Workshop

WORKSHOP ON THE USE AND THE MATHEMATICS OF THE ASTROLABE

Wilfred de Graaf [a] & Michel Roelens[b]

[a]Utrecht University, [b]University Colleges Leuven-Limburg

For more than one thousand years the astrolabe was one of the most used astronomical instruments in both the Islamic World and Europe. It was used to locate and predict the positions of the sun and stars, for instance to schedule prayer times, and to determine the local time. In the first part of the workshop, the participants learn how to use (a cardboard model of) the astrolabe. In the second part they study the mathematical and astronomical principles on which the astrolabe is based. We explain the mathematical properties of stereographic projection and we show how the lines and circles on the astrolabe can be computed.

INTRODUCTION

Background of the workshop

The astrolabe workshop is based on an idea by Prof. Dr. Jan P. Hogendijk, University of Utrecht. The workshop has been held in recent years on several occasions in Iran, Tajikistan, The Netherlands, Turkey, Syria and United Kingdom by Wilfred de Graaf. In Belgium the workshop has been held by Michel Roelens for high school students and (future) teachers. Recently a detailed instruction on the use of the astrolabe and on the mathematical method of stereographic projection has been published by Michel and Wilfred in *Uitwiskeling*, a Belgian journal for high school mathematics teachers.

Classroom use

The workshop as a whole is suitable as an interdisciplinary project for high school students aged between roughly 15 and 18 years that have a keen interest in the school subjects of physics and mathematics. The first part of the workshop, on the actual use of the astrolabe, can also be given to a wider audience with interest in history, geography and culture, and younger pupils (13-14 years old). For mathematics education the astrolabe is of particular interest since the instrument is based on the mathematical concept of stereographic projection. In a classroom situation the pupils could for example be asked to derive certain formulas related to this projection, thereby using such things as Thales' theorem and the inscribed angle theorem of plane geometry. About the use of the astrolabe in the classroom, see also de Graaf and Roelens (2013) and Merle (2009).

The original astrolabe

Based on mathematical principles which date back to Greek antiquity, the astrolabe flourished in the Islamic World from the year 800 CE onwards. In this workshop the participants learn how to use the astrolabe of the renowned mathematician and astronomer Abū Maḥmūd Khujandī.

Figure 1: The astrolabe of Abū Maḥmūd Khujandī

He constructed this astrolabe in the year 985 CE at the observatory in Baghdad. It is one of the oldest and most beautiful decorated astrolabes extant today. It is currently displayed at the Museum of Islamic Art in Doha, Qatar. The distributed astrolabe model has been recalculated for the latitude of Antwerp, i.e., 51° N.

Principles

The astrolabe is based on the mathematical principles of the *celestial sphere* and *stereographic projection*. The celestial sphere is an imaginary sphere concentric with the earth on which the stars and the apparent one year path of the sun are projected from the centre of the earth. Stereographic projection is a method to map a sphere onto a plane. In this case the celestial sphere is mapped from the celestial south pole onto the plane of the celestial equator.

FIRST PART: THE USE OF THE ASTROLABE

The astrolabe model consists of two parts.

On the overhead slide: the spider

The spider contains the stereographic projections of the *ecliptic*, which is the apparent one year trajectory of the sun along the sky, and of 33 stars. These stars are the same as on the astrolabe of Khujandī. The positions of the stars are recomputed for the year 2000 CE, showing the effect of precession of the equinoxes if the model is compared to the original astrolabe. The precession is about 15 degrees in a 1000 year interval. In the model, the position of a star is indicated by a dot in the middle of a small circle.

On the sheet of paper: the plate

The plate has been combined with the rim, which is a circular scale divided into 360 degrees. The plate displays (parts of the) the stereographic projections of the following points and circles.

- The centre of the plate is the *celestial north pole*, which is the centre of three concentric circles: the Tropic of Cancer, the celestial equator and the Tropic of Capricorn.
- The *horizon*, whose projection is visible on the plate in Eastern, Northern and Western directions. The *twilight line* is 18° below the horizon.
- The *almuqanṭarāt* (altitude circles) are the nearly concentric circles 3°, 6°, 9°… above the horizon.
- The *zenith* is the point directly above the head of the observer, i.e. 90° above the horizon.
- The *azimuthal circles* or circles of equal direction. Its projections are drawn for 5° intervals and are numbered at their intersections with the horizon. The *first vertical* is the azimuthal circle through the East and the West point. It is the reference circle for the other azimuthal circles. Note that all azimuthal circles pass through the zenith.

