scholar Fulvia Furinghetti (2000) introduced a general process to integrate history of mathematics into mathematics teaching. Based on her idea, we made some adaptations and set the following teaching process: choosing a teaching subject → investigating related history → selecting suitable materials → analyzing classroom requirements → developing classroom activities → implementing teaching design → evaluating the course.

The key to success in HPM teaching design is selecting proper materials of history of mathematics. In our opinion, the historical materials selected must be interesting, scientific, effective, learnable and innovative. **Interesting**: the historical materials should raise students' interests in study, that's why we need to select stories closely related to the teaching design. **Scientific**: we mean the materials must comply with facts or historical backgrounds. As the HPM teaching is not only using history of mathematics for the sake of history of mathematics. **Effective**: we mean that the materials should serve for the objectives of the teaching. **Learnable**: we mean that the materials should be provided in accordance with students' cognition level and could be readily accepted by them. **Innovative**: we mean that the materials should be new to students, the teaching design has distinguishing features, and can promote teachers' professional development.

Approaches of integrating history of mathematics into classroom instruction

One of the important questions in HPM is the study on approaches to integrate history of mathematics into mathematics education. Under the frame of mathematics education, researchers have constructed many integrating approaches, considering the relations between history of mathematics and teaching factors. Fauvel (1991) has generalized 10 ways. Tzanakis and Arcavi (2000) have 3 different approaches, including (1) "Learning history, by the provision of direct historical information", (2) "Learning mathematical topics, by following a teaching and learning approach inspired by history", and (3) "Developing deeper awareness, both of mathematics itself and of the social and cultural contexts in which mathematics has been done". Jankvist (2009) outlined another three ways, illumination, the modules and the history-based approaches. Professor Wang Xiaoqin (2012), by integrating and adapting the above-mentioned two grouping methods, labels them as complementation, replication, accommodation, and reconstruction, (Table 1).

Table 1: Approaches of using history of mathematics in teaching

Approaches	Description	Tzanakis&Arcavi	Jankvist
complementation	Display mathematicians' pictures, give an account of related stories,etc.	Direct historical information	Illumination approach
Replication	Directly using mathematical problems, methods, etc.	Direct historical information	Illumination approach; modules.
Accommodation	Problems adapted from historical ones or based upon historical materials	–	–
Reconstruction	Genesis of knowledge based on or inspired by the history of mathematics	Teaching approach inspired by history (Genetic approach)	History-based approaches

Direct using of the historical materials is the first level of using the history of mathematics, while mathematics teaching and learning in the perspective of HPM is the second level, which means to learn from, replay, and reconstruct the history of mathematics.

The next we'd like to present the teaching of the area of a circle as an example to better elaborate the above mentioned problems.

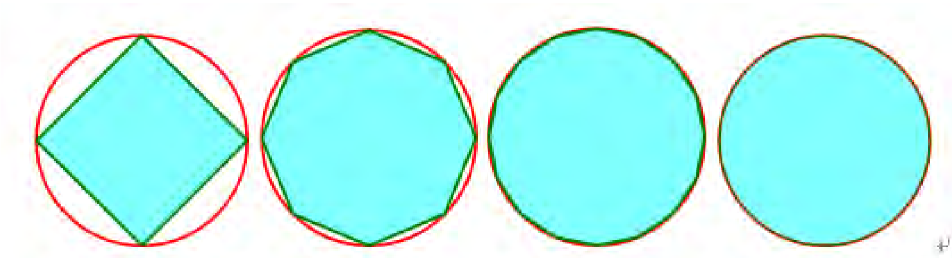
TEACHING THE AREA OF A CIRCLE INTEGRATED WITH HISTORY OF MATHEMATICS

The area of a circle is a knowledge point in the 6th graders' mathematics textbook in Shanghai. Previously the students have had a rough idea of the circle and the circumference, after having studied the areas of the linear graphics such as rectangles, squares, triangles, parallelograms and trapezoids. In this teaching design, Kepler's method of calculating the area of a circle is programmed and detailed in classroom teaching. This is the approach of accommodation in HPM teaching.

