Oral Presentation

ON THE UNDERSTANDING OF THE CONCEPT OF NUMBERS IN EULER'S "ELEMENTS OF ALGEBRA"

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Many problems encountered by students during the time of transition from arithmetic to algebra are based on different understandings and usages of the concept of number and variable. The analysis of the historical development of algebra and within this field the nature of the discussed objects can be helpful to understand the problems for students nowadays. In the following article Leonhard Euler's understanding of algebra in his textbook "Elements of Algebra" will be discussed. Unlike modern mathematics Euler considers numbers as objects grounded in an empirical subject area. Numbers are defined as the ratio of measurable quantities. In conclusion Euler's understanding of the concept of number will be discussed with the help of the idea of empirical theories.

"ELEMENTS OF ALGEBRA"

The analysis of the understanding of algebra in course of the historical development is based on Euler's textbook "Elements of Algebra". The choice of this textbook is firstly justified by Euler's position in the development of mathematics in general and his contribution to teaching of mathematics in particular and secondly by the significance of the textbook itself. The importance of his textbook has to be seen in the chronological context in which it was written. Most likely Euler started to write the "Elements of Algebra" in Berlin. It was published 1768 in St. Petersburg at first in a Russian translation before it was released 1770 in the original German version. The textbook was translated and reprinted several times, especially with the additions of Lagrange. 1774 the "Elements of Algebra" appeared in a French translation of Johann III Bernoulli. This French edition became a source of the English version. The following analysis applies to this English translation by John Hewlett from 1828. Even though Euler's Textbook had only a small positive impact on science after the appearance (Schubring, 2005, p. 258), it was widely read. In the German edition of Reclam, the textbook was printed from 1883 till 1942 in 108.000 copies. Therefore, Euler's Algebra was really a bestseller (Fellmann, 2007, p.120f). Because of this great demand and the many translations in other languages the "Elements of Algebra" played a major role for the learning of algebra.

The circumstances of the appearance of the textbook are affected by Euler' blindness. Euler needed the help of his servant to write the book. In accordance to an anecdote Euler's non-skilled servant understood the mathematics Euler dictated to him and was in the end able to do algebra by himself (Euler, 1828, Advertisement).

The textbook is addressed to a mathematical interested audience. According to the Advertisement, Euler's intention was to

> "compose an Elementary Treatise, by which a beginner, without any other assistance, might make himself a complete master of Algebra." (Euler, 1828, Advertisement)

Judging from today's point of view the standard set in this textbook and also the treated subjects are beyond the capability of an untrained learner. Nevertheless, the textbook "Elements of Algebra" is a progressive introduction from the natural numbers to Diophantine equations. As set out by Fellmann, the textbook is still

> "-in the judgment of today's foremost mathematicians – the best introduction into the realm of algebra for a "mathematical infant." (Fellmann, 2007, p.121)

The "Elements of Algebra" are a systematic introduction into the arithmetic and elementary algebra. The book is subdivided in two parts. The first part contains the initiation of different kinds of numbers, the basic arithmetic operations, the calculation with variables and the calculation of interests. The second part deals mainly with solving equations of different degrees.

The following analysis is a systematic and text-based approach in order to obtain an understanding of the concept of numbers within the "Elements of Algebra". The achieved insights will be used to discuss the broader understanding of algebra as presented in this textbook. These results will indicate what kind of understanding the reader of Euler's textbook will possibly develop.

THE CONCEPT OF NUMBER

Euler starts his presentation with an ontological explanation of mathematics and the processed objects. He writes at the beginning of the first chapter:

> "[…] And this is the origin of the different branches of the Mathematics, each being employed on a particular kind of magnitude. Mathematics, in general, is the *science of quantity*; or, the science which investigates the means of measuring quantity." (Euler, 1828, part 1 § 2)

This definition of mathematics can be seen as a programmatic fundament for the following contents in the textbook. Contrary to today's understanding of mathematics as an abstract formal science, Euler considers mathematics as a science of concrete measurable quantities. A quantity is defined as follows.