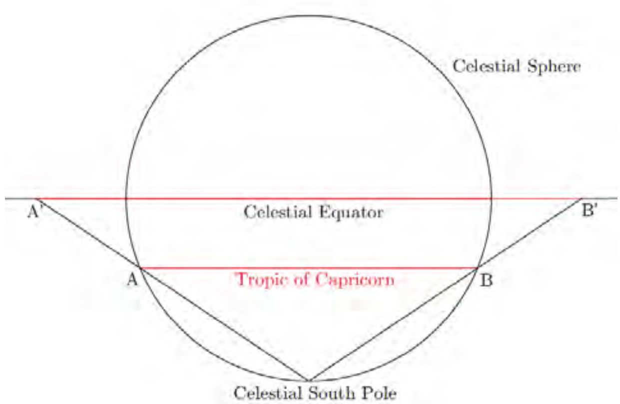

Figure 2: The stereographic projection of the Tropic of Capricorn from the celestial sphere onto the plane of the celestial equator

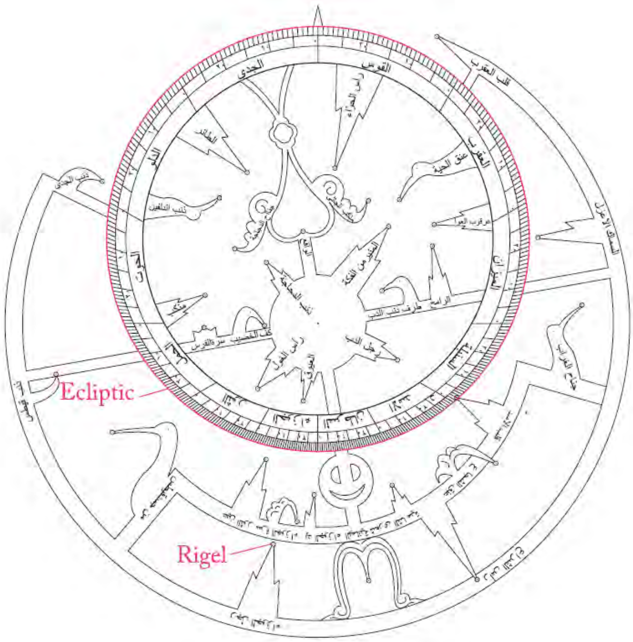

Figure 3: The spider with the ecliptic and the star Rigel highlighted

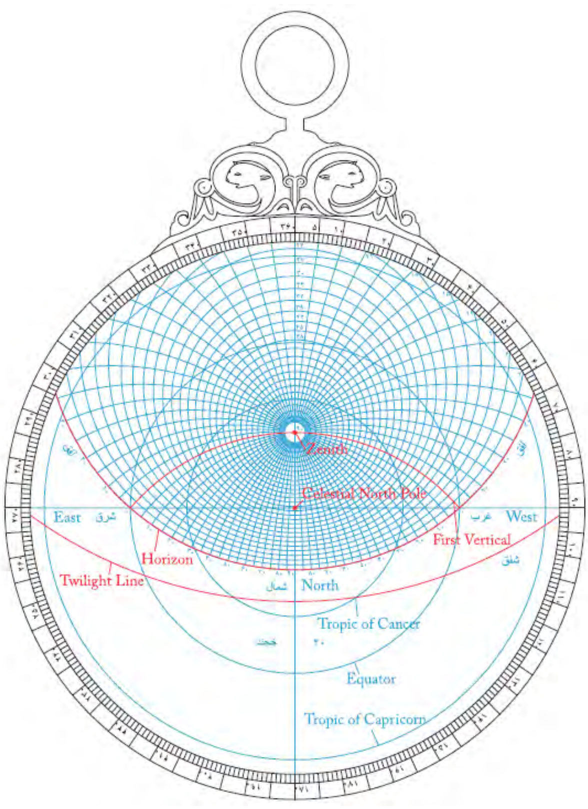

Figure 4: The plate for 40° N (i.e. the latitude of Khujand, Tajikistan)

The back side of the astrolabe is not shown in this model. It contains the alidade: a metal strip with two sights and a pointer. An alidade can be used to measure the altitude of the sun or a star in degrees, if the astrolabe is suspended vertically. The altitude can be read off on a circular scale.