Investigating the history of the area of a circle

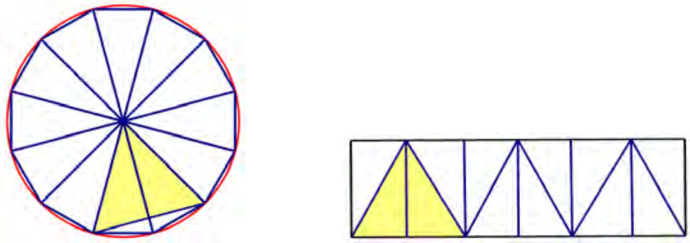
The ancient Babylonians and Egyptians encountered the problem of the area of a circle when measuring land, but they did not produce a calculation formula (Liang, 1995, p164-165). As shown on the Babylonian tablets YBC 7302, the area of a circle would be 1/12 times the square of its circumference. The Egyptian Rhind papyrus of 1800 BC gives the area of a circle as (64/81) d^2, where d is the diameter of the circle. In ancient Greece, Antiphon (c.480-411BC) originated the idea of squaring a circle with an inscribed regular polygon (Liang, 1995, p255). As in Figure 1, we inscribe a square in a circle, and then double its number of sides repeatedly. When the sides are infinite, this regular polygon eventually 'becomes' a circle. Then we have the area of a circle. Learning from Antiphon's idea, Archimedes (287-212BC) used inscribed and circumscribed regular polygons, applying the method of exhaustion to prove that the area of a circle is half its circumference times its radius (Heath, 1949).

Figure 1: Antiphon's method of squaring a circle



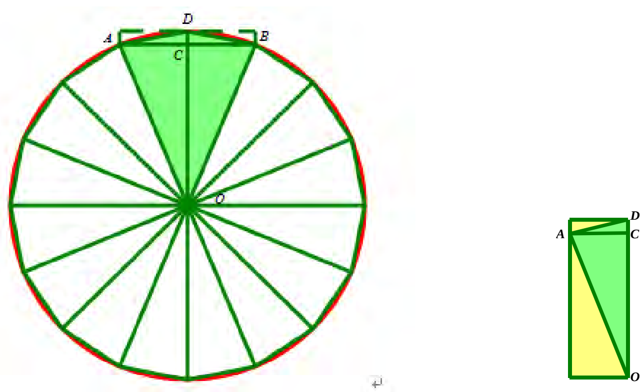
In China, a book named *The Nine Chapters on the Mathematical Art* written before the 2nd century BC also tells that the area of a circle is half its circumference times its radius. However, ancient Chinese mathematicians used a quite rough method to compute the area of a circle. They took the circumference of an inscribed regular 6-gon as a circle's circumference, and the area of an inscribed regular 12-gon as that of a circle, applying *the Out-In Complementary Principle* to patch the regular 12-gon into a rectangular which has half of the regular 6-gon's circumference as its length and radius of the circle as its width (Figure 2). Hence the area of a circle is $3r^2$, where r is the radius of the circle. Here π is 3, very roughly in terms of its real value (Guo , 2007).

Figure 2: Ancient Chinese Mathematicians' method of calculating the area of a circle



To find a formula for the area of a circle, Liu Hui (c.225-295AD) used the cyclotomic method (Wang, 2013), (Figure 3). The area of inscribed regular 2n-gon is added up by *n* times of the deltoid OADB. As every deltoid is a rectangular consisting of 4 parts, its area is $\frac{1}{2}a_nR$. We then have the formula for the area of the regular 2n-gon $S_{2n}=\frac{1}{2}na_nR$. Liu Hui commented that when the sides of the inscribed regular n-gon increase, its circumference is more approximate to the circumference of the circle and its area is more approximate to the area of the circle. In today's mathematical language, we have $S=\lim_{n\to\infty}S_{2n}=\lim_{n\to\infty}\frac{1}{2}na_n R=\frac{1}{2}CR$.