> "Whatever is capable of increase or diminution, is called *magnitude*, or *quantity*." (Euler, 1828, part 1 § 1) [1]

Euler introduces quantities not as an element of a formal axiomatic structure, but as quantity founded empirically. As examples for quantities Euler names weight, length and the sum of money. The given examples indicate that Euler refers quantities to real subject area. Euler's definition of quantity can be traced back to Euclid. Thiele describes the introduction and use of the concept of quantities in Euclid's Elements in this way:

> "There is no definition of the concept of magnitude (Greek $\mu\varepsilon\gamma\varepsilon\theta o\varsigma$, megathos) because there is no superior concept for this fundamental concept. Nevertheless, Euclid is dealing with magnitudes throughout the Elements; […] Magnitudes are generally characterized by the property of being able to increase and decrease." (Thiele, 2003, p. 6)

Like Euclid Euler defines quantities in reference to their capability of increase and diminution. Therefore, Euler assumes a definite order of the quantities, which he does not discuss explicitly. The same applies for the properties of an axiomatic domain of quantities, as transitive and irreflexive. The domain of quantities should be considered as an algebraic structure with an operation addition and an order relation less-than. The quantities in Euler's "Elements of Algebra" are given by empirical examples and it seems like the properties of the quantities are also given based on the empirical foundation and require no formal definition.

To compare and calculate with quantities it is necessary to be able to measure or determine a quantity. Euler remarks to this:

> "Now, we cannot measure or determine any quantity, except by considering some other quantity of the same kind as known, and point out their mutual relation." (Euler, 1828, part 1 § 3)

The determination of a quantity requires a unit, a quantity of the same kind, which can be put in a ratio to the proposed quantity. The following given examples are once again real quantities as weight, length and the sum of money.

Natural Number, Whole Numbers and Rational Numbers

Based on the concept of quantity Euler defines numbers as the ratio of one quantity to another:

> "So that a number is nothing but the proportion of one magnitude to another arbitrarily assumed as the unit."(Euler, 1828, part 1§ 4)

The definition of numbers by Euler is based on the fundamental idea of partition and measurement. Nowadays in mathematics school courses numbers are defined as cardinal numbers, ordinal numbers or measure values. However, Euler introduces natural numbers as the ratio of quantities of the same kind. Therefore, natural numbers are characterized according to the empirical origin of the underlying quantities.

After the introduction to numbers Euler initiates the basic arithmetic calculation for the new objects. He starts with an explanation of the symbols + and – and the use of these symbols related to the natural numbers. Within this approach Euler mixes the symbols as operation signs and as algebraic signs of a number. He states:

> "Hence it is absolutely necessary to consider what sign is prefixed to each number: for in Algebra, simple quantities are numbers considered with regard to the signs which precede or affect them. Farther we call those *positive quantities,* before which the sign + is found; and those are called *negative quantities*, which are affected by the sign –." (Euler, 1828, part 1 § 16) [2]

It can be seen that a change of the ontological state of the signs happens here. Before the sign stood for an operation, which connects two numbers with each other. In the context of positive and negative numbers the sign is part of the name of the quantity itself. Euler pays no particular attention to this fact.

Also nowadays the whole numbers are defined as difference a – b of to natural numbers a and b. The subtraction of the field ℤ is introduced with the inverse element regarding to the addition.

In Euler's approach this is just a step further towards the extension of the number system to whole numbers. Based on the characterising of negative quantities Euler introduces negative numbers with regard to the empirical quantities:

> "The manner in which we generally calculate a person's property, is an apt illustration of what has just been said. For we denote what a man really possesses by positive numbers, using, or understanding the sign +; whereas his debts are represented by negative numbers, or by using the sign –." (Euler, 1828, part 1 § 17)

It becomes clear at this point, that Euler does not strictly distinguish between a quantity and a number, which is defined as ratio of quantities. Euler considers negative number as quantity itself. In this sense Euler also proceeds with numbers as if they were quantities of a material world. According to the relation of whole numbers to the domain of quantities of a sum of money, numbers can be ordered linearly on the number line. Euler argues that:

> "Since negative numbers may be considered as debts, because positive numbers represent real possessions, we may say that negative numbers are less than nothing." (Euler, 1828, part 1 § 18)

In this explanation zero stands for the case when someone has no property of his own or in other words it represents nothing. Euler himself does not name zero directly as number in this chapter, but includes it in the series of natural numbers and also in the series of negative numbers. Nevertheless, in the summarization of whole numbers Euler does not name zero as a possible value of numbers. The given context and also the handling in the subsequent chapters show that zero as a possible value has to be included. He describes whole numbers as follows:

> "All these numbers, whether positive or negative, have the known appellation of whole numbers, or integers, which consequently are either greater or less than nothing." (Euler, 1828, part 1 § 20)

This characterisation corresponds to the law of trichotomy for ℝ or more generally for ordered sets.