The use of the astrolabe

If one knows the position of the sun in the ecliptic on a given day, the astrolabe can be used to tell the local time. It can then also be used as a compass. The position of the sun can be estimated using the fact that the sun moves through the twelve zodiacal signs, into which the ecliptic is divided, in the course of one year. Every sign is divided into 30 degrees. The sun moves with a velocity of approximately one degree per day.

For any day, the position of the sun in the ecliptic can be marked on the spider by a non-permanent marker. The altitude of the sun can be measured using the alidade on the back side of the astrolabe. The spider can now be set to represent the actual position of the celestial constellations with respect to the horizon. By means of the azimuthal circles one can read off the direction of the sun, for example 10° S or E. To determine the local time, note that the pointer of the spider indicates a number on the rim. A full rotation of the spider corresponds to 24 hours, so 1 degree of rotation corresponds to 4 minutes of time. By rotating the spider, one can determine the interval of time between the moment of observation and, for example, sunset, noon, and sunrise.

Aries	March 21	- April 19
Taurus	April 20	- May 20
Gemini	May 21	- June 20
Cancer	June 21	- July 22
Leo	July 23	- August 22
Virgo	August 23	- September 22
Libra	September 23	- October 22
Scorpio	October 23	- November 21
Sagittarius	November 22	- December 21
Capricornus	December 22	- January 19
Aquarius	January 20	- February 18
Pisces	February 19	- March 20

Table 1: The signs of the zodiac and their corresponding dates

The assignments of the workshop are divided into three levels: the calculation of the length of daylight on a given day of the year (level 1), the use of the astrolabe as a clock and as a compass (level 2) and the determination of the direction of Mecca (bonus level).

Workshop on the use of the astrolabe, level 1

1. The date of your anniversary is: day .... month ....
2. Then the sun is in the sign of the zodiac: ....
3. And in the degree: .... In case the degree is 31, write 30.

Now mark the position of the sun on the ecliptic on the spider. Be sure to mark it on the outer rim of the ecliptic!

4. At sunrise on your anniversary, the position of the pointer is: ....

Recall that the sun rises on the Eastern horizon.

5. At sunset on your anniversary, the position of the pointer is: ....
6. The difference between the position of the pointer at sunset and the position of the pointer at sunrise is: .... degrees. When encountering a negative difference, add 360 degrees to the position of the pointer at sunset.
7. The length of daylight on your anniversary is: ....

Recall that 15 degrees corresponds to 1 hour.

Workshop on the use of the astrolabe, level 2

Suppose you have measured with the alidade on the back side of the astrolabe that the sun is 9 degrees above the horizon. You have done the measurement in the afternoon of your anniversary date.

8. The position of the pointer at that moment is: ....
9. The position of the pointer at noon (12.00 true local solar time) is: ....
10. The difference between the position of the pointer at the moment that the sun is 9 degrees above the horizon and the position of the pointer at noon is: .... degrees.
11. The true local solar time at the moment that the sun is 9 degrees above the horizon is: ....
12. The direction of the sun at that moment is: ....

Workshop on the use of the astrolabe, bonus level

Suppose you are at a place with the same latitude as Antwerp. The geographical longitude of this place is 15 degrees East of Mecca. You know that the sun passes through the zenith of Mecca on the days when it is in 7 Gemini and in 23 Cancer.

13. Use the astrolabe to find the direction of prayer, *qibla*, at your place.

SECOND PART: THE LINES ON THE ASTROLABE

Stereographic projection

Each line on the astrolabe is the stereographic projection onto the equatorial plane of a circle on the celestial sphere. We project from the south pole of the celestial sphere. This means that the circle that we want to project is connected by straight lines with the celestial south pole so that an oblique[1] circular cone is created. The stereographic projection is the intersection of this cone with the plane of the equator.

A major advantage of the stereographic projection is that the circles are projected as circles (we will prove this later). The lines of the astrolabe can thus be drawn with a compass!

Figure 5 shows the stereographic projection of a set of circles on the celestial sphere that lie in a set of planes that are parallel to each other, but not parallel to the plane of the equator. This is for example the case with circles at a fixed altitude above the horizon (e.g. all the points 20°,40°… above the horizon).

