Oral Presentation

THE DIFFERENCE AS AN ANALYSIS TOOL OF THE CHANGE OF GEOMETRIC MAGNITUDES: THE CASE OF THE CIRCLE

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In this paper we present a didactical proposal focused on the study of some of the existing relationships between the radius, area and circumference of a circle. The proposal is inspired by historic elements of the genesis of calculus and makes use of the software GeoGebra. Although the proposal could add dynamism to the teaching of geometry and even have some motivational value for students, it would be necessary to do some field research to illustrate its scope and limitations.

INTRODUCTION

The use of history in the teaching and learning of mathematics is a well-established area of research within the international community of mathematics educators; this can be verified through the many publications, conferences, and study groups that have specialized in this area in recent years. The sixth volume of ICMI studies (Fauvel & Van Maanen, 2002), and the groups History and Pedagogy of Mathematics (HPM) and History in Mathematics Education at the European conference CERME are just a few examples that illustrate the interest of our community in the use of history of mathematics as an element of mathematics instruction. However, not all the uses made of the history of mathematics in teaching are of the same nature. In his categorization of the "whys" and "hows" of using history in mathematics education, Jankvist (2009) proposes three categories to organize the different uses of history in mathematics education: (1) the illumination approaches, (2) the modules approaches, and (3) the history-based approaches.

In this manuscript we present a didactical proposal framed in the category of history-based approaches; it is a proposal inspired by the historical development of mathematics. In particular, we consider the idea of the *difference* between two quantities x_1 and x_2, which was a tool for analysing the variation of quantities that was used during the genesis of calculus. Our proposal is inspired by the interpolation method called *methodus differentialis* which was first used by Isaac Newton as a tool for predicting the behaviour of some celestial bodies (Newton, 1686, pp. 287-288).

One aim of our proposal is to help students to discover some existing relationships between the length of the radius of a circle, and the area and perimeter of that circle. As we will see, the proposal includes the use of the software GeoGebra, which is perceived as an instrument of knowledge mediation.

In the next section we explain the context that gave rise to our proposal, referring to a recent reform of secondary education in Mexico. Next, we briefly introduce the *methodus differentialis*. In the next two sections we detail the purpose and operation of the proposal, but we also clarify the links between the proposal and Newton's *methodus differentialis*. We conclude the manuscript with some reflections on the didactical proposal.

THE CONTEXT OF THE PROPOSAL

The didactical proposal originates in the context of a recent reform of secondary education in Mexico. The implementation guidelines of the reform recommend the use of dynamic geometry software to support the study of geometric bodies (see Secretaría de Educación Pública, 2006).

In the guidelines, the following instruction caught our attention:

> As with the study of the other figures, the aim is not only is to calculate the area and perimeter but also, given the perimeter and the area, to calculate the length of the radius or diameter, as well as to find areas of shaded regions (annulus); **the relationship between the length of the radius and the area of the circle must also be analysed, and compared with the relationship between the length of the diameter and the length of the circumference**. (Secretaría de Educación Pública, 2006, p. 54, our translation, our emphasis)

These guidelines tell the teacher what to teach, but do not clarify how it should be taught. For this particular lesson, the guidelines recommend consulting the supplementary material "Geometría dinámica" (Dynamic Geometry) (Secretaría de Educación Pública, 2000). One would expect this material to include specific instructions on how the teacher can implement the required exploration using a dynamic geometry software; however, as stated on pages 68–70 of the supplementary material (which deals with the analysis of the magnitudes of a circle), only a pen-and-paper activity is proposed, but not one using a dynamic geometry package. Our didactical proposal was designed with the intention of filling this gap.

ILLUSTRATION OF THE OPERATION OF THE METHODUS DIFFERENTIALIS

The *methodus differentialis* is a mathematical method originally used by Isaac Newton as a tool for predicting the behaviour of some celestial bodies; it is essentially an interpolation arithmetic method in which a finite set of points in a plane, *A*, *B*, *C*, *D*, *E*, *F*, etc., is considered. From these points the line segments *AH*, *BI*, *CK*, *DL*, *EM* and *FN* are drawn. These segments are perpendicular to another line segment *HN* (see Figure 1).