> "If $x \in S$ and $y \in S$ then one and only one of the statements $x < y$, $x = y$, $y < x$ is true." (Rudin, 1964, p.3)

A formal introduction of negative numbers does not occur. Euler's justification of negative numbers and the existence of the order of whole numbers is based on the presented quantities.

In the same manner Euler initiates rational numbers. Here again Euler refers to a concrete domain of quantities to justify the new numbers. Special attention should be given to the fact, that Euler defines rational numbers not directly as ratio of two natural numbers, but rather introduces rational numbers by the help of lengths. The example leads Euler to an idea of the concept of rational numbers and justifies the ontological existence of the number at the same time. Euler states:

> "When a number, as 7, for instance, is said not to be divisible by another number, let us suppose by 3, this only means, that the quotient cannot be expressed by an integer number; but it must not by any means be thought that it is impossible to form an idea of that quotient. Only imagine a line of 7 feet in length; nobody can doubt the possibility of dividing this line into 3 equal parts, and of forming a notion of the length of one of those parts." (Euler, 1828, part 1 § 68)

The "number" we gain by dividing a quantity by a number is a quantity with a given unit, which is contrasted by Euler's definition of numbers in general. The given problem is based on the fundamental idea of distribution and not of the proposed fundamental idea of partition and measurement, as indicated by the definition of number. Euler just introduces numbers by empirical examples and does not define the ordered field ($\mathbb{Q}$, <,+). But like for the whole numbers Euler presupposes a natural order of the rational numbers.

Euler's formulation "nobody can doubt" emphasises the self-evident character of his explanation. Euler uses the knowledge and laws of the everyday life to introduce and also justify new contents. Vollrath points out that also for today's students it is obvious that a division of a distance leads to another distance. The recurrent problem is only to determine the length of the parts (Vollrath, Weigand, 2007, p. 40).

In summary, the numbers underlying concrete quantities are the basic concepts, which require no definition. They are defined by the capability of increase and diminution and are clarified by examples. Euler's understanding of mathematics is highly related to science.

Properties of Numbers

The properties of numbers are gained by the interpretation of the numbers as quantities of an empirical subject area. The justification of the properties relies on empirical examples. Euler refers to obvious characteristics of the quantities, which are transferred to the numbers, equally to his approach by the extension of the number systems. This is clearly evidenced in Euler's explanation of density:

> "For instance, 50 being greater by an entire unit than 49, it is easy to comprehend that there may be, between 49 and 50, an infinity or intermediate number, all greater than 49, and yet all less than 50. We need only to imagine two lines, one 50 feet, the other 49 feet long, and it is evident that an infinity number of lines may be drawn, all longer than 49 feet, and yet shorter than 50." (Euler, 1828, part 1§ 20)

Euler does not only introduce new concepts and properties referring to empirical quantities, but also justifies operational rules and laws by reference to concrete quantities. For the justification of the operational rule (+)(–) = – Euler observes:

> "Let us begin by multiplying –a by 3 or +3. Now since –a may be considered as debt, it is evident that if we take the debt three times, it must thus becomes three time greater, and consequently the required product is –3a." (Euler, 1828, part 1 § 32)

The phrase "evident" suggests for the student an implicitness of the obtained rule. Euler apparently considers this rule as an evident statement, which is an extract from an empirical observation and for this reason does not require a formal proof.

The validation of the commutative law is illustrated by empirical examples. Contrary to the previous rule Euler explains the commutative law not by referring directly to quantities but under the specification of concrete number values. He argues:

> "It may be farther remarked here, that the order in which the letters are joined together is indifferent; thus ab is the same thing as ba; for b multiplied by a is the same as a multiplied by b. To understand this, we have only to substitute, for a and b, known numbers, as 3 and 4; and the truth will be self-evident; for 3 times 4 is the same as 4 times 3." (Euler, 1828, part 1 § 27)

Euler's example is so simple and common in the everyday life, that he also states this fact as self-evident. Similar to the other presented introductions and explanations Euler abdicates a formal derivation or proof.