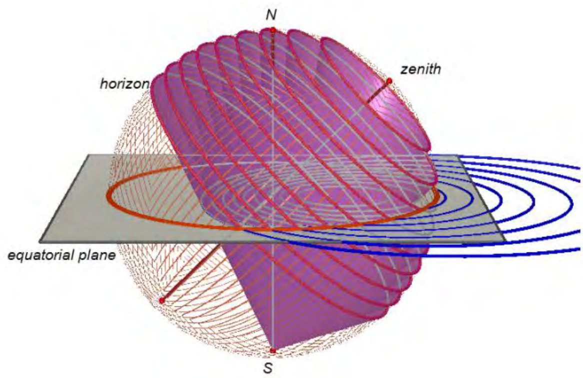

Figure 5: Stereographic projection of circles parallel to the horizon

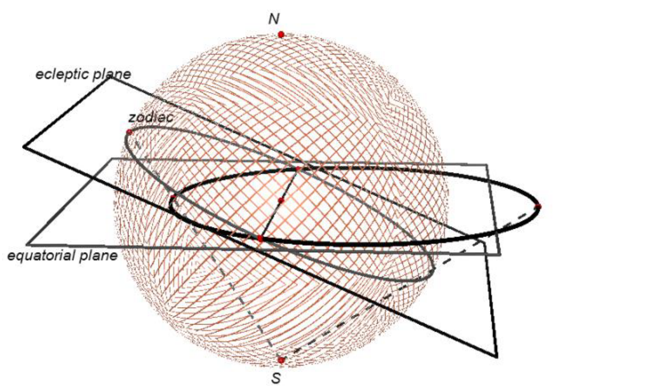

Figure 6: Stereographic projection of the ecliptic

Figure 6 shows the stereographic projection of the zodiac. The zodiac is the apparent path of the sun around the earth in one year. It is the intersection of the celestial sphere with the ecliptic plane. Since the rotation axis of the earth is not perpendicular to the ecliptic plane, the projection of the zodiac is decentred with respect to the centre of the astrolabe.

Circles on the celestial sphere remain circles on the astrolabe

We want to prove that the stereographic projection on the equatorial plane of a circle on the celestial sphere is again a circle.

Take any circle c on the celestial sphere. The stereographic projection of c is the intersection c' of the cone of base c and apex S (the south pole of the celestial sphere) with the plane of the equator. We want to prove that c' is also a circle.

Apollonius of Perga (3rd century BCE) proves in *Conica* the following two propositions.

(Conica I.4) The intersection of an oblique circular cone with a plane parallel to the basis is a circle.

(Conica I.5) The intersection of an oblique circular cone with a 'subcontrary' plane is also a circle.

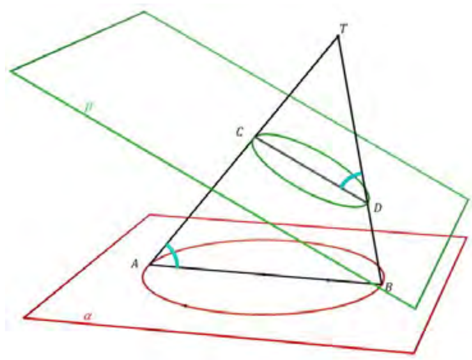

Figure 7: Intersecting a cone with a 'subcontrary' plane

Apollonius explains what he means by 'subcontrary'. For this purpose, he uses the intersection ABT of the cone with the plane perpendicular to the basis and containing T and the centre of the basis (see figure 7). In this plane, AB is the diameter of the basis and CD is the diameter of the intersection, with C on AT and D on BT. If we cut the cone parallel to the basis, then the angle $\hat{C}$ is equal to the angle $\hat{A}$. Now, cutting with a subcontrary plane means cutting in such a way that the angle $\widehat{D}$ is equal to the angle $\hat{A}$.