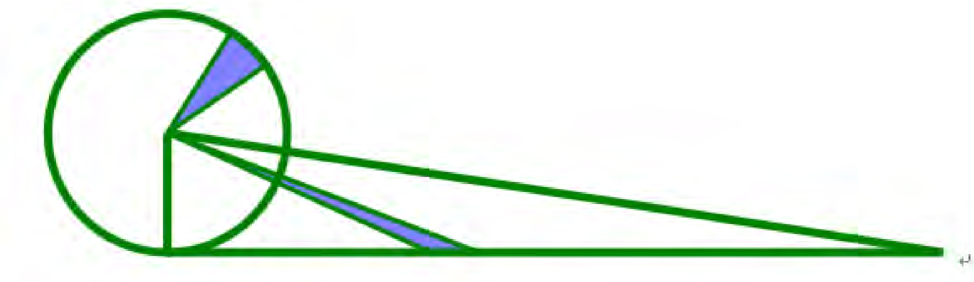
Figure 3: Inscribed regular 2n-gon is made up of n deltoids



It is worthwhile to note that the ancient Greeks had no notions of limit. In their opinion, the inscribed polygon approximates a circle, as close as one's mind can reach. However, there must be tiny parts missing between the area of a circle and that of a polygon. They proved their thinking with the technique of double reduction to absurdity, not the method of limit (Boyer, 1977). Liu Hui's cyclotomic method, in his opinion, will eventually lose no parts of the circle. This idea is quite close to modern concept of limit (Guo, 1983).

The German mathematician Johannes Kepler (1571-1630) came out an idea to compute the area of a circle on his second wedding while calculating the volume of a wine barrel. As in Figure 4, he divided a circle into countless small triangles with vertexes at the circle centre and radius as their heights. In fact, these triangles are small sectors. As the number of the circle divided is getting increasingly greater, the sector is coming closer to a triangle (Struik, 1948). If we change these small triangles into triangles with same base and height, they would form a right triangle. Hence we have $S = \sum \frac{1}{2} c_i r = \frac{1}{2} Cr = \pi r^2$.

Figure 4: Kepler's method of calculating the area of a circle



Selecting proper teaching materials

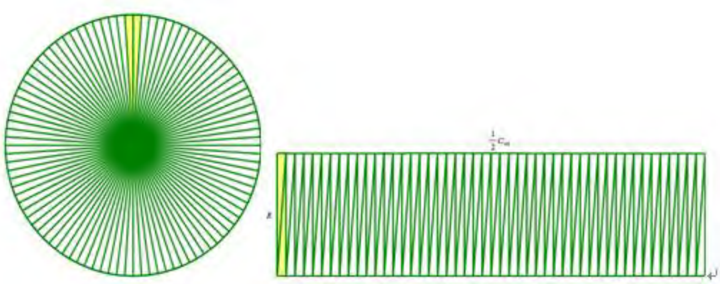
For the sixth graders, the area of a circle, upgraded from linear graph to curve graph, is a qualitative leap in both learning contents and methods, especially the inference of the area of a circle. Antiphon's method of squaring a circle is just an idea. Archimedes' calculation uses the double reduction to absurdity and the method of exhaustion, which is quite a complicated process. Liu Hui's cyclotomic method connecting 4 parts of a deltoid into a rectangular is also a little bit complex compared to Kepler's forming of triangles. These methods are not suitable for teaching sixth graders.

In Kepler's method, the area of a circle is the sum of all countless small triangles with vertexes at the center of the circle and base at its circumference. Hence the area of a circle is equal to the half circumference multiplied by radius. This method cannot be called precise if we have no concept of limit. Besides, it's not easy for a sixth grader to understand. For that reason, we use the method of accommodation， make some adaptations to Kepler's method and program it as follows: Substitute the inscribed n-gon with right triangles with the same area, and continually increase n, thus the area of the right triangle will get closer to that of a circle.

Kepler's method meets the five principles of our teaching design. The story itself is fun. The method Kepler used is in line with that of Archimedes and Liu Hui, and in compliance with sixth graders' cognition level, which means scientific. Kepler's method, without complicated calculation, makes it easier for the students to understand the formula of the area of a circle, which means our teaching process and objective can be effectively realized. As the students need only to know that triangles with same base and height have the same area, it is easy for them to learn Kepler's area equation process. And last but not least, Kepler's method itself is quite innovative.

Besides, the textbook introduces a way to calculate the area of a circle, by connecting little sectors into a rectangle in Shanghai (Shanghai Education Publishing House, 2011). As in Figure 5, the circle is equally divided into small sectors, and connected to create a graph that is approximately a rectangle. The more sectors the circle is divided into, the closer the area of the rectangle is to that of the circle. Then we can have the formula of the area of a circle.