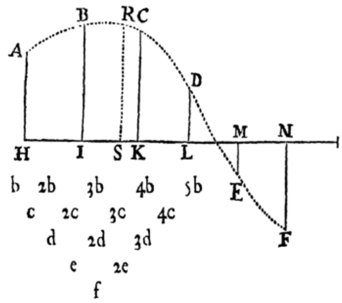

Figure 1: Graph illustrating the operation of the *methodus differentialis* (Newton, 1686, p. 288).

The main purpose of this method is to find the length or height corresponding to an unknown point that is located in any intermediate position between the points *A*, *B*, *C*, *D*, *E*, and *F*. In Figure 1 this unknown height is represented by the line segment *RS*. Clearly the represented lengths could be interpreted today as the ordinates corresponding to elements *H*, *I*, *K*, *L*, *M* and *N* in the domain of a function.

The method makes use of differences and quotients of these differences. Such quotients are represented in Figure 1 by the expressions including the lowercase letters *a*, *b*, *c*, *d*, *e* and *f*. The quotients are defined as follows:

$$b = \frac{AH - BI}{HI}, 2b = \frac{BI - CK}{IK}, 3b = \frac{CK - DL}{KL}, 4b = \frac{DL - ME}{LM}, etc.$$

$$c = \frac{b - 2b}{HK}, 2c = \frac{2b - 3b}{IL}, 3c = \frac{3b - 4b}{KM}, etc.$$

$$d = \frac{c - 2c}{HL}, 2d = \frac{2c - 3c}{IM}, etc.$$

$$e = \frac{d - 2d}{HM}$$

It is important to clarify that the terms 2*b*, 3*b*, …, 2*c*, 3*c*, …, 2*d*, etc., do not carry the meaning of multiplication by 2, 3, etc., but rather they carry the meaning of the modern day notation of subscripts like b_2, b_3, etc.

Our main interest in this manuscript is not to detail the operation of the interpolation method, as it is not the underlying principle upon our didactical proposal is based; instead we want to emphasize an idea behind our proposal, namely that the *n*th order differences of a polynomial of degree *n* are constant. Later we will illustrate this principle in our proposal (see for example figures 3 and 4).

THE DIDACTICAL PROPOSAL TO ANALYZE RELATIONSHIPS BETWEEN SOME MAGNITUDES OF THE CIRCLE

The GeoGebra software allows dynamic linking of the geometric and numeric contexts of representation, which is ideal for analysing the relationships between the magnitudes of the circle specified in the reform. The aim of the didactical proposal is twofold: firstly, it aims to help students to discover that the area and length of the circumference increase when the length of the radius increases, but not in the same way or with the same speed; on the other hand, it is also intended to help mathematics teachers to explore these kind of mathematical relationships along with their students in a dynamic way.

We begin with the following situation:

> *Consider a circle whose initial radius is 1 unit long. As the length of the radius increases in steps of 1, what happens to the area of the circle? What about the length of the circumference?*

This situation is represented by an animation made with GeoGebra in which the length of the radius of a circle varies discretely, while the magnitudes of the area and the circumference when the radius changes are recorded. Column A in Figure 2 shows the values of the radius *r*, while column B shows the respective areas and column C the values of the circumference.

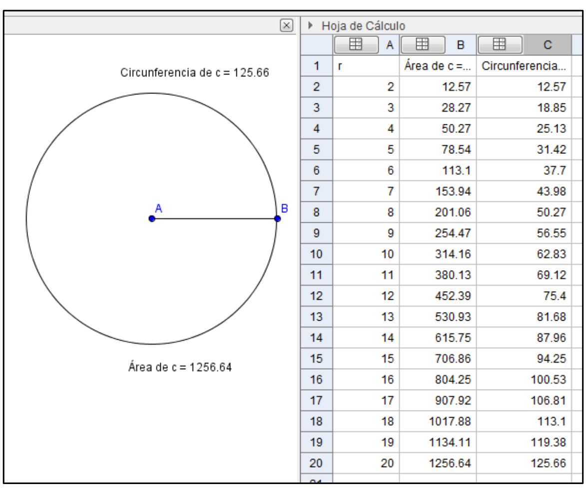

Figure 2: Variation of the radius of a circle and the magnitudes of the area and the circumference using the graphical and spreadsheet capabilities of GeoGebra.

Column D in Figure 3 shows that the calculation of the differences applied to the values of the circumference during the animation generates the constant value of 6.28 ≈ 2π. In the animation, it can be seen that as the radius grows, so do the area and circumference, but how do the area and circumference grow? To address this question we use the spreadsheet application in GeoGebra to calculate the differences between consecutive numbers in a similar way as performed in the *methodus differentialis*. We do this for both values: the values of the circumference and the values of the area of the circle.