Proof based on Apollonius' theorem

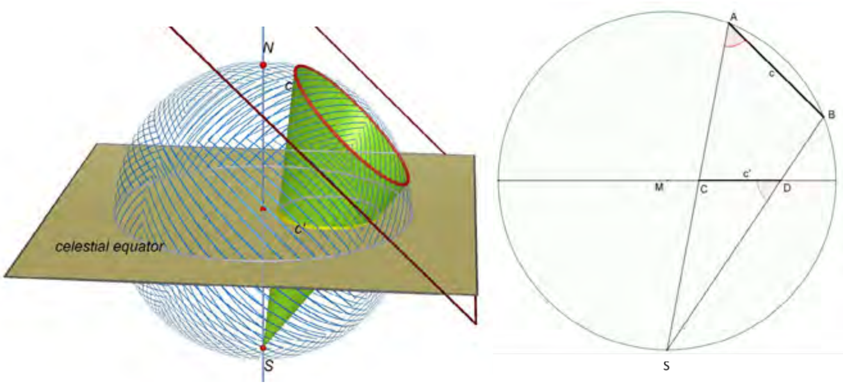

Figure 8

We operate in the plane passing through the centre M of the sphere, the celestial south pole S and the centre of the circle c. This plane is then automatically perpendicular to the equatorial plane. We have to prove that the angles $\hat{A}$ and $\widehat{D}$ are equal (figure 8). Indeed, if this is the case, it follows from the theorem of Apollonius that the intersection *c'* of the cone with the equator plane is also a circle.

Using figure 9, we can prove the equality of the angles $\hat{A}$ and $\widehat{D}$. We have: $\hat{A}$ is equal to $\widehat{N}$ because they are inscribed in the same circle. Now, $\widehat{N}$ is the complement of $\hat{S}$ because $\widehat{B}$ is inscribed in a semicircle. Finally, $\hat{S}$ is the complement of $\widehat{D}$ in the right angled triangle *DMS*. This proves that $\hat{A} = \widehat{D}$.

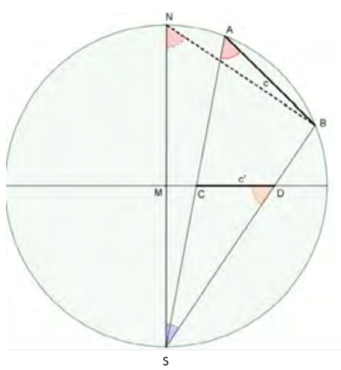

Figure 9

With theorem I.5 of the Conica we have proved that the stereographic projection of a circle on the celestial sphere is a circle on the equatorial plane (and thus on the astrolabe). We now give a proof of Apollonius' theorem I.5.

Proof of Apollonius' theorem

Given is an oblique cone, with circular base in the plane α and apex *T*. This cone is cut by a plane that is perpendicular to the plane *ABT*, in such a way that the angle $\widehat{D}$ is equal to the angle $\hat{A}$, as in figure 7. We have to prove that the intersection with β is a circle too.

Apollonius takes an arbitrary point *P* on this intersection and he proves that the angle $C\hat{P}D$ is right. He considers the intersection of the cone with a plane α' parallel to α through *P*. In a previous theorem (Conica I.4), Apollonius proved that this intersection is a circle. Since the planes α' and β are both perpendicular to the plane *ABT*, their intersection line is also perpendicular to this plane. Denote by *M* the intersection point of this line with the plane *ABT* (figure 10). The triangles *CEM* and *DFM* are similar. So we have:

$$CM \cdot MD = EM \cdot MF$$

In the right angled triangle *EPF*, we have

$$EM \cdot MF = PM^2$$

Hence

$$CM \cdot MD = PM^2$$

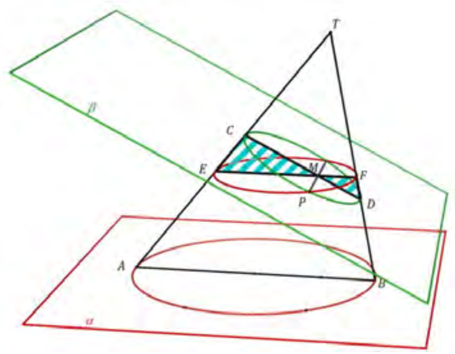

Figure 10

From this it follows that the triangle *CPD* is rectangular in *P*. Note the use in the two directions of the property "the triangle *CPD* is rectangular in *P if and only if* the height on *CD* is equal to the product of the length of the segments *CM* and *MD* in which it divides *CD* ".

This proves Apollonius' theorem.