Euler's approach resembles the methods and access in nowadays school mathematics. As Padberg points out, in the primary school the properties are obviously not formulated in an abstract way. The students will rather experience them as computational advantageous. The justification of the properties can be obtained by example-attached strategies of proof (Padberg, 2009, p.125). The given explanations refer mainly to dot patterns or arrangements of objects.

For Euler numbers simply have their properties because of the fact, that the underlying relevant quantities have these properties and this is so obvious and common knowledge, that there is no need for any kind of proof. It seems as if the numbers inherit the characteristics from the basic empirical entities.

Imaginary Numbers

Of great importance for Euler in the "Elements of Algebra" is the concept of the imaginary number. It should be pointed out here that it should be distinguished between the complex number as an element of the field $\mathbb{C}$, like it is understood today, and the square root of a negative number as imaginary number of Euler's days. In Euler's days there was no theory of complex numbers and therefore there had not been an axiomatic approach, on which it could fall back on. The first documents bringing up complex numbers date back to the Renaissance. In 1645 Cardano published his book "Ars Magna", where the process to solve cubic equations had been generalised by the help of the square root of negative numbers (Remmert, 1991). By means of some equation the presented process provides imaginary numbers as solutions. Cardano suggested that square roots of negative numbers have a "sophisticated nature" since they are neither near the "nature of a number" nor near the "nature of a quantity". Cardano concluded that the results of equations, which include square roots of negative numbers, are useless (Cardano, 1545, p. 288). Even years later, when imaginary numbers were used in calculations a systematic analysis of the imaginary numbers is missing. Although Euler handled imaginary numbers in calculations and actually invented *i* as notation for $\sqrt{-1}$, the ontological state of imaginary number was undetermined. Only 1831, Gauss was able to interpret imaginary numbers as points in a plane and founded them in geometry. Several years later Hamilton described imaginary numbers as an ordered pairs (x,y) of real numbers and defined for them arithmetic calculations as addition and multiplication (Remmert, 1991).

As written above, Euler was well aware of imaginary numbers, but nevertheless

> "Euler had great difficulty in explaining and defining just what the imaginary numbers, which he had been handling so masterfully during the past forty years and more, really were." (Remmert, 1991, p. 59)

For Euler the imaginary number represents an expression without any relation to the real subject area. Nevertheless, this expression has to exist, due to the fact, that he gains them by applying allowed calculation rules on negative numbers. Thus the term of the square root of a negative number appears in this sense in a natural way. But the new term is not compatible with Euler's understanding of a number, since the definition of number, as ratio of quantities of the same kind, does not apply to the square root of negative numbers. Especially the properties of numbers that result from their definition do not refer to the new terms. Euler notes that:

> "All such expressions, as $\sqrt{-1}, \sqrt{-2}, \sqrt{-3}, \sqrt{-4}$, … are consequently impossible, or imaginary numbers, since they represent roots of negative quantities; and of such numbers we may truly assert that they are neither nothing, or greater then nothing, nor less than nothing; which necessarily constitutes them imaginary, or impossible." (Euler 1828, part 1 § 143)

As pointed out above, a main characteristic of numbers is that they can be ordered linearly on a number line. The law of trichotomy must be fulfilled. Thus every kind of number has to be less than zero, equal to zero or greater than zero as condition to be a possible number. The square root of negative numbers, however, does not follow any characteristic of a linear order. Moreover, the unknown expressions cannot even be approximated. The value of the square root of a negative number can not be qualified. Euler points this out as follows:

> ""[…] whereas no approximation can take place with regard to imaginary expressions, such as $\sqrt{-5}$; for 100 is as far from being the value of the root as 1, or any other number." (Euler, 1828, part 1 § 702)

The square root of a negative number is a result of solving an equation, but is not even considered as possible number. Imaginary numbers do not refer to empirical objects and therefore, they are not part of our material world. There is no empirical quantity, which is expressed by the imaginary number, as it is the case for the negative number and the debts.