Hoja de Cálculo

	A	B	C	D
1		Área de c ...	Circunferencia...	Diferencias Circunferencia
2	2	12.57	12.57	6.28
3	3	28.27	18.85	6.28
4	4	50.27	25.13	6.28
5	5	78.54	31.42	6.28
6	6	113.1	37.7	6.28
7	7	153.94	43.98	6.28
8	8	201.06	50.27	6.28
9	9	254.47	56.55	6.28
10	10	314.16	62.83	6.28
11	11	380.13	69.12	6.28
12	12	452.39	75.4	6.28
13	13	530.93	81.68	6.28
14	14	615.75	87.96	6.28
15	15	706.86	94.25	6.28
16	16	804.25	100.53	6.28
17	17	907.92	106.81	6.28
18	18	1017.88	113.1	6.28
19	19	1134.11	119.38	6.28
20	20	1256.64	125.66	

$$C_3 - C_2 = 18.85 - 12.57 = 6.28$$
$$C_4 - C_3 = 25.13 - 18.85 = 6.28$$
$$C_5 - C_4 = 31.42 - 25.13 = 6.28$$
$$C_6 - C_5 = 37.7 - 31.42 = 6.28$$
$$C_7 - C_6 = 43.98 - 37.7 = 6.28$$
$$C_8 - C_7 = 50.27 - 43.98 = 6.28$$
$$C_9 - C_8 = 56.55 - 50.27 = 6.28$$
$$C_{10} - C_9 = 62.83 - 56.55 = 6.28$$
$$C_{11} - C_{10} = 69.12 - 62.83 = 6.28$$
$$C_{12} - C_{11} = 75.4 - 69.12 = 6.28$$
$$C_{13} - C_{12} = 81.68 - 75.4 = 6.28$$
$$C_{14} - C_{13} = 87.96 - 81.68 = 6.28$$
$$C_{15} - C_{14} = 94.25 - 87.96 = 6.28$$
$$C_{16} - C_{15} = 100.53 - 94.25 = 6.28$$
$$C_{17} - C_{16} = 106.81 - 100.53 = 6.28$$
$$C_{18} - C_{17} = 113.1 - 106.81 = 6.28$$
$$C_{19} - C_{18} = 119.38 - 113.1 = 6.28$$
$$C_{20} - C_{19} = 125.66 - 119.38 = 6.28$$

Figure 3: Calculating the differences between the values of the circumference using the spreadsheet application in GeoGebra.

The constant value of the differences (6.28) indicates that the circumference of the circle changes linearly when the length of the radius increases; if we plot the radius-circumference ratio, we obtain a linear function graph. Since we calculated the differences of the data once, we can say that we obtained a variation of first order and in this case the result is a constant value.

A purpose of our proposal is to illustrate how the arithmetic difference may be used as a tool to analyse the variation of geometrical magnitudes, and we believe such mathematical technique is accessible to lower secondary school pupils; however, some aspects of the relationship between the length of the radius of a circle and its circumference and area can serve as an introduction to more advanced concepts belonging to differential calculus. For example, if we proceed similarly for the case of the radius-area ratio, when calculating the second differences (i.e. the differences of the differences contained in column B in Figure 3), we will obtain again the constant value 6.28 ≈ 2π. In this case we have a second order variation (Figure 4).

E	F
Diferencias de área	Seg. Diferencias de área
15.71	6.28
21.99	6.28
28.27	6.28
34.56	6.28
40.84	6.28
47.12	6.28
53.41	6.28
59.69	6.28
65.97	6.28
72.26	6.28
78.54	6.28
84.82	6.28
91.11	6.28
97.39	6.28
103.67	6.28
109.96	6.28
116.24	

Figure 4: Calculating the first differences (column E) and second differences (column F) for the values of the area of the circle using the spreadsheet application in GeoGebra.

The fact that we obtain a constant value (approximately equal to 2π) after calculating the first differences of the length of the circumference and the second differences of the values of the area is related to the mathematical concept of the derivative, since the idea of difference is the foundation of the structure of this mathematical concept: it is known that the derivative of a function $C = C(r)$ with respect to r is $C'(r) = \lim_{h \to 0} \frac{C(r+h)-C(r)}{h}$, provided that this limit exists; if we deliberately omit applying the limit as h approaches 0, then an approximation of the derivative is obtained.