Drawing the Horizon

Using some trigonometry, students can draw some circles on the astrolabe themselves. In the workshop below, we will do this for the special case of the horizon.

Workshop: Drawing the Horizon

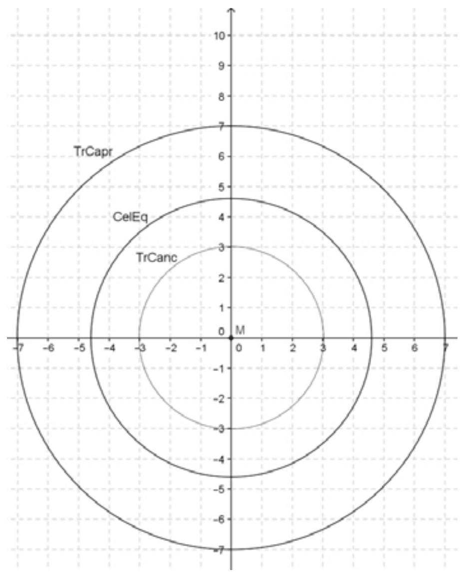

Figure 11: Blank astrolabe

We want to draw the horizon on the plate of the 'blank' astrolabe of figure 11. The projections of the celestial equator and the two tropics are already drawn. Just like on the model, we have taken r = 4.6 cm as the radius of the celestial equator. Using that the latitude of the tropics is at 23°26'16" N and S, we can calculate that the radius of the Tropic of Cancer on this model is 3.0 cm and that of the Tropic of Capricorn is 7.0 cm.

The horizon of an observer at a certain latitude on earth is projected on the plate. We assume that the observer is at the latitude of Antwerp, 51° N. All circles on the plate are stereographic projections of circles on the celestial sphere. In order to draw the horizon, we first identify what circle on the celestial sphere represents the horizon; then we determine its stereographic projection. Because we know that the stereographic projection is a circle again, it suffices to determine its centre and radius.

On an earth globe, we locate Antwerp at 51° N. The plane tangent to the earth at this point is the plane of the horizon for an observer in Antwerp (figure 12).

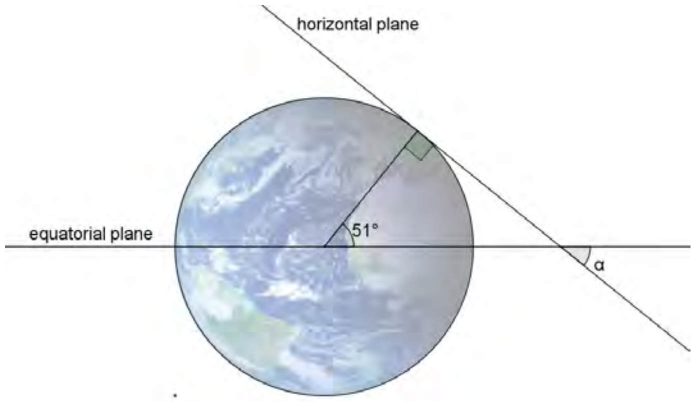

Figure 12: Earth globe with plane of the horizon

Exercise 1 What is the angle α between the plan of the horizon and the plane of the equator?

Because the earth is negligibly small compared to the celestial sphere, we can regard the plane of the horizon going through the centre *M* and having an angle α with the plane of the equator. The horizon is the intersection of this plane with the celestial sphere. The earth is represented as the point *M* (figure 13).