Figure 5: Method in textbook (the circle is divided into 96 equal sectors)



Though the method of joining rectangles in the textbook is simple, straightforward, and easy to understand, it might still mislead students into believing that the formula is a rough and ready one,. If we do not make it clear, it would be puzzling for some students to comprehend this method.

In view of this, based on the above-mentioned five principles and an easy-to-difficult logical approach, we design our teaching by combining the methods mentioned in the mathematics textbook and used by Kepler. We start by using the method in the textbook, employing intuitive means to activate students' thinking. Then we bring in Kepler's method, preparing them with the idea that the number of a circle divided is increasingly greater, the sector is coming closer to a triangle. Then we teach them the method of connecting regular right triangles. In this way the students would be much more ready to accept that the formula of the area of a circle is an exact value.

The combination of these two methods enables the students to know not only the area of a circle, but also how that is realized. Apart from revealing knowledge's internal relations, it can also help students to understand the concepts of infinite approximation, to develop their spatial imagination, to motivate them to pick up more scientific methods, to acquire new knowledge with what they have learnt, to deepen their understanding of surroundings, and improves their learning ability.

Developing and implementing teaching design

This teaching project, the area of a circle, was designed by HPM team from Shanghai East China Normal University and teachers from a local middle school, under the guidance of Project for HPM and Professional Development of Mathematic Teachers in Junior High Schools by Shanghai Hutai Road Education Development. It was carried out in a common grade 6 class of 47 students by a young teacher under 5-year teaching experience. The teacher gave a simulated lecture to the project team, and then the design was discussed by all participants. At last, the teacher carried out the teaching project in the class. The teaching process is as follows.

(1) Introduction

Play an animation about a goat tying to a woodpile eating grass; introduce the topic of the area of a circle.

Design objective

If a teacher creates life situations in classroom to merge intangible emotions with tangible situations, the students would feel that what they study are from the life and for the life. This would boost their interest for learning, and lead them into the first step of studying a new topic, i.e., the concept of the area of a circle.

(2) Exploratory I

Introduce the concept of cut-supplement method; students assemble the jigsaw puzzle prepared beforehand; deduce the formula of the area of a circle. This is the method used in the textbook.

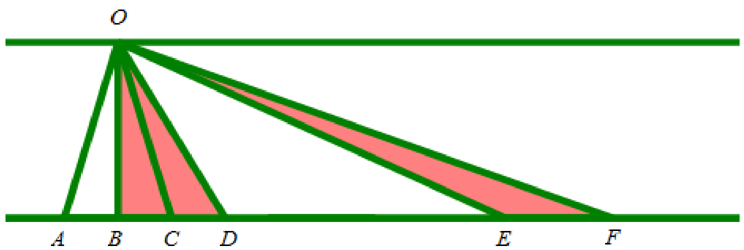
Design objective

Students participate, research and discover by themselves. they go through the development of the mathematical knowledge, and experience mathematical ideas and methods.

(3) Exploratory II

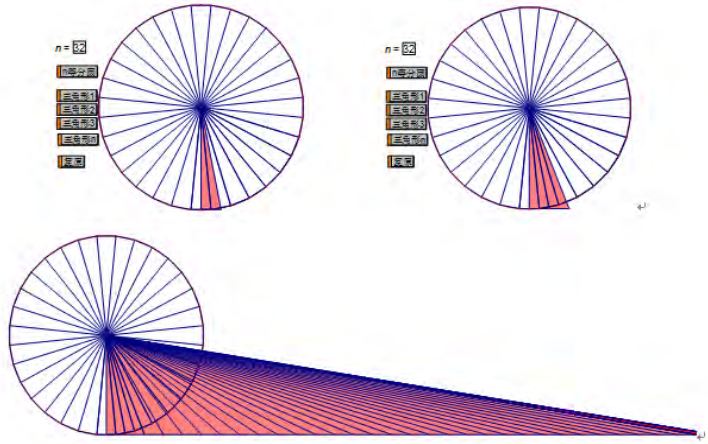
Narrate the story of Kepler calculating the area of a circle. Create graphs to show that triangles with same base and same height have same areas. As in Figure 6, triangles OAC, OBD and OEF have same base and same height, so their areas are also the same.