In the case of the values of the circumference analysed in this work, $h = 1$ and the expression $\frac{C(r+1)-C(r)}{1}$ is equivalent to the differences $C_{r+1} - C_r$ calculated in the spreadsheet. In the case of the area we would have $B_{r+1} - B_r$ for r = 2, 3, ... 19; these are the calculations that we performed in the spreadsheet and that are shown in Figure 3 and Figure 5. A similar rationale is used for the second differences: the result of these calculations is 6.28 ≈ 2π. Furthermore, it is known that the area of a circle is $A(r) = \pi r^2$ and its derivative is $A'(r) = 2\pi r$; the result of the second derivative is $A''(r) = 2\pi \approx 6.28$. If both functions are graphed, the results are a parabola and a straight line.

LINK BETWEEN THE METHODUS DIFFERENTIALIS AND THE DIDACTICAL PROPOSAL

As previously mentioned, our proposal to explore how the changes in the magnitudes of a circle is based on the *methodus differentialis,* but we consider only the first part of the method in which quotients of difference are calculated. Next we illustrate the application of this part of the method to the values of the areas that we have worked in GeoGebra (these values are shown in Figure 3).

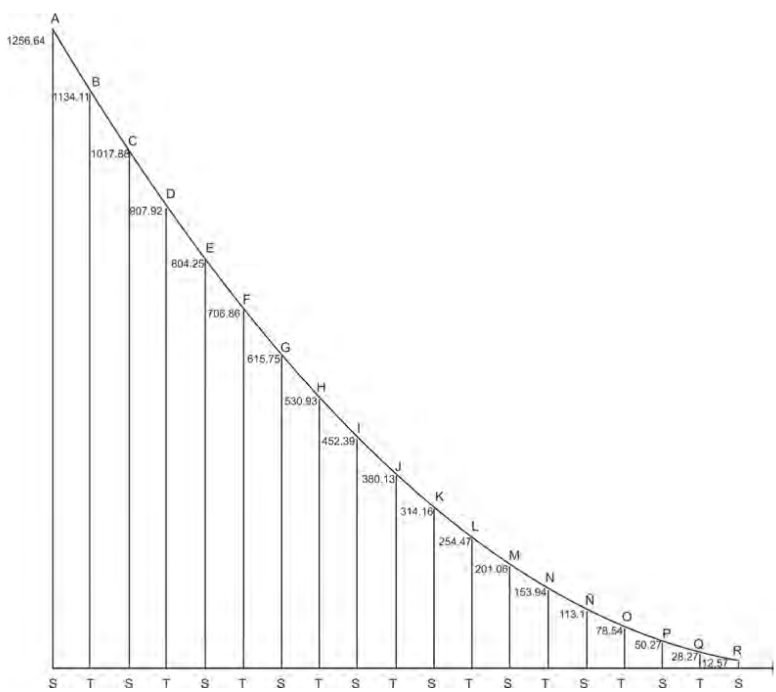

Figure 5: Representation of the variations of the area similar to the graphical representation used in the *methodus differentialis* (see Figure 1).

In this case we would have:

$$b = \frac{AS - BT}{ST} = \frac{1256.64 - 1134.11}{1} = 122.53$$

$$2b = \frac{BT - CS}{TS} = \frac{1134.11 - 1017.88}{1} = 116.23$$

$$3b = \frac{CS - DT}{ST} = \frac{1017.88 - 907.92}{1} = 109.96$$

$$4b = \frac{DT - ES}{TS} = \frac{907.92 - 804.25}{1} = 103.67, etc.$$

FINAL CONSIDERATIONS

This combination of historical elements of mathematics with the use of technological tools is not new; there are proposals like the one by Kidron (2004), in which software with graphical and algebraic capabilities is used to teach the topics of approximation and interpolation according to their historical development. We believe that these types of proposals that combine history and the use of technology should be further developed since, on the one hand, the use of history in the teaching of geometry can have a motivational value to students (Gulikers & Blom, 2001), and, on the other hand, the use of technology can add meaning to the concepts studied in the mathematics classroom by making evident the relationship between the different contexts of representation (such as the numerical, algebraic, and geometric). It is necessary that proposals such as the one presented in this paper be tested in real mathematics classrooms to learn more about the scope and limitations of this type of teaching approach.

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