Nevertheless, Euler considers imaginary numbers to be important for addressing algebra. Euler justifies his considerations regarding the imaginary numbers against the widespread opinion that they are useless expressions and do not need to be discussed. Euler identifies the benefit of imaginary numbers as indicator whether an equation is solvable or not. He states:

> "For the calculation of imaginary quantities is of the greatest importance, as questions frequently arise, of which we cannot immediately say whether they include any thing real and possible, or not; but when the solution of such a question leads to imaginary numbers, we are certain that what is required is impossible." (Euler, 1828, part 1§ 151)

This opinion on the imaginary numbers as an indicator for solvability of problems is not new. Before Euler, Newton only had understood the imaginary expression as symbol for the impossibility to solve the equation (Remmert, 1991, p. 58) and Descartes had actually understood imaginary numbers as geometric impossibility:

> "To see how Descartes understood the association of imaginary numbers with geometrical impossibility, consider his demonstration on how to solve quadric equation with geometric constructions. He began with the equation $z^2 = az - b^2$, where a and b^2 both non-negative, […]." (Nahin, 1998, p. 34)

Another point in which imaginary numbers show themselves to be of use for Euler is as provisional result, since after operating with them they can lead to possible numbers. Thus Euler uses imaginary numbers later on to find the factorisation of the equation $ax^2 + bxy + cy^2$. The integration of imaginary number in the calculation is one further step to a theory of complex numbers.

Euler's solution in handling the imaginary expressions is to transfer the well-known operations and calculation rules from the real numbers to the new expressions. This is done without a formal definition of the potential operations regarding imaginary numbers. For Euler it is natural that the normal calculation rules also apply to imaginary numbers due to the fact that we can have an idea of them:

> "But notwithstanding this, these numbers present themselves to the mind; they exist in our imagination, and we still have a sufficient idea of them; since we know that by $\sqrt{-4}$ is meant a number which, multiplied by itself, produces -4; for the reason also, nothing prevents us from making use of these imaginary numbers, and implying them in calculation." (Euler, 1828, part 1§ 145)

Euler's proceeding resembles his examination of real numbers. Real numbers also appear by extracting the square root of numbers, which are no square themselves. In the same way as for the imaginary numbers Euler gains an idea of the real numbers. He writes as follows:

> "These irrational quantities, though they cannot be expressed by fractions, are nevertheless magnitudes of which we may form an accurate idea; since, however

> concealed the square root of 12, for example, may appear, we are not ignorant that it must be a number, which, when multiplied by itself, would exactly produce 12; and this property is sufficient to give us an idea of the number, because it is in our power to approximate towards its value continually." (Euler, 1828, part 1§ 129)

Despite the parallels in these two remarks the differences are obvious. Although Euler is not able to obtain a concrete perception of the square root of 12, he may approximate the value of the real number by rational numbers and especially he can order the real numbers linearly. Both qualities do not apply for the imaginary numbers, as pointed out above.

The application of the empirically founded calculation methods to imaginary numbers without a proper definition is problematic. The missing definition of the basic arithmetic operation for the square root of negative numbers leads to an ambiguity of the multiplication. On the one hand Euler writes:

> "In general, that by multiplying $\sqrt{-a}$ by $\sqrt{-a}$, or by taking the square of $\sqrt{-a}$ we obtain –a." (Euler, 1828, part 1 § 146)

On the other hand Euler states two paragraphs later:

> "Moreover, as $\sqrt{a}$ multiplied by $\sqrt{b}$ makes $\sqrt{ab}$, we shall have $\sqrt{6}$ for the value of $\sqrt{-2}$ multiplied by $\sqrt{-3}$;" (Euler, 1828, part 1 § 148)

Like Neumann points out, the attentive reader will have to ask himself how $\sqrt{-a}\sqrt{-a}$ has to be determined (Neumann, 2008, p. 118). Firstly it can be calculated $\sqrt{-a}\sqrt{-a}$ = $(\sqrt{-a})^2$ = –a, and secondly like this: $\sqrt{-a}\sqrt{-a}$ = $\sqrt{(-a)^2}$ = $\sqrt{a^2}$ = –a. Euler does not clarify this issue. [3] Remmert remarks to this problem, that "Euler occasionally makes some mistakes" (Remmert, 1991, p. 59). This is, however, not tenable, because it implies the existence of the definition of the multiplication of imaginary numbers. But an algebraic definition of the multiplication did not exist until Hamilton.