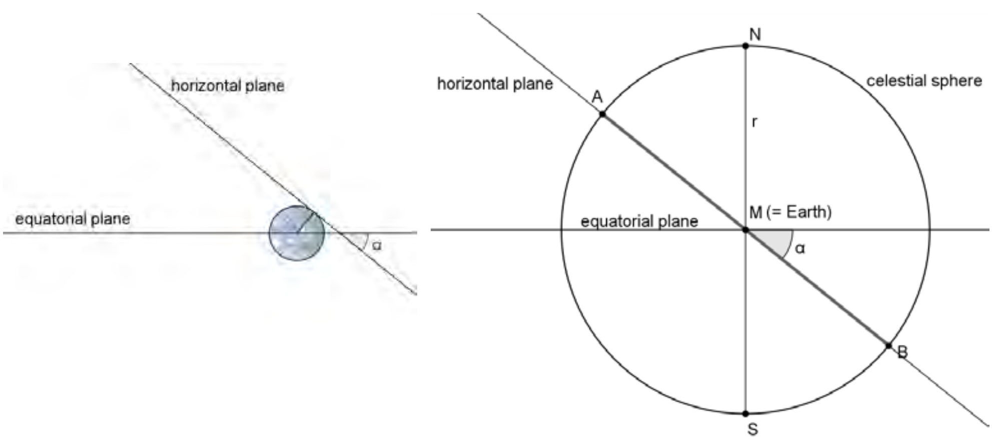

Figure 13: Horizon in the celestial sphere

We have to determine the stereographic projection of the horizontal circle. In figure 14, the horizontal circle is represented by the line segment *AB* and its stereographic projection by the line segment *A'B'*.

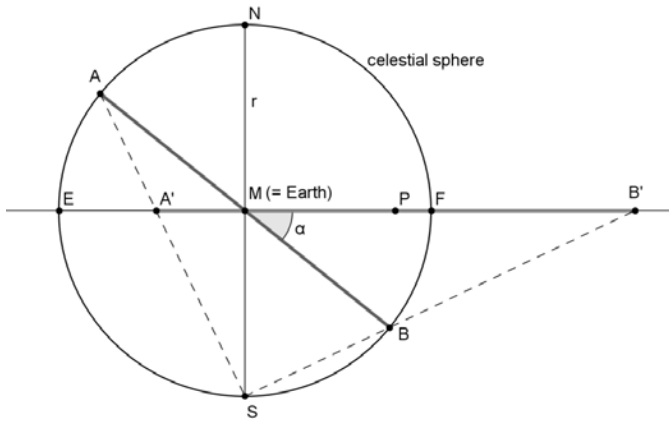

Figure 14: Constructing the projection of the horizon

Let us now write r for the radius of the celestial sphere in general. For the astrolabe drawing we will take r = 4.6 cm at the end of the computation.

Exercise 2 Express the distance *A'M* and the distance *B'M* in terms of the radius r. Make use of the right angled triangles *A'MS* and *B'MS*. What is the radius r_h of the projection of the horizon on the astrolabe? How far from the centre *M* of the astrolabe should the centre *P* of the projection of the horizon be drawn?

Did you find

$$r_h = \frac{r}{2}(\tan 64.5° + \tan 25.5°) \approx 5.9 \text{ cm};$$

$$|PM| = \frac{r}{2}(\tan 25.5° - \tan 64.5°) \approx 3.7 \text{ cm?}$$

Exercise 3 Draw the projection of the horizon on the blank astrolabe. Note that *P* should be 'above' the midpoint *M* (that is southward) on the astrolabe.

In an analogous manner (other) altitude circles can be drawn. You can do this at home for example for the altitude circle 30° above the horizon (figure 15). It is more complicated to draw the azimuthal circles. (It involves another feature of stereographic projection, namely that it preserves angles.)

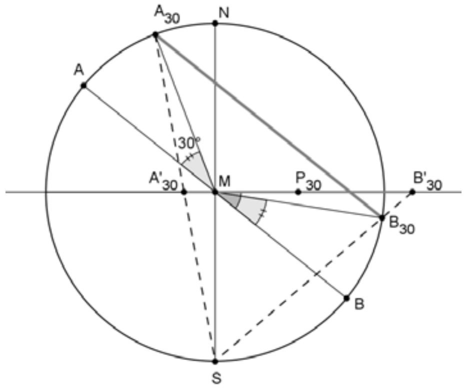

Figure 15: Construction for the projection of the altitude circle 30° above the horizon

FINAL REMARK

We believe the astrolabe is a very powerful didactic instrument to learn on the one hand about the movements of the earth, the sun and the stars, and on the other hand about the mathematics that is behind the method of stereographic projection. Also, we believe, the astrolabe is a wonderful historical tool to enthuse young students for the study of mathematics and natural sciences.

NOTES

1. An oblique (circular) cone is a cone of which the apex is not situated directly above the centre of the (circular) base. It may also occur that the cone is not oblique but right. This is the case when the circle on the celestial sphere happens to be in a plane parallel to the equatorial plane.

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http://www.ibttm.org (Istanbul Museum for History of Science and Technology in Islam)

http://www.jphogendijk.nl/ (website of Jan Hogendijk with many relevant links)