Figure 6: triangles with same base and same height have same areas



After that, we use authalic triangles to replace small triangles of the inscribed regular n-gon, and connect them one by one. In the end we will get a right triangle which has the same area as the inscribed regular n-gon. And we come up with the formula of the area of a circle: (Figure 7).

Figure 7: The process of Kepler's calculating of the area of a circle, n= 32



Design objective

These two methods have different starting points. The former guides the students to form an approximate rectangle, while the latter a regular right triangle. It enables the students to understand the nature of mathematical activities from different perspectives, and also to develop the fluency, flexibility and uniqueness of their thinking.

(4) Examples and exercises

(Omitted)

(5) Summary

(Omitted)

Teaching feedbacks

After the teaching, all 47 students took part in a survey, while some of them were interviewed.

Question: Do you have any trouble in understanding Kepler's method of calculating the area of a circle taught by your teacher?

45 students (96%) said they could understand fairly well or completely. Only a tiny minority of the teachers present at the class doubt that some students could understand Kepler's method, which was refuted by the survey.

Question: In comparison with previous teaching styles, do you like this one integrated with the history of mathematics?

46 students (98%) said they liked this style or liked it very much.

Question: Is there any difference in your attitude toward mathematical study after this teaching project on the origin of the formula for the area of a circle?

39 students (83%) said they were more interested or much more interested in mathematics.

As to subjective question: What impressed you most in this teaching project? Why?

Over one third of the students mentioned that the story of Kepler and his method impressed them most. Here are some of the students' responses.

Student A: I like mainly that part with mathematical history, for it is interesting and informative. (Figure 8)

Student B: What impressed me most is the method Kepler used to find the formula of the area of a circle. The story makes me more engaged to this class. (Figure 9)

Student C: I was most impressed by Kepler's method. It's simple and easy to understand.

Student D: How to calculate the area of a circle impresses me most. It is interesting, comprehensible and unique.

Figure 8: Feedback from Student A on this teaching design

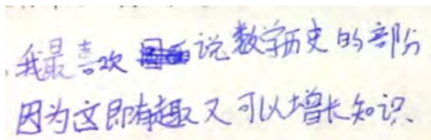
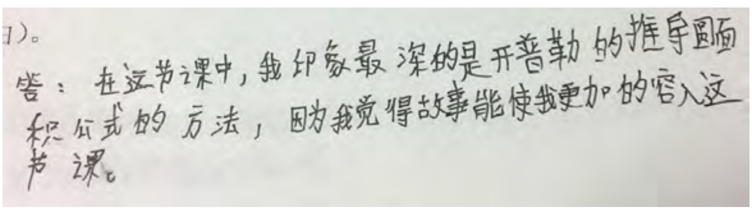


Figure 9: Feedback from Student B on this teaching design



Generally speaking, the following four benefits of integrating the history of mathematics into classroom teaching are identified from the students' feedbacks:

(1) The history of mathematics is very interesting and can capture students' attention;

(2) The methods integrated with history of mathematics are easy to understand for the students;

(3) The history of mathematics can extend students' knowledge of mathematics;

(4) The history of mathematics tells the students that there are a variety of methods to solve a problem.

CONCLUSIONS

This teaching design integrated with the history of mathematics is quite a success. We rearranged the historical materials related to Kepler's method. And we also used the computer to demonstrate its process, which makes it more comprehensible and acceptable. Besides, Kepler's optimistic attitude toward his hard life also had positive effect on the students.

The survey and interview reveal that most students like the teaching with the history of mathematics. They are interested in Kepler’s story and are impressed deeply by his method, and understand well the thinking of cyclotomic method.

This project proves that teaching from the perspective of the HPM can well serve China’s Three-Dimension Instructional Objectives. It also brings some inspirations on mathematical teaching and textbook compilation. For example, we can create lively teaching situations with historical materials to make the learning more interesting, enable students to know about the development of essential mathematics, and strengthen students’ understanding and improve their problem-solving ability by means of appropriate computer aided teaching. With the help of a textbook integrated with proper historical materials, right arrangement of teaching resources, and an intuitive-to-abstract, easy-to-difficult process, it will be of great use for students to reflect what they have learned and acquire new cognitive experience by analogy.

In a word, it’s worthwhile for us to further study, explore, and apply HPM teaching into our classroom instructions.

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