The question of the ontological status of imaginary numbers was not sufficiently answered. Scholz points out correctly, that the question has to be whether the knowledge about calculation methods is reason enough to award imaginary numbers with their own ontological status (Scholz, 1990, p. 294). It seems in the "Elements of Algebra" that any kind of algebraic expression based on empirically founded arithmetic operation, are ontologically justified due to the fact that these exist simply because of this operation. Besides the investigation of imaginary numbers Euler discusses algebraic expression such like $\frac{1}{0}$ in his Algebra. His statement has to been seen critically:

> "For $\frac{1}{0}$ signifying a number infinitely great and $\frac{2}{0}$ being incontestably the double of $\frac{1}{0}$, it is evident that a number, though infinitely great, may still become twice, thrice, or any number of times greater." (Euler, 1828, part 1 § 84)

In this regard, Jahnke draws attention to the fact that an abstract quantity simply can be determined by its occurrence as a variable in a formula. And due to this fact also objects, which cannot be interpreted empirically, can be referred under this concept (Jahnke, 2003, p. 106).

In contrary Kvasz does not believe that for imaginary numbers a complete detachment from the empirical subject area is possible. He writes:

> "Thus for Euler too these quantities exist only in our imagination. But this subjective interpretation of the complex numbers cannot explain how it is possible for computations involving these non-existent quantities to lead to valid results about the real world. […] If the complex numbers make it possible to disclose new knowledge about the world, they must be related to the real world in some way. A purely subjective interpretation is therefore unsatisfactory." (Kvasz, 2008, p. 182)

Euler's discussion of the imaginary numbers clearly shows that the ontological status must not be fully determined and that an axiomatic access to operate with expression as objects is not necessarily required. The known and established operations, which were initiated on the basis of empirical quantities, can be transferred to new, undefined expressions. The imaginary numbers do not belong to any known and empirically justified number system. Nevertheless, they exist, since they result by taking the square root of a negative number.

As it has been made clear in this chapter, Euler deals with symbolic expressions without referring directly to a real subject area. It is indeed wrong to assume that Euler justifies each step in his Algebra by referring to empirical objects. Furthermore, Euler introduces new concepts with regard to his basic objects, the empirical quantities, but subsequently handles them without reference to the domain of quantities. Euler handles and uses the concepts algebraically. At the beginning of his textbook he points out:

> "In Algebra, then we consider only numbers, which represent quantities, without regarding the different kinds of quantity." (Euler 1828, part 1 § 6)

The foundation of his approach remains the localisation of Algebra in the context of empirical quantities.

THE CONCEPT OF VARIABLE

The word "variable" is a term from the present day and is not used by Euler in the "Elements of Algebra". Euler describes the variables in terms of a sought number, unknown quantity or known numbers. Since for Euler a number is the ratio of two quantities, it could be expected that the unknown itself is no abstract entity to him.

Euler introduces the variables at the very beginning of the Algebra during the initiation of the basic arithmetic operations. Euler discusses arithmetical laws in this manner generally. He characterises the variable as follows:

> "All this is evident; and we have only to mention, that in Algebra, in order to generalise numbers, we represent them by letters, as a, b, c, d etc." (Euler, 1828, part 1 § 10)

In Euler's Algebra variables represent numbers. The use of these variables is to generalise a proposition and to be able to examine equations. Therefore, Euler needs a general symbolism and syntactic rules to operate with the letters. After demonstrating every arithmetic operation for examples they are applied to letter as variables.

Euler does not specify the conditions for arithmetical operations and laws. Therefore, he does not introduce a set to which the operation or law applies. He neither discusses the closure under the operation. Even for the generalisation of a ratio Euler does not limit the domain for the variable. In this context, Heuser mentioned in his introduction to real analysis that the calculation with letters can be handled as used from school, since there does not exist something newly learned regarding to the basic arithmetical operations (Heuser, H. (2009), p. 40). It can be said that in the same way, in Euler's Algebra, the transfer of the operations to the variables is familiar, because the variables just represent the numbers or quantities sought.

In German secondary schools nowadays the variable as concept is usually introduced as representation of a number or quantity. Also, known operations from the presented number system are transferred to the variables without further formal explanation, but with a visualisation of the validation regarding concrete quantities.

Euler uses letters as variables not only for the number sought, but also for given unknown numbers. During the discussion of solving quantities Euler states:

> "And, in general, if we have found x + a = b, where *a* and *b* express any known number, […]." (Euler 1828, part 1 § 574)

In order to solve the equation, Euler demonstrates the calculation methods for exemplified problems. During the problem solving Euler handles the variables as if they were concrete numbers. This can be clearly seen by this example:

> "In order to resolve this question, let us suppose that the number of men is = x; and, considering this number as known, we shall proceed in the same manner as we wished to try whether it corresponded with the conditions of the equation." (Euler, 1828, part 1 § 567)

Euler does not justify every transformation step during a calculation with regard to empirical quantities. As already described above, Euler discusses mental representations of empirical objects and uses empirically founded operations. Thus, a justification is given implicitly all the time by the nature of the processed objects.

EULER'S UNDERSTANDING OF ALGEBRA

The manner in which Euler introduced the concepts in the textbook as well as the introduction of properties provides justified conclusion about Euler's understanding of algebra. The previously gained insights into the understanding of the concept of numbers and variables shall be discussed with the help of the idea of empirical theories. [4]

Contrary to the modern understanding of algebra, which is focussed on the structure, Euler's ambition in the "Elements of Algebra" is to describe and explain empirical phenomena. He wants to develop a theory of algebra, which can help to solve problems of the natural environment. Since the "Elements of Algebra" is constituted as a textbook with the intention that an unskilled student can learn the algebra without further help, Euler starts his description with the basic objects of his theory. The basic objects are concrete, measurable quantities, because Euler defines the quantities through their empirical characteristic of the capability of increase and diminution. A natural number is defined through the ratio of quantities of the same kind. Based on this concept of numbers Euler extends the number system to the whole numbers and also the rational numbers with reference to a domain of quantities. Thus Euler introduces the numbers as representation of empirical objects. They are Elements of a real subject area. In the introduction of other concepts and laws Euler refers to the underlying empirical quantities. His justifications as shown above are intuitive. He calls on the common knowledge of the reader of empirical quantities and numerical examples. Thus it can be said that Euler fulfils in his Algebra the characteristics of an empirical algebraic theory regarding to a subject area.

Euler does not define his theory of algebra like modern mathematicians. The objects in this textbook are not composed abstract elements of a set, but rather representation of empirical objects. The properties of the numbers are not deduced from stated axioms but are derivated from the properties of the quantities. Similar the calculation laws are constituted related to a subject area. Thus, a formal proof or logical derivation from axioms is not required. Euler's characterisation of the imaginary numbers as indication of insolvability of a problems shows the necessity of verifying the statements empirically. This contrast modern understanding of mathematics, in which verification of a statement can only be attained by a formal proof.

He experiments with symbols like a scientist. Fraser's opinion about analysts can be transferred to Euler:

> For the 18^{th} century analyst, functions are things that are given ‚out there', in the same way that the natural scientist studies plants, insects or minerals, given in nature." (Fraser (2005), p. 246)

In the "Elements of Algebra" natural numbers are defined by the ratio of two quantities. In this sense the numbers are for Euler given objects of his Algebra with which he can experiment.

Imaginary numbers have a unique status in Euler's Algebra. They are not possible numbers, since they are not less than, equal to or more than zero. Imaginary numbers do not represent empirical objects and therefore they are imaginary. Nevertheless, they are created by the application of allowed operations on negative numbers. Thus, imaginary numbers exist and Euler applies the current operation on the undefined expressions. Imaginary numbers have no independent ontological status and thus cannot be discussed isolated from the operation, which creates them. Imaginary numbers only have a meaning in the context of the theory of algebra, since they exist through the defined operation within the Euler's theory of algebra. In this manner it can be said that Euler's approach in Algebra is empirical. [5]

NOTES

1. In this article the terms magnitude and quantity are used as synonyms.

2. It should be pointed out here, that in the original German version the formulation of the sentence leads to a stricter interpretation of Euler's understanding of numbers. The understanding of numbers as quantities themselves is more clearly expressed. Euler writes: "Hence, they used to consider in the algebra numbers with the preceding sign as a single quantity." (Ibid. Euler, 1770)

3. Indeed Neumann notes correctly that an ambiguity of the multiplication exists here, but he himself makes a mistake by formulating the two different ways to handle the equation. Instead of taking the square of $\sqrt{-a}$, he calculates $\sqrt{-a}\sqrt{-a} = \sqrt{(-a)^2} = -a$ for the first possibility.

4. Compare the approach of empirical theories by Balzer & Moulines & Sneed (1987) and Burscheid & Struve (2010).

5. For further discussion compare Reimann & Witzke (2013